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Chebyshev Polynomial Approximation to Solutions of Ordinary Differential Equations

Amber Sumner Robertson
University of Southern Mississippi

The University of Southern Mississippi

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ABSTRACT

CHEBYSHEV POLYNOMIAL APPROXIMATION TO SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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In this thesis, we develop a method for finding approximate particular solutions for second order ordinary differential equations. We use Chebyshev polynomials to approximate the source function and the particular solution of an ordinary differential equation. The derivatives of each Chebyshev polynomial will be represented by linear combinations of Chebyshev polynomials, and hence the derivatives will be reduced and differential equations will become algebraic equations. Another advantage of the method is that it does not need the expansion of Chebyshev polynomials. This method is also compared with an alternative approach for particular solutions. Examples including approximation, particular solution, a class of variable coefficient equation, and initial value problem are given to demonstrate the use and effectiveness of these methods.

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF TABLES

LIST OF ABBREVIATIONS

ODE - Ordinary Differential Equations
PDE - Partial Differential Equations
MPS - Method of Particular Solutions
IVP - Initial Value Problem

1 Introduction

In this thesis, we consider a linear second order ordinary differential equation (ODE),

$$L(y) = ay''(x) + by'(x) + cy(x) = f(x) \tag{1}$$

where $f(x)$ is a continuous function, a, b, and c are given constants, and $a \neq 0$. We are interested to find a particular solution y_p for Equ. (1).

1.1 Applications of Second Order Ordinary Differential Equations

The second order ordinary differential equation (1) can model many different phenomena that we encounter every day. For example, the equation

$$\frac{d^2y}{dt^2} = -ky, \text{ with } k > 0$$

represents what happens when an object is subject to a force towards an equilibrium position with the magnitude of the force being proportional to the distance from equilibrium. The above equation can be considered as an approximation to the equation of motion of a particular point on the basilar membrane, or anywhere else along the chain of transmission between the outside air and the cochlea [10]. Thus the solution of the differential equation can help explain the perception of pitch and intensity of musical instruments by human ears.

We may also use a second order ODE to model at what altitude a skydiver's parachute must open before he reaches the ground so that he lands safely. To model this phenomenon, we let y denote the altitude of the skydiver. We consider Newton's second law

$$F = ma \tag{2}$$

where F represents the force, m represents the mass of the skydiver and a represents the acceleration. We assume that the only forces acting on the skydiver are air resistance and gravity when the skydiver falls through the air toward the earth. We also assume that the air resistance is proportional to the speed of the skydiver with b being the positive constant of proportionality known as the damping constant. The Newton's second law (2) then translates to

$$y'' + \frac{b}{m}y' = g. \tag{3}$$

If we know the initial height y_0 and velocity v_0 at the time of jump, we have an initial value problem which is the equation (3) together with the initial conditions,

$$y(0) = y_0,$$
$$y'(0) = v_0.$$

As another application, the second order ODE can model a damped mass-spring oscillator that consists of a mass m that is attached to a spring fixed at one end [?, 6]. Taking into account the forces acting on the spring due to the spring elasticity, damping friction, and other external influences, the motion of the mass-spring oscillator is governed by the differential equation

$$my'' + by' + ky = F_{ext}(t) \tag{4}$$

where b (≥ 0) is the damping coefficient and k (≥ 0) is known as the stiffness of the spring. The differential equation (4) is derived by using Newton's second law and Hooke's law.

Yet another application the equation (1) can model is an electrical circuit consisting of a resistor, capacitor, inductor, and an electromotive force [6]. With charge being the function, we can obtain an initial value problem of the form,

$$\begin{aligned} Lq''(t) + Rq'(t) + \frac{q(t)}{C} &= E(t), \\ q(0) &= q_0, \\ q'(0) &= I_0, \end{aligned}$$

where L is the inductance in henrys, R is the resistance in ohms, C is the capacitance in farads, $E(t)$ is the electromotive force in volts, and $q(t)$ is the charge in coulombs on the capacitor at time t.

We can easily see that the applications of the second order ODE has a wide scope that it is applicable in many fields of study. A general form of these equations is given by Eq. (1). When the right hand side function $f(x)$ is zero, it is a homogeneous equation; otherwise, it is a nonhomogeneous equation.

1.2 Particular Solution of an Ordinary Differential Equation

We hope to be able to find a particular solution for Eq. (1) that is nonhomogeneous. A particular solution of Eq. (1) is a function that satisfies Eq. (1). The particular solution to an ordinary differential equation can be obtained by assigning numerical values to the parameters in the general solution [3]. We note that there are many possible answers for a particular solution.

A particular solution y_p will allow us to reduce, for example, an initial value problem

$$\left\{ \begin{array}{c} L(y) = ay''(x) + by'(x) + cy(x) = f(x), \\ y(x_0) = y_0,\ y'(x_0) = y_1, \end{array} \right. \tag{5}$$

to a homogeneous equation that is subject to a different initial data

$$\left\{ \begin{array}{c} L(y_h) = ay_h''(x) + by_h'(x) + cy_h(x) = 0, \\ y_h(x_0) = y_0 - y_p\left(x_0\right),\ y_h'(x_0) = y_1 - y_p'\left(x_0\right), \end{array} \right. \tag{6}$$

with $y_h = y - y_p$. The method of particular of solutions (MPS) allows us to split the solution y into a particular solution y_p and a homogeneous solution y_h.

In this thesis, we use Chebyshev polynomials for approximating equations and their particular solutions. Chebshev polynomials [1] of the first kind are solutions to the Chebyshev differential equations

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y=0 \text{ for } |x|<1,$$

and Chebshev polynomials of the second kind are solutions to the Chebyshev differential equations

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+n\left(n+2\right)y=0 \text{ for } |x|<1.$$

The Chebyshev polynomials of either kind are a sequence of orthogonal polynomials that can also be defined recursively. The motivation for Chebyshev interpolation is to improve control of the interpolation error on the interpolation interval [9]. The MPS and Chebyshev polynomials have been used for solving partial differential equations (PDE). For example, the paper [13] solved elliptic partial differential equation boundary value problems; the paper [11] studied two dimensional heat conduction problems and the authors used Chebyshev polynomials and the trigonometric basis functions to approximate their equations for each time step. In their two-stage approximation scheme, the use of Chebyshev polynomials in stage one is because of the high accuracy (spectral convergence) of Chebyshev interpolation.

If the right hand side function $f(x)$ in Eq. (1) is a polynomial of degree n, i.e.,

$$\begin{aligned} f(x) &= P_n(x) \\ &= d_n x^n + d_{n-1}x^{n-1} + ... + d_1 x + d_0 \end{aligned}$$

where $d_n \neq 0$, we can find a particular solution of Eq. (1) that is a polynomial.

If $f(x)$ is not a polynomial, we approximate it using Chebyshev polynomials. We then look for a particular solution that is expressed as a linear combination of Chebyshev polynomials. Our choice of Chebyshev polynomials is because of their high accuracy. The Chebyshev polynomial is very close to the **minimax polynomial** which (among all polynomials of the same degree) has the smallest maximum deviation from the true function $f(x)$. The **minimax criterion** is that $P_n(x)$ is the polynomial of degree n for which the maximum value of the error, which is defined by $e_n(x) = f(x) - P_n(x)$, is a minimum within the specified range of $-1 \leq x \leq 1$ [1]. This is extremely ideal for polynomial approximations.

In our approach, the derivatives of each Chebyshev polynomial will be represented by the linear combinations of Chebyshev polynomials, and hence the differential equations will become algebraic equations. In Chapter 2, we introduce the approximation method and illustrations using Chebyshev polynomials. In Chapter 3, we describe our method for finding a particular solution or an approximate particular solution by the approximation and reduction of order. An alternative method by superposition principle is used in Chapter 4 for ODEs. These two methods are compared through examples. Conclusions are made in Chapter 5.

2 Approximation using Chebyshev Polynomials

The Chebshev polynomials of the first and second kind are denoted by $T_n(x)$ and $U_n(x)$ respectively. The subscript n is the degree of these polynomials. The Chebshev polynomials of the first and second kind are closely related. For example, a Chebyshev polynomial of first kind can be represented as a linear combination of two Chebyshev polynomials of second kind,

$$T_n(x) = \frac{1}{2}\left(U_n\left(x\right) - U_{n-2}\left(x\right)\right),$$

and the derivative of a Chebyshev polynomial of first kind can be written in terms of a Chebyshev polynomial of second kind,

$$T_n'(x) = nU_{n-1}(x),\ n = 1,\ 2,\ \cdots.$$

In this paper, we direct our attentation to the Chebyshev polynomials of first kind and we use them for approximating a function and a particular solution for second order ODEs.

2.1 Method of Chebyshev Polynomial Approximation

In this section, we give an introduction to the Chebyshev polynomials and their basic properties. See the references [1, 9, 5, 2] for more details.

Chebyshev polynomials of the first kind are denoted by T_n and the first several polynomials are listed below:

$$\begin{aligned}
T_0(x) &= 1,\\
T_1(x) &= x,\\
T_2(x) &= 2x^2 - 1,\\
T_3(x) &= 4x^3 - 3x,\\
T_4(x) &= 8x^4 - 8x^2 + 1,\\
T_5(x) &= 16x^5 - 20x^3 + 5x,\\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1,\\
&\cdots.
\end{aligned}$$

The first six Chebyshev polynomials T_j, $j = 0, 1, \cdots, 6$ are shown in Figure 1.

A Chebyshev polynomial can be found using the previous two polynomials by the recursive formula

$$T_{n+1} = 2xT_n(x) - T_{n-1}(x),\ \text{for}\ n \geq 1.$$

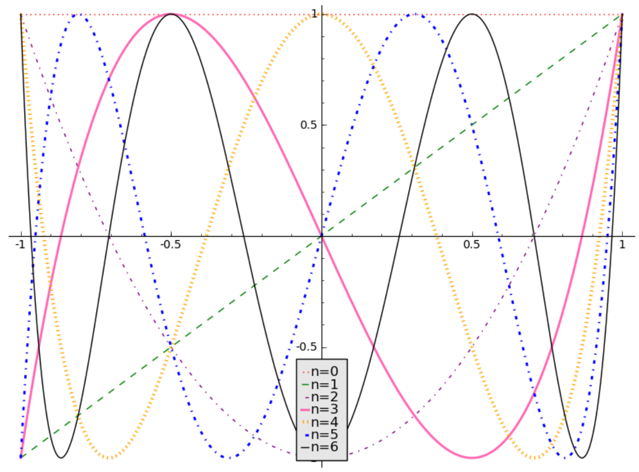

Figure 1: Plot of Chebyshev polynomials T_n, $n = 1, 2, \cdots, 6$

[1] The leading coefficient of the $T_n(x)$ is 2^{n-1} for $n \geq 1$ as we can see from the recursive formula shown above. These polynomials are orthogonal with respect to the weight function $\dfrac{1}{\sqrt{1-x^2}}$ on the interval $[-1, 1]$,

$$\int_{-1}^{1} T_i(x) T_j(x) \frac{1}{\sqrt{1-x^2}} dx = \begin{cases} 0, & i \neq j, \\ \pi, & i = j = 0, \\ \frac{\pi}{2}, & i = j \neq 0. \end{cases}$$

As we have mentioned, $T_n(x)$ denotes the Chebyshev polynomial of degree n. It has n roots, which are also known as Chebyshev nodes. These nodes [5] can be calculated by the formula

$$x_i = cos\left(\frac{i-\frac{1}{2}}{n}\right)\pi, \text{ for } i = 1, 2, \ldots, n.$$

A function $f(x)$ can be approximated [4] by an n-th degree polynomial $P_n(x)$ expressed in terms of $T_0, \ldots, T_n$,

$$P_n(x) = C_0T_0(x) + C_1T_1(x) + \ldots + C_nT_n(x) - \frac{1}{2}C_0 \tag{7}$$

where

$$C_j = \frac{2}{n}\sum_{k=1}^{n+1} f(x_k)T_j(x_k), \; j = 0, 1, \cdots, n. \tag{8}$$

and x_k, $k = 1, \ldots, n+1$ are zeros of T_{n+1}.

Since

$$T_j(x) = \cos(j \arccos x),$$

we have

$$T_j(x_k) = \cos(j \arccos x_k)$$
$$= \cos\left(\frac{j\left(k-\frac{1}{2}\right)}{n+1}\pi\right).$$

Let $f(x)$ be a continuous function defined on the interval $[-1, 1]$, that we want to approximate by a polynomial P_n defined by (7) and (8). We can measure how good an approximation is of $f(x)$ by the uniform norm,

$$||f - P_n|| = \max_{-1\leq x\leq 1} |f(x) - P_n(x)|.$$

This means that the measure of the error of the approximation is given by the greatest distance between $f(x)$ and $P_n(x)$ with x going through the interval $[-1, 1]$[5].

For m that is much less than n, $f(x) \approx C_0T_0(x)+C_1T_1(x)+\ldots+C_mT_m(x)-\frac{1}{2}C_0$, is a truncated approximation with $C_{m+1}T_{m+1}(x) + \ldots + C_nT_n(x)$ being the truncated part of the sum. The error in this approximation is dominated by the leading term of the truncated part $C_{m+1}T_{m+1}(x)$ since typically the coefficients C_k are repidly decreasing. We know that the $m + 2$ equal extrema of $T_{m+1}(x)$ spread out smoothly over the interval $[-1, 1]$, thus the error spreads out smoothly over the interval $[-1, 1]$.

The equations. (7) and (8) are used to approximate a function $f(x)$ defined on the interval $[-1, 1]$. For a function $f(x)$ that is defined on an interval $[a, b]$, we can obtain $f(y)$ with $y \in [-1, 1]$ by a change of variable

$$y \equiv \frac{x - \frac{1}{2}(b + a)}{\frac{1}{2}(b - a)}.$$

2.2 Illustrations of Chebyshev Polynomial Approximation

Let P_n be the n^{th} degree polynomial approximation to the function $f(x)$. Example 2.2.1. Let $f(x) = \frac{3}{125}x^3 - \frac{x}{80} + 1$. The function $f(x)$ and its Chebyshev polynomial approximation P_3 are shown in Figure 2. The approximation polynomial is identical to the function $f(x)$ as we expected.

Example 2.2.2. Let $f(x) = x^2 \sin(10x)$. Its Chebyshev polynomial approximation P_n with $n = 1$, 4, 9, 16 are shown in Figures 3-6. As n increases, the error of the approximation polynomial decreases.

3 An Approximate Particular Solution

3.1 Method of Reduction of Order

We recall that we can reduce an initial value differential equation problem to an albegraic equation through the Laplace transform [12, 6]. Here we propose

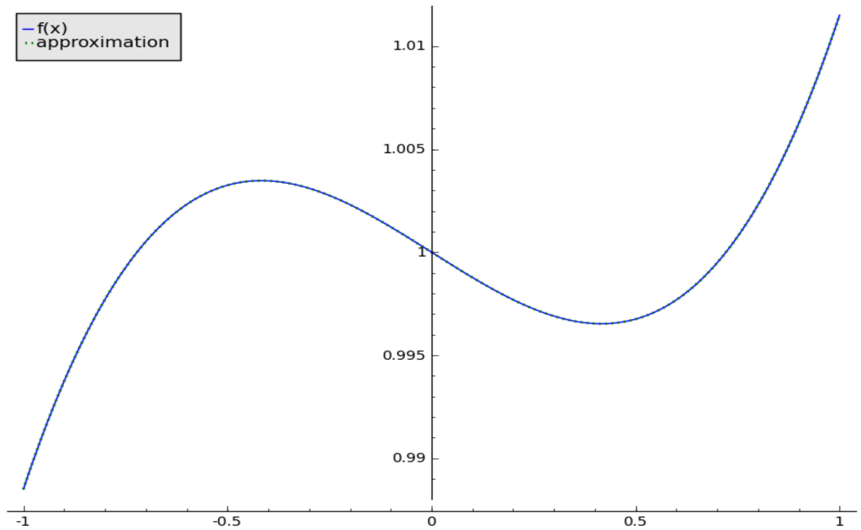

Figure 2: Plot of P_3

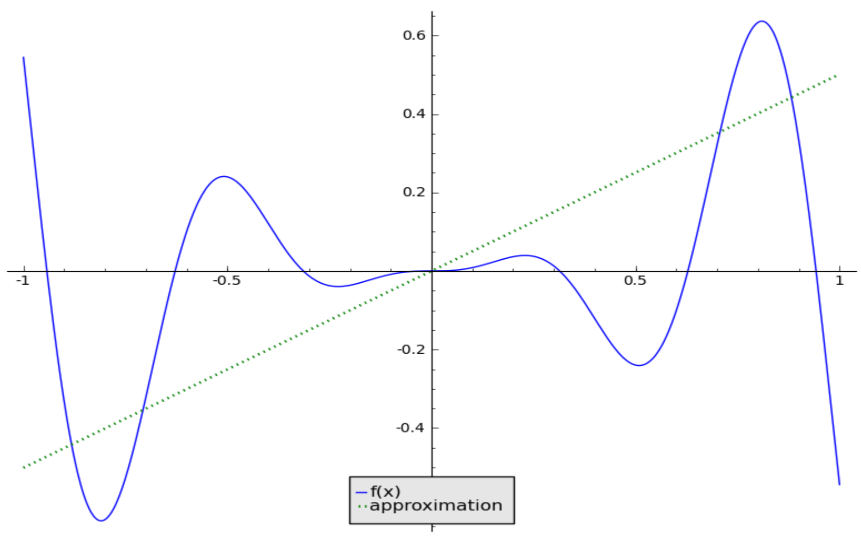

Figure 3: Plot of $f(x)$ with T_1

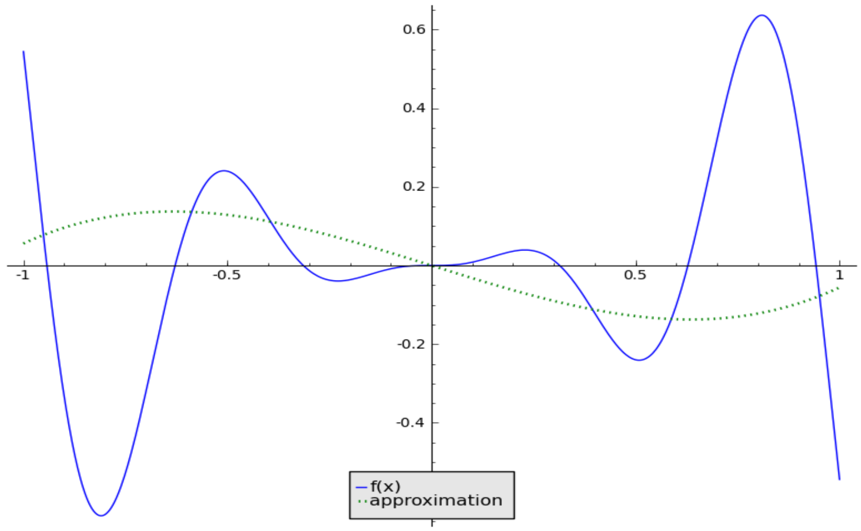

Figure 4: the plot of $f(x)$ with T_4

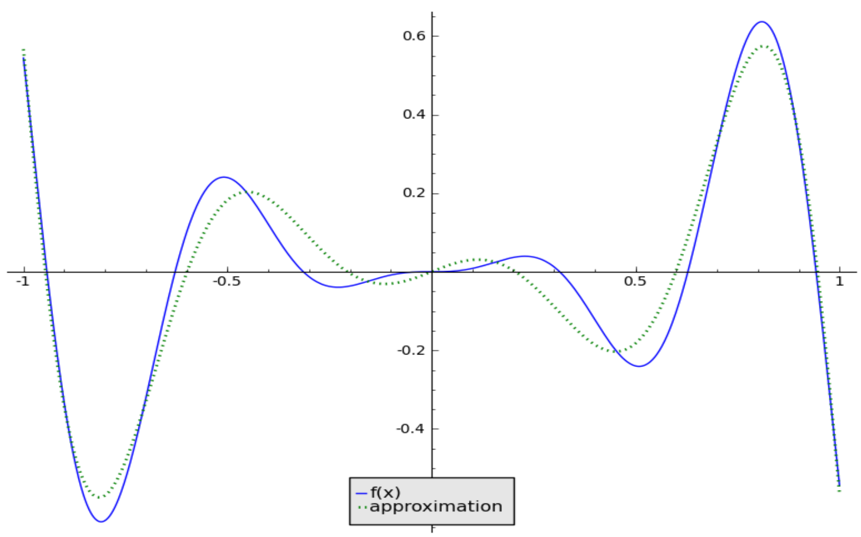

Figure 5: the plot of $f(x)$ with T_9

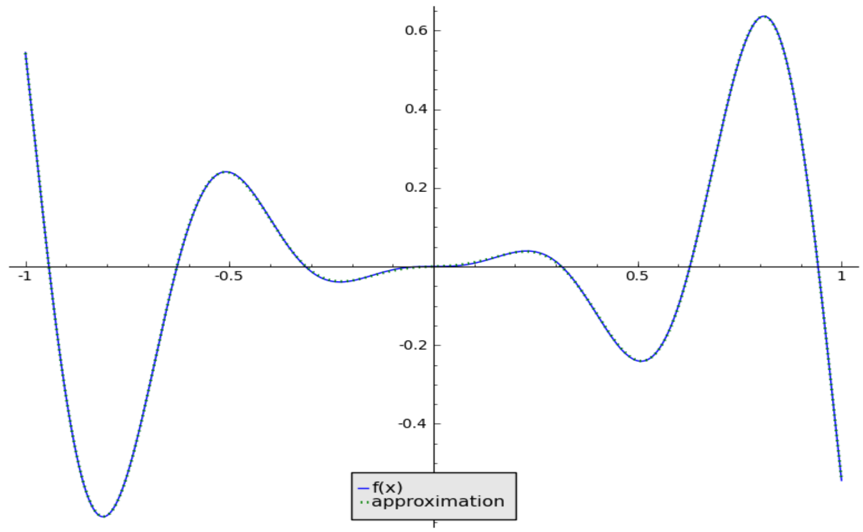

Figure 6: the plot of $f(x)$ with T_{16}

a method that achieves the similar goal using Chebyshev polynomial approximation and order reduction techniques.

By defining the Laplace transform to $f(t)$,

$$\mathcal{L}\left\{f(t)\right\} = \int_0^{\infty} e^{-st} f(t)dt = F(s),$$

we get a new function $F(s)$. Assume that the functions f, f', $\ldots$, $f^{(n-1)}$ are continuous and that $f^{(n)}$ is piecewise continous on any interval $0 \le t \le A$. Suppose further that there exist constants K, a, and M such that $|f(t)| \le Ke^{at}$, and $|f'(t)| \le Ke^{at}$, $\cdots$, $|f^{(n-1)}(t)| \le Ke^{at}$ for $t \ge M$. Then $\mathcal{L}\left\{f^{(n)}(t)\right\}$ exists for $s > a$ and is given by,

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n \mathcal{L}\left\{f(t)\right\} - s^{n-1}f(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

If the solution $y(t)$ of the equation (1) **,** when substituted for f, satisfies the above condition for $n = 2$, we can apply the Laplace transform to (1) to get the following algebraic equation,

$$a[s^2Y(s) - sy(0) - y'(0)] + b[sY(s) - y(0)] + cY(s) = F(s)$$

where $Y(s)$ and $F(s)$ are respectively the Laplace transforms of $y(t)$ and $f(t)$. By solving the above equation for $Y(s)$, we find that

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2+bs+c} + \frac{F(s)}{as^2+bs+c}.$$

With the given initial conditions $y(0)$ and $y'(0)$, we can find a unique solution $Y(s)$. The inverse Laplace transform $\mathcal{L}^{-1}\{Y(s)\}$ will give the solution to the original initial value problem.

When employing the Laplace transform for an initial value problem, we first transform a differential equation into an algebraic one by taking into account the given initial data. Then we obtain the solution by inverse Laplace transform. In our proposed MPS, we express a particular solution and the derivatives of the particular solution by Chebyshev polynomials. Then using identities to obtain a system of algebraic equations the coefficients should satisfy. Thus we find a particular solution by solving algebraic equations. To get the solution of an IVP, we need to get the solution of the corresponding homogenous problem.

We first approximate $f(x)$ in Eq. (1) by $P_n(x)$ using Chebyshev polynomials. Then we find a particular solution y_p of the equation

$$ay''(x) + by'(x) + cy(x) = P_n(x). \tag{9}$$

We let the particular solution be in the form of

$$y_p = \sum_{j=0}^{m} q_j T_j(x). \tag{10}$$

The coefficients q_j for (10) are to be determined. We let $m = n$ if $c \neq 0$ in Equ. (9); $m = n+1$ if $c = 0$, $b \neq 0$; $m = n+2$ if $c = 0$, $b = 0$.

Substituting (10) into (9),

$$a\sum_{j=0}^{m} q_j T_j''(x) + b\sum_{j=0}^{m} q_j T_j'(x) + c\sum_{j=0}^{m} q_j T_j(x) = P_n(x).$$

Our next goal is to use linear combinations of Chebyshev polynomials to represent $T_j'(x)$ and $T_j''(x)$ for each j. That is, the first and second order derivatives of Chebyshev polynomials $T_j'(x)$ and $T_j''(x)$ are represented in terms of $T_k(x)$, $k = 0, \cdots, j$. Then we arrive at a system of algebraic equations of q_j by eqating coefficients to find the solution y_p given by (10).

According to [8] and its tables for representation coefficients, we reduce the first and second order derivatives as follows

$$T_j'(x) = \sum_{k=0}^{j-1} b_k T_k(x), \tag{11}$$

where

$$b_{2l} = 0, \text{ for } l = 0, 1, \ldots, \frac{j}{2} - 1,$$
$$b_{2l+1} = 2j, \text{ for } l = 0, 1, \ldots, \frac{j}{2} - 1,$$

for even j, and

$$b_0 = j,$$
$$b_{2l} = 2j, \text{ for } l = 1, \ldots, \frac{j-1}{2},$$
$$b_{2l+1} = 0, \text{ for } l = 0, 1, \ldots, \frac{j-3}{2},$$

for odd j.

Next, we expand the second order derivative $T_j''(x)$ in terms of Chebyshev polynomials $T_i(x)$, $i = 0, \ldots, j-2$. With c_i being the representation coefficients for $T_j''(x)$, that is,

$$T_j''(x) = \sum_{i=0}^{j-2} c_i T_i(x). \tag{12}$$

We use the tables provided in [8] to calculate c_i, for $i = 0, \ldots, j-2$. By Eq. (11),

$$\begin{aligned} T_j''(x) &= \left(T_j'(x)\right)' \\ &= \left(\sum_{k=0}^{j-1} b_k T_k(x)\right)' \\ &= \sum_{k=0}^{j-1} b_k T_k'(x) \\ &= \sum_{k=0}^{j-1} b_k \left(\sum_{i=0}^{k-1} b_i T_i(x)\right) \\ &= \sum_{k=0}^{j-1} \sum_{i=0}^{k-1} b_k b_i T_i(x) \end{aligned} \tag{13}$$

3.2 Examples of Approximation

Example 3.1.1. We consider the equation

$$y''(x) + 2y(x) = P_4(x)$$

with $P_4(x) = 2T_1(x) + 5T_4(x)$. We look for a particular solution $y_p = \sum_{j=0}^{4} q_j T_j(x)$.

By (11)-(13), we list the coefficients for $T_j'(x)$, $j = 0, 1, \cdots, 4$ in Table 1,which is equivalent to the following,

$$\begin{aligned} (T_0)' &= 0, \\ (T_1)' &= T_0(x), \\ (T_2)' &= 4T_1(x), \\ (T_3)' &= 3T_0(x) + 6T_2(x), \\ (T_4)' &= 8T_1(x) + 8T_3(x). \end{aligned}$$

j=0				
j=1	$b_0 = 1$			
j=2	$b_0 = 0$	$b_1 = 4$		
j=3	$b_0 = 3$	$b_1 = 0$	$b_2 = 6$	
j=4	$b_0 = 0$	$b_1 = 8$	$b_2 = 0$	$b_3 = 8$

Table 1: Coefficients for $T_j'(x)$, $j = 0, 1, \cdots, 4$

$j = 0$	
$j = 1$	$b_0 = 1,$
$j = 2$	$b_0 = 0$ $b_1 = 4,\ b_1 \cdot b_0 = 4 \cdot 1$
$j = 3$	$b_0 = 3,$ $b_1 = 0,\ b_1 \cdot b_0 = 0 \cdot 1$ $b_2 = 6,\ b_2 \cdot b_0 = 6 \cdot 0,\ b_2 \cdot b_1 = 6 \cdot 4$
$j = 4$	$b_0 = 0,$ $b_1 = 8,\ b_1 \cdot b_0 = 8 \cdot 1$ $b_2 = 0,\ b_2 \cdot b_0 = 0 \cdot 0,\ b_2 \cdot b_1 = 0 \cdot 4$ $b_3 = 8,\ b_3 \cdot b_0 = 8 \cdot 3,\ b_3 \cdot b_1 = 8 \cdot 0,\ b_3 \cdot b_2 = 8 \cdot 6$

Table 2: Coefficients for $T_j''(x)$, $j = 0, 1, \cdots, 4$

Thus,

$$y_p' = (q_1 + 3q_3)T_0(x) + (4q_2 + 8q_4)\,T_1(x) + 6q_3T_2(x) + 8q_4T_3(x). \tag{14}$$

The coefficients for second order derivatives $T_j''(x)$, $j = 0, 1, \cdots, 4$ are listed in Table 2 below,

Now we obtain the second order derivatives for T_j, $j = 0, 1, \cdots, 4$. We list the results as follows:

$$\begin{aligned}
(T_0)'' &= 0, \\
(T_1)'' &= 0, \\
(T_2)'' &= 4T_0(x), \\
(T_3)'' &= 24T_1(x), \\
(T_4)'' &= 32T_0(x) + 48T_2(x).
\end{aligned}$$

We notice, for example, $c_0 = 32$, $c_1 = 0$, and $c_2 = 48$ for (12) when $j = 4$.

Therefore,

$$\begin{aligned}
y_p'' &= 4q_2T_0(x) + 24q_3T_1(x) + 32q_4T_0(x) + 48q_4T_2(x) \\
&= (4q_2 + 32q_4)\,T_0(x) + 24q_3T_1(x) + 48q_4T_2(x).
\end{aligned}$$

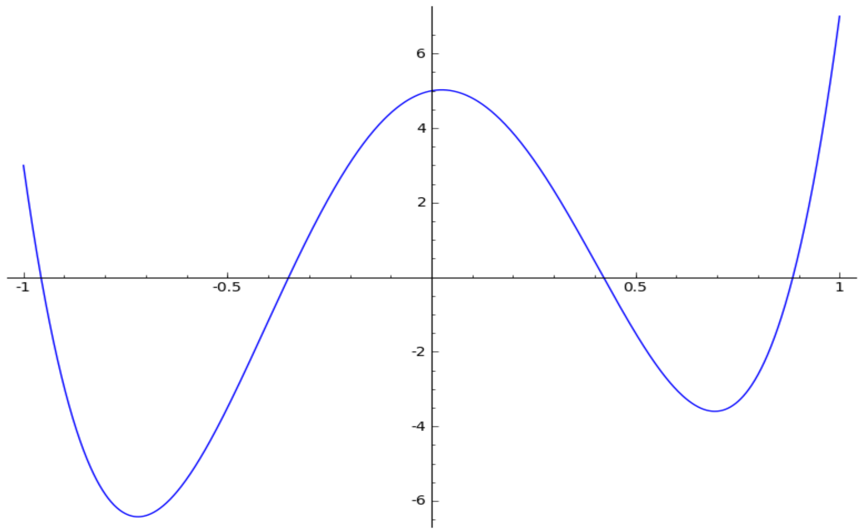

Figure 7: Plot of the right hand side function in Example 3.1.1

We obtain the following linear system of equations by comparing coefficients of T_j, $j = 0, 1, \cdots, 4$.

$$\begin{cases} 2q_0 + 4q_2 + 32q_4 = 0, \\ 2q_1 + 24q_3 = 2, \\ 2q_2 + 48q_4 = 0, \\ 2q_3 = 0, \\ 2q_4 = 5. \end{cases} \tag{15}$$

The solution for (15) is $q_4 = 5/2$, $q_3 = 0$, $q_2 = -60$, $q_1 = 1$, $q_0 = 80$. The plots of the right hand side function and the particular solution are shown in Figures 7-8.

Assume we use P_n defined by (7)-(8) with $n = 4$ to approximate the function $f(x)$ in Eq. (1). Without loss of generality, we assume $c \neq 0$ in (1). Using our method of reduction of order, the coefficients q_j, $j = 0, 1, \cdots, 4$ of the particular solution satisfy the following system of equations,

$$\begin{aligned} cq_0 + bq_1 + 4aq_2 + 3bq_3 + 32aq_4 &= \frac{1}{2}C_0, \qquad (16) \\ cq_1 + 4bq_2 + 24aq_3 + 8bq_4 &= C_1, \\ cq_2 + 6bq_3 + 48aq_4 &= C_2, \\ cq_3 + 8bq_4 &= C_3, \\ cq_4 &= C_4. \end{aligned}$$

Example 3.1.2. In this example, we consider

$$y''(x) + y'(x) + 1.25y(x) = 3x^3 + x^2 + 2x + 7.$$

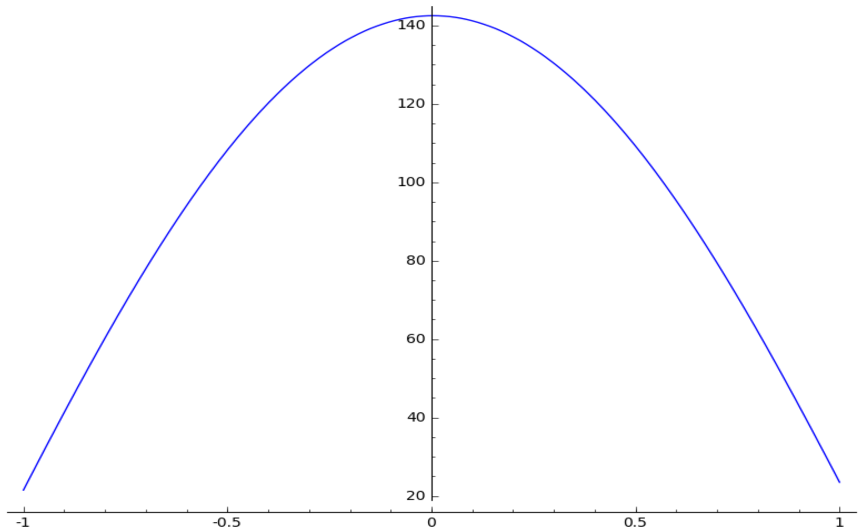

Figure 8: Plot of the particular solution for Example 3.1.1

We notice here the function $f(x) = 3x^3 + x^2 + 2x + 7$. We can still look for a particular solution in the form of $y_p = \sum_{j=0}^{4} q_j T_j(x)$, but q_4 will be 0 as expected. The Chebyshev polynomial representation for $f(x)$ is

$$f(x) = \sum_{j=0}^{3} p_j T_j(x)$$

with $p_0 = 15/2$, $p_1 = 17/4$, $p_2 = 1/2$, $p_3 = 3/4$. According to the system of equations (16), we obtain the following system for q_j, $j = 0, \ldots, 4$,

$$\begin{aligned}
1.25q_0 + q_1 + 4q_2 + 3q_3 + 32q_4 &= \frac{15}{2}, \\
1.25q_1 + 4q_2 + 24q_3 + 8q_4 &= \frac{17}{4}, \\
1.25q_2 + 6q_3 + 48q_4 &= \frac{1}{2}, \\
1.25q_3 + 8q_4 &= \frac{3}{4}, \\
1.25q_4 &= 0.
\end{aligned}$$

Using backward substitution, we get $q_4 = 0$, $q_3 = 3/5$, $q_2 = -62/25$, $q_1 = -23/125$, $q_0 = 7902/625$.

Example 3.1.3. In this example, we consider the Cauchy-Euler equations that are expressible in the form of

$$ax^2y''(x) + bxy'(x) + cy(x) = h(x),$$

where a, b and c are constants. This important class of variable coefficient differential equations can be solved by the particular solutions method of reduction of order. As an illustration, we solve a specific Cauchy-Euler equation as follows,

$$x^2y''(x) - 2xy'(x) + 2y(x) = x^3. \tag{17}$$

By a change of variable $x = e^t$, the equation (17) can be transformed into the constant coefficient equation in the new independent variable t,

$$y''(t) - 3y'(t) + 2y(t) = e^{3t}.$$

Using the method described above, we approximate the right hand side function by Chebyshev polynomial approximation, then we use the reduction of order method to solve for the approximate particular solution.

The approximation to $f(t) = e^{3t}$ by P_4 is

$$f(t) = \frac{C_0}{2}T_0(x) + C_1T_1(x) + C_2T_2(x) + C_3T_3(x) + C_4T_4(x),$$

where the numerical values of $C_0, \ldots, C_4$ are given as follows,

$$\begin{aligned}
\frac{C_0}{2} &= 4.88075365707809, \\
C_1 &= 7.90647046809621, \\
C_2 &= 4.48879683378407, \\
C_3 &= 1.9105629481, \\
C_4 &= 0.608043176705983.
\end{aligned}$$

Solving the resulting system of equations,

$$\begin{aligned}
2q_0 - 3q_1 + 4q_2 - 9q_3 + 32q_4 &= \frac{1}{2}C_0, \\
2q_1 - 12q_2 + 24q_3 - 24q_4 &= C_1, \\
2q_2 - 18q_3 + 48q_4 &= C_2, \\
2q_3 - 24q_4 &= C_3, \\
2q_4 &= C_4,
\end{aligned}$$

we get

$$\begin{aligned}
q_0 &= 344.481628331765, \\
q_1 &= 170.637487061598, \\
q_2 &= 36.3797465297907, \\
q_3 &= 4.60354053428590, \\
q_4 &= 0.304021588352991,
\end{aligned}$$

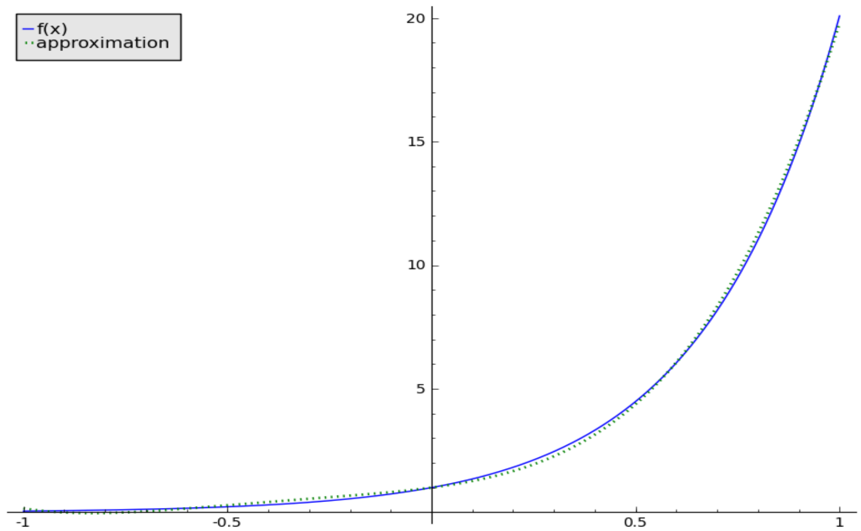

Figure 9: Plot of $f(t)$ and P_4 of Example 3.1.3

which are coefficients of $T_j(x)$ for an approximate particular solution $y_p(t)$ of the constant coefficient equation. The plots of the right hand side function and the particular solution are shown in Figures 9-10. An approximate particular solution of the Cauchy-Euler equation is $y_p(\ln x)$.

Example 3.1.4. As another example we consider the initial value problem,

$$y'' + 3y' - 4y = \sin(2x),$$

with initial data,

$$y(0) = 1,$$
$$y'(0) = 2.$$

First, the Chebyshev approximation of $\sin(2x)$ by P_4 is,

$$\sin(2x) \approx \frac{C_0}{2}T_0(x) + C_1T_1(x) + C_2T_2(x) + C_3T_3(x) + C_4T_4(x),$$

where the coefficients $C_0, \ldots, C_4$ are

$$\frac{C_0}{2} = 5.55111512312578e^{-17},$$
$$C_1 = 1.15344467691257,$$
$$C_2 = 0,$$
$$C_3 = 0.257536611098051,$$
$$C_4 = 2.22044604925031e^{-16}.$$

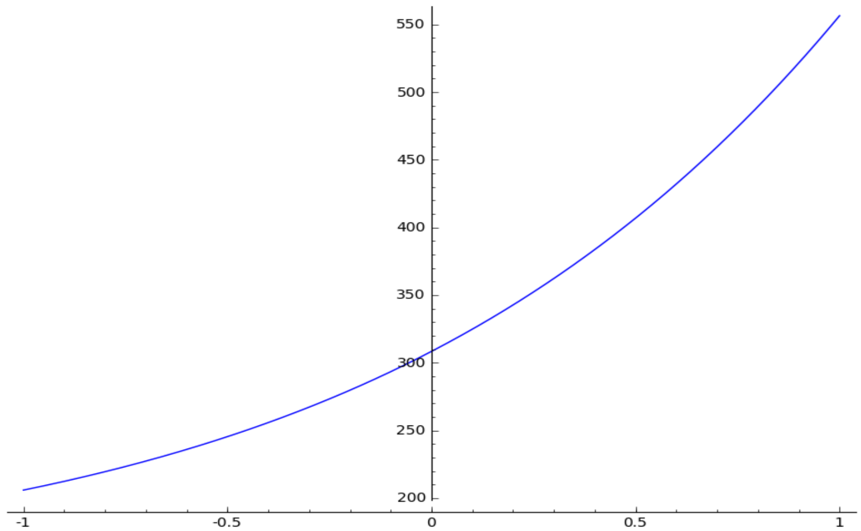

Figure 10: Plot of an approximate particular solution of Example 3.1.3

The resulting system by the reduction of order method with $n = 4$ gives

$$\begin{aligned}
-4q_0 + 3q_1 + 4q_2 + 9q_3 + 32q_4 &= \frac{1}{2}C_0, \\
-4q_1 + 12q_2 + 24q_3 + 24q_4 &= C_1, \\
-4q_2 + 18q_3 + 48q_4 &= C_2, \\
-4q_3 + 24q_4 &= C_3, \\
-4q_4 &= C_4.
\end{aligned}$$

The coefficients $q_0, \ldots, q_4$ for the particular solution of the IVP is given by

$$\begin{aligned}
q_0 &= 1.15994038863409 \\
q_1 &= 0.9671298098748495 \\
q_2 &= 0.289728687485306 \\
q_3 &= 0.0643841527745125 \\
q_4 &= -5.55111512312578e^{-17}
\end{aligned}$$

So we arrive at the particular solution,

$$y_p(x) = q_0T_0(x) + q_1T_1(x) + q_2T_2(x) + q_3T_3(x) + q_4T_4(x).$$

The derivative of y_p is given by (14) and thus

$$\begin{aligned} y_p'(0) &= (q_1 + 3q_3)T_0(0) + (4q_2 + 8q_4)\, T_1(0) + 6q_3T_2(0) + 8q_4T_3(0) \\ &= (q_1 + 3q_3) * 1 + (4q_2 + 8q_4) * 0 + 6q_3 * (-1) + 8q_4 * 0 \\ &= q_1 - 3q_3. \end{aligned}$$

This together with

$$\begin{aligned} y_p(0) &= q_0T_0(0) + q_1T_1(0) + q_2T_2(0) + q_3T_3(0) + q_4T_4(0) \\ &= q_0 * 1 + q_1 * 0 + q_2 * (-1) + q_3 * 0 + q_4 * 1 \\ &= q_0 - q_2 + q_4 \end{aligned}$$

gives the initial data for the corresponding homogeneous problem,

$$y'' + 3y' - 4y = 0,$$

that is subject to the initial data,

$$y(0) = 1 - y_p(0) = 1 - q_0 + q_2 - q_4 =,$$

$$y'(0) = 2 - y_p'(0) = 2 - q_1 + 3q_3 = .$$

We need to find the solution y_h to the above homogeneous problem so that we can obtain the numerical solution to the original IVP. The homogenous solution is,

$$y_h = c_1e^{-4x} + c_2e^x$$

where $c_1 = -0.219246869919495$ and $c_2 = 0.349035168770711$.

Now we obtain the numerical solution of the IVP as follows,

$$\widetilde{y}(x) = y_p - 0.219246869919495e^{-4x} + 0.349035168770711e^x$$

The exact solution of the problem is

$$y(x) = \frac{1}{20}\cos(2x) - \frac{1}{20}\sin(2x) - \frac{9}{40}e^{-4x} + \frac{6}{5}e^x.$$

Figure 11 shows the right hand side function and its approximation, and Figure 12 shows an approximate particular solution together with the exact solution of the IVP.

4 An Alternative Method

4.1 Method of Superposition

In this section, we use the idea of paper [13] to solve our 1-D problem. We want to be able to find a particular solution for Eq. (1). Assume that $f(x)$ has the Chebyshev polynomial approximation $P_n(x)$ given by (7) and (8), i.e.,

$$Ly = ay'' + by' + cy = P_n(x). \tag{18}$$

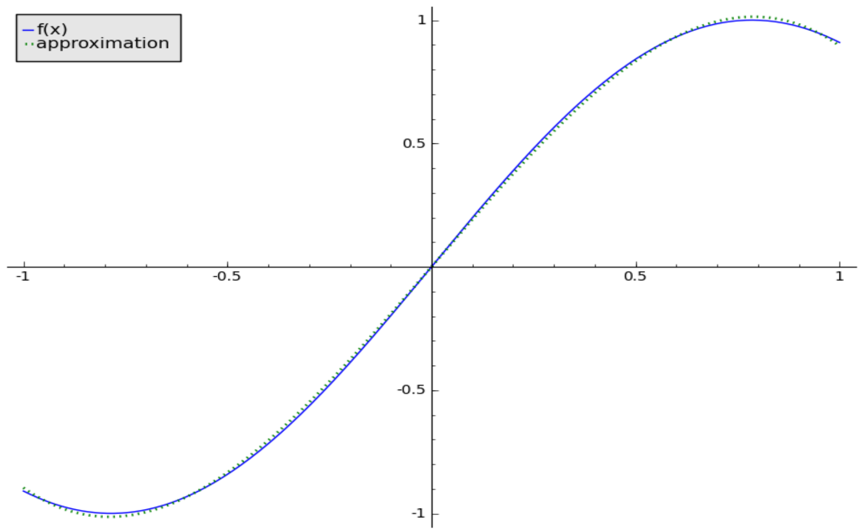

Figure 11: Plot of $f(t)$ and P_4 for Example 3.1.4.

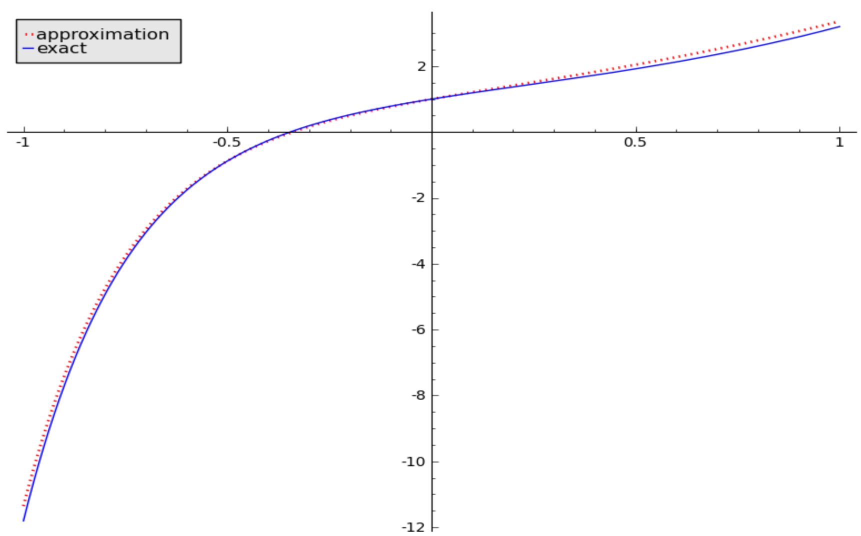

Figure 12: Plot of the numerical solution and the exact solution for Example 3.1.4.

We first look for a particular solution y^k for

$$ay'' + by' + cy = T_k(x), \tag{19}$$

with $k = 0, 1, \cdots, n$.

To find a particular solution for (19), we expand the polynomial $T_k(x)$ in terms of the polynomial basis $\{1,\ x,\ x^2,\ \cdots,\ x^k\}$. That is, $T_k(x) = d_k x^k + d_{k-1}x^{k-1} + \cdots + d_1 x + d_0$.

For the equation (19), we consider three cases:

Case 1) If $c \neq 0$, we look for a particular solution that is in the form of

$$y_p = e_k x^k + e_{k-1}x^{k-1} + \cdots + e_1 x + e_0; \tag{20}$$

Case 2) If $c = 0$ and $b \neq 0$, we look for a particular solution that is in the form of

$$y_p = e_{k+1}x^{k+1} + e_k x^k + \cdots + e_1 x + e_0;$$

Case 3) If $c = b = 0$, the particular solution is

$$y_p = e_{k+2}x^{k+2} + e_{k+1}x^{k+1} + \cdots + e_1 x + e_0.$$

We consider case 1) with the assumption $c \neq 0$ in the following since other cases can be handled similarly. We The particular solution (20) is substituted into (19),

$$a\left(\sum_{j=0}^{k} e_j x^j\right)'' + b\left(\sum_{j=0}^{k} e_j x^j\right)' + c\left(\sum_{j=0}^{k} e_j x^j\right) = \sum_{j=0}^{k} d_j x^j.$$

By comparing coefficients, the e_j should satisfy the following system of equations

$$\begin{aligned} ce_k &= d_k, \\ ce_{k-1} + bke_k &= d_{k-1}, \\ ce_i + b\,(i+1)\,e_{i+1} + a\,(i+2)\,(i+1)\,e_{i+2} &= d_i, \text{ for } i = k-2,\ \cdots,\ 0. \end{aligned}$$

Using backward substitution, we obtain the coefficients

$$\begin{aligned} e_k &= \frac{d_k}{c}, \qquad (21) \\ e_{k-1} &= \frac{d_{k-1} - bke_k}{c}, \\ e_i &= \frac{d_i - b\,(i+1)\,e_{i+1} - a\,(i+2)\,(i+1)\,e_{i+2}}{c}, \text{ for } i = k-2,\ \cdots,\ 0. \end{aligned}$$

Since y^k is the particular solution corresponding to the Chebyshev polynomial T_k and $L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c$ is a linear differential operator,

$$L\left(\sum_{k=0}^{n} c_k y^k\,(x)\right) = \sum_{k=0}^{n} c_k L(y^k\,(x))$$

which is,

$$L(\sum_{k=0}^{n} c_k y^k(x)) = \sum_{k=0}^{n} c_k T_k(x), \tag{22}$$

The equation (22) means that

$$y_p = \sum_{k=0}^{n} c_k y^k(x)$$

is a particular solution of $Ly = P_n(x)$, with $P_n(x) = \sum_{k=0}^{n} c_k T_k(x)$.

4.2 Examples of Approximation

Example 4.2.1 We consider the differential equation

$$y''(x) + y'(x) + 1.25y(x) = 3x^3 + x^2 + 2x + 7.$$

Due to the polynomial form of the right hand side function, the coefficients for particular solution can be directly determined by (21), that is,

$$\begin{aligned} e_3 &= \frac{d_3}{c} = \frac{12}{5}, \\ e_2 &= \frac{d_2 - 3be_3}{c} = -\frac{124}{25}, \\ e_1 &= \frac{d_1 - 2be_2 - 6ae_3}{c} = -\frac{248}{125} \\ e_0 &= \frac{d_0 - be_1 - 2ae_2}{c} = \frac{9452}{625} \end{aligned}$$

Thus, a particular solution for the equation is $y_p = e_0 + e_1 x + e_2 x^2 + e_3 x^3$. This verifies the result we obtained in Example 3.1.2.

Example 4.2.2 We consider

$$y''(x) + y'(x) + 1.25y(x) = f(x)$$

where $f(x)$ has Chebyshev polynomial approximation $P_3(x) = \frac{15}{2}T_0 + \frac{17}{4}T_1 + \frac{1}{2}T_2 + \frac{3}{4}T_3$. We must find particular solutions $y^{(0)}$, $y^{(1)}$, $y^{(2)}$, $y^{(3)}$ with respect to the corresponding Chebyshev polynomials T_0, T_1, T_2, T_3. The coefficients for $y^{(3)}$ is

$$\begin{aligned} e_3 &= \frac{d_3}{c} = \frac{4}{1.25} = \frac{16}{5}, \\ e_2 &= \frac{d_2 - 3be_3}{c} = \frac{0 - 3e_3}{1.25} = -\frac{192}{25}, \\ e_1 &= \frac{d_1 - 2be_2 - 6ae_3}{c} = \frac{-3 - 2e_2 - 6e_3}{1.25} = -\frac{684}{125}, \\ e_0 &= \frac{d_0 - be_1 - 2ae_2}{c} = \frac{0 - e_1 - 2e_2}{1.25} = \frac{10416}{625}. \end{aligned}$$

The coefficients for $y^{(2)}$ is

$$\begin{aligned} e_2 &= \frac{d_2}{c} = \frac{2}{1.25} = \frac{8}{5}, \\ e_1 &= \frac{d_1 - b2e_2}{c} = \frac{0 - 2e_2}{1.25} = -\frac{64}{25}, \\ e_0 &= \frac{d_0 - be_1 - 2ae_2}{c} = \frac{-1 - e_1 - 2e_2}{1.25} = -\frac{164}{125}. \end{aligned}$$

The coefficients for $y^{(1)}$ is

$$\begin{aligned} e_1 &= \frac{d_1}{c} = \frac{1}{1.25} = \frac{4}{5}, \\ e_0 &= \frac{d_0 - be_1}{c} = \frac{0 - e_1}{1.25} = -\frac{16}{25}. \end{aligned}$$

The coefficients for y^0 is

$$e_0 = \frac{1}{1.25} = \frac{4}{5}.$$

We list these particular solutions as follows:

$$\begin{aligned} y^0 &= \frac{4}{5} \\ y^1 &= \frac{4}{5} - \frac{16}{25}x \\ y^2 &= \frac{-164}{125} - \frac{64}{25}x + \frac{8}{5}x^2 \\ y^3 &= \frac{10416}{625} - \frac{685}{125}x - \frac{192}{25}x^2 + \frac{16}{5}x^3 \end{aligned}$$

Hence we find an approximate particular solution for the equation as

$$\begin{aligned} y_p &= \frac{15}{2}y^0 + \frac{17}{4}y^1 + \frac{1}{2}y^2 + \frac{3}{4}y^3 \\ &= \frac{9452}{625} - \frac{248}{125}x - \frac{124}{25}x^2 + \frac{12}{5}x^3. \end{aligned}$$

The alternative approach further verifies the result obtained by the reduction of order method in Example 3.1.2.

5 Conclusions

The Chebyshev polynomials have been in existence for over a hundred years and they have been used for solving many different problems. In this thesis, we have a new strategy using chebyshev polynomials for a common problem. We obtain satisfactory results because of the excellent convergence rate of Chebyshev approximation, which is very close to the minimax polynomial which minimizes the maximum error in approximation. We find approximate particular solutions by Chebyshev polynomial approximation and the reduction of order for the derivatives of Chebyshev polynomials. For comparison purpose, we solve same problems using an existing approach as an alternative approach. The first approach does not need the expansion of Chebyshev polynomials. We end up with an algebraic system of equations by expressing the derivatives of Chebyshev polynomials in terms of these polynomials. We can compare the coefficients of the Chebyshev polynomials in the resulting system of equations so that we can determine the coefficients of the particular solution. The second approach uses the approximation of a right hand side function, which we need to do expansion for each Chebyshev polynomial basis function. We use the particular solutions corresponding to each Chebyshev polynomial to construct a particular solution of the differential equation by superposition principle. Since we are approximating a particular solution with polynomials in both cases, a particular solution by either approach will be exact if the right hand function is already a polynomial.

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