

5-2013

## Chebyshev Polynomial Approximation to Solutions of Ordinary Differential Equations

Amber Sumner Robertson  
*University of Southern Mississippi*

Follow this and additional works at: [https://aquila.usm.edu/undergraduate\\_theses](https://aquila.usm.edu/undergraduate_theses)



Part of the [Mathematics Commons](#)

---

### Recommended Citation

Robertson, Amber Sumner, "Chebyshev Polynomial Approximation to Solutions of Ordinary Differential Equations" (2013). *Undergraduate Theses*. 1.  
[https://aquila.usm.edu/undergraduate\\_theses/1](https://aquila.usm.edu/undergraduate_theses/1)

This Article is brought to you for free and open access by The Aquila Digital Community. It has been accepted for inclusion in Undergraduate Theses by an authorized administrator of The Aquila Digital Community. For more information, please contact [Joshua.Cromwell@usm.edu](mailto:Joshua.Cromwell@usm.edu).

The University of Southern Mississippi

CHEBYSHEV POLYNOMIAL APPROXIMATION TO  
SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

By

Amber Sumner Robertson

May 2013

## ABSTRACT

### CHEBYSHEV POLYNOMIAL APPROXIMATION TO SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

By

Amber Sumner Robertson

May 2013

In this thesis, we develop a method for finding approximate particular solutions for second order ordinary differential equations. We use Chebyshev polynomials to approximate the source function and the particular solution of an ordinary differential equation. The derivatives of each Chebyshev polynomial will be represented by linear combinations of Chebyshev polynomials, and hence the derivatives will be reduced and differential equations will become algebraic equations. Another advantage of the method is that it does not need the expansion of Chebyshev polynomials. This method is also compared with an alternative approach for particular solutions. Examples including approximation, particular solution, a class of variable coefficient equation, and initial value problem are given to demonstrate the use and effectiveness of these methods.

COPYRIGHT

BY

AMBER SUMNER ROBERTSON

2013

The University of Southern Mississippi

CHEBYSHEV POLYNOMIAL APPROXIMATION TO SOLUTIONS OF  
ORDINARY DIFFERENTIAL EQUATIONS

by

Amber Sumner Robertson

Approved by

-----  
-----  
-----

May 2013

## ACKNOWLEDGEMENTS

I would like to reserve this section to thank all of those who made this paper possible. I owe a huge debt of gratitude to my advisor, Dr. Haiyan Tian, for guiding me through this endeavor. Without her, this would not have been possible. I would like to thank Dr. Joseph Kolibal for encouraging me to pursue an undergraduate research project and for reminding me that education is truly about the pursuit of knowledge. I would also like to thank Dr. James Lambers for his interest in this project, and his helpful comments, suggestions, and encouragement along the way. I would like to thank Dr. Karen Kohl who taught me how to do mathematical computing and helped in this endeavor as well. I owe thanks to all of the professors and instructors whom I have studied under here at the University of Southern Mississippi. Lastly, I would like to thank Mr. Corey Jones and Mrs. Kerri Pippin from Jones County Junior College for sparking a lifelong interest in learning Mathematics.

## TABLE OF CONTENTS

<b>ABSTRACT</b> .....	ii
<b>ACKNOWLEDGEMENTS</b> .....	v
<b>LIST OF ILLUSTRATIONS</b> .....	vii
<b>LIST OF TABLES</b> .....	viii
<b>LIST OF ABBREVIATIONS</b> .....	ix
<b>1. INTRODUCTION</b> .....	1
1.1 Applications of Second Order Ordinary Differential Equations .....	1
1.2 Particular Solution of an Ordinary Differential Equation .....	2
<b>2. APPROXIMATION USING CHEBYSHEV POLYNOMIALS</b> .....	3
2.1 Method of Chebyshev Polynomial Approximation .....	4
2.2 Illustrations of Chebyshev Polynomial Approximation .....	6
<b>3. AN APPROXIMATE PARTICULAR SOLUTION</b> .....	9
3.1 Method of Reduction of Order .....	9
3.2 Examples of Approximation .....	11
<b>4. AN ALTERNATIVE METHOD</b> .....	20
4.1 Method of Superposition.....	20
4.2 Examples of Approximation.....	21
<b>5. CONCLUSIONS</b> .....	23

## LIST OF ILLUSTRATIONS

1. Plot of Chebyshev polynomials $T_n$ , $n = 1, 2, \dots, 6$ .....	5
2. Plot of $P_3$ .....	7
3. Plot of $f(x)$ with $T_1$ .....	7
4. Plot of $f(x)$ with $T_4$ .....	8
5. Plot of $f(x)$ with $T_9$ .....	8
6. Plot of $f(x)$ with $T_{16}$ .....	9
7. Plot of the right hand side function in Example 3.1.1 .....	7
8. Plot of the particular solution for Example 3.1.1 .....	14
9. Plot of $f(t)$ and $P_4$ with $T_1$ for Example 3.1.3 .....	16
10. Plot of an approximate particular solution for Example 3.1.3 .....	17
11. Plot of $f(t)$ and $P_4$ for Example 3.1.4 .....	19
12. Plot of an approximate particular solution for Example 3.1.4 .....	20



## LIST OF TABLES

1. Coefficients for  $T'_j(x)$ ,  $j = 0, 1, \dots, 4$  ..... 12
2. Coefficients for  $T''_j(x)$ ,  $j = 0, 1, \dots, 4$  ..... 12

## **LIST OF ABBREVIATIONS**

**ODE** - Ordinary Differential Equations  
**PDE** - Partial Differential Equations  
**MPS** - Method of Particular Solutions  
**IVP** - Initial Value Problem

# 1 Introduction

In this thesis, we consider a linear second order ordinary differential equation (ODE),

$$L(y) = ay''(x) + by'(x) + cy(x) = f(x) \quad (1)$$

where  $f(x)$  is a continuous function,  $a$ ,  $b$ , and  $c$  are given constants, and  $a \neq 0$ . We are interested to find a particular solution  $y_p$  for Equ. (1).

## 1.1 Applications of Second Order Ordinary Differential Equations

The second order ordinary differential equation (1) can model many different phenomena that we encounter every day. For example, the equation

$$\frac{d^2y}{dt^2} = -ky, \text{ with } k > 0$$

represents what happens when an object is subject to a force towards an equilibrium position with the magnitude of the force being proportional to the distance from equilibrium. The above equation can be considered as an approximation to the equation of motion of a particular point on the basilar membrane, or anywhere else along the chain of transmission between the outside air and the cochlea [10]. Thus the solution of the differential equation can help explain the perception of pitch and intensity of musical instruments by human ears.

We may also use a second order ODE to model at what altitude a skydiver's parachute must open before he reaches the ground so that he lands safely. To model this phenomenon, we let  $y$  denote the altitude of the skydiver. We consider Newton's second law

$$F = ma \quad (2)$$

where  $F$  represents the force,  $m$  represents the mass of the skydiver and  $a$  represents the acceleration. We assume that the only forces acting on the skydiver are air resistance and gravity when the skydiver falls through the air toward the earth. We also assume that the air resistance is proportional to the speed of the skydiver with  $b$  being the positive constant of proportionality known as the damping constant. The Newton's second law (2) then translates to

$$y'' + \frac{b}{m}y' = g. \quad (3)$$

If we know the initial height  $y_0$  and velocity  $v_0$  at the time of jump, we have an initial value problem which is the equation (3) together with the initial conditions,

$$\begin{aligned} y(0) &= y_0, \\ y'(0) &= v_0. \end{aligned}$$

As another application, the second order ODE can model a damped mass-spring oscillator that consists of a mass  $m$  that is attached to a spring fixed at one end [?, 6]. Taking into account the forces acting on the spring due to the spring elasticity, damping friction, and other external influences, the motion of the mass-spring oscillator is governed by the differential equation

$$my'' + by' + ky = F_{ext}(t) \quad (4)$$

where  $b$  ( $\geq 0$ ) is the damping coefficient and  $k$  ( $\geq 0$ ) is known as the stiffness of the spring. The differential equation (4) is derived by using Newton's second law and Hooke's law.

Yet another application the equation (1) can model is an electrical circuit consisting of a resistor, capacitor, inductor, and an electromotive force [6]. With charge being the function, we can obtain an initial value problem of the form,

$$\begin{aligned} Lq''(t) + Rq'(t) + \frac{q(t)}{C} &= E(t), \\ q(0) &= q_0, \\ q'(0) &= I_0, \end{aligned}$$

where  $L$  is the inductance in henrys,  $R$  is the resistance in ohms,  $C$  is the capacitance in farads,  $E(t)$  is the electromotive force in volts, and  $q(t)$  is the charge in coulombs on the capacitor at time  $t$ .

We can easily see that the applications of the second order ODE has a wide scope that it is applicable in many fields of study. A general form of these equations is given by Eq. (1). When the right hand side function  $f(x)$  is zero, it is a homogeneous equation; otherwise, it is a nonhomogeneous equation.

## 1.2 Particular Solution of an Ordinary Differential Equation

We hope to be able to find a particular solution for Eq. (1) that is nonhomogeneous. A particular solution of Eq. (1) is a function that satisfies Eq. (1). The particular solution to an ordinary differential equation can be obtained by assigning numerical values to the parameters in the general solution [3]. We note that there are many possible answers for a particular solution.

A particular solution  $y_p$  will allow us to reduce, for example, an initial value problem

$$\begin{cases} L(y) = ay''(x) + by'(x) + cy(x) = f(x), \\ y(x_0) = y_0, y'(x_0) = y_1, \end{cases} \quad (5)$$

to a homogeneous equation that is subject to a different initial data

$$\begin{cases} L(y_h) = ay_h''(x) + by_h'(x) + cy_h(x) = 0, \\ y_h(x_0) = y_0 - y_p(x_0), y_h'(x_0) = y_1 - y_p'(x_0), \end{cases} \quad (6)$$

with  $y_h = y - y_p$ . The method of particular of solutions (MPS) allows us to split the solution  $y$  into a particular solution  $y_p$  and a homogeneous solution  $y_h$ .

In this thesis, we use Chebyshev polynomials for approximating equations and their particular solutions. Chebyshev polynomials [1] of the first kind are solutions to the Chebyshev differential equations

$$(1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + n^2 y = 0 \text{ for } |x| < 1,$$

and Chebyshev polynomials of the second kind are solutions to the Chebyshev differential equations

$$(1 - x^2) \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + n(n + 2)y = 0 \text{ for } |x| < 1.$$

The Chebyshev polynomials of either kind are a sequence of orthogonal polynomials that can also be defined recursively. The motivation for Chebyshev interpolation is to improve control of the interpolation error on the interpolation interval [9]. The MPS and Chebyshev polynomials have been used for solving partial differential equations (PDE). For example, the paper [13] solved elliptic partial differential equation boundary value problems; the paper [11] studied two dimensional heat conduction problems and the authors used Chebyshev polynomials and the trigonometric basis functions to approximate their equations for each time step. In their two-stage approximation scheme, the use of Chebyshev polynomials in stage one is because of the high accuracy (spectral convergence) of Chebyshev interpolation.

If the right hand side function  $f(x)$  in Eq. (1) is a polynomial of degree  $n$ , i.e.,

$$\begin{aligned} f(x) &= P_n(x) \\ &= d_n x^n + d_{n-1} x^{n-1} + \dots + d_1 x + d_0 \end{aligned}$$

where  $d_n \neq 0$ , we can find a particular solution of Eq. (1) that is a polynomial.

If  $f(x)$  is not a polynomial, we approximate it using Chebyshev polynomials. We then look for a particular solution that is expressed as a linear combination of Chebyshev polynomials. Our choice of Chebyshev polynomials is because of their high accuracy. The Chebyshev polynomial is very close to the **minimax polynomial** which (among all polynomials of the same degree) has the smallest maximum deviation from the true function  $f(x)$ . The **minimax criterion** is that  $P_n(x)$  is the polynomial of degree  $n$  for which the maximum value of the error, which is defined by  $e_n(x) = f(x) - P_n(x)$ , is a minimum within the specified range of  $-1 \leq x \leq 1$  [1]. This is extremely ideal for polynomial approximations.

In our approach, the derivatives of each Chebyshev polynomial will be represented by the linear combinations of Chebyshev polynomials, and hence the differential equations will become algebraic equations. In Chapter 2, we introduce the approximation method and illustrations using Chebyshev polynomials. In Chapter 3, we describe our method for finding a particular solution or an approximate particular solution by the approximation and reduction of order. An

alternative method by superposition principle is used in Chapter 4 for ODEs. These two methods are compared through examples. Conclusions are made in Chapter 5.

## 2 Approximation using Chebyshev Polynomials

The Chebyshev polynomials of the first and second kind are denoted by  $T_n(x)$  and  $U_n(x)$  respectively. The subscript  $n$  is the degree of these polynomials. The Chebyshev polynomials of the first and second kind are closely related. For example, a Chebyshev polynomial of first kind can be represented as a linear combination of two Chebyshev polynomials of second kind,

$$T_n(x) = \frac{1}{2} (U_n(x) - U_{n-2}(x)),$$

and the derivative of a Chebyshev polynomial of first kind can be written in terms of a Chebyshev polynomial of second kind,

$$T'_n(x) = nU_{n-1}(x), \quad n = 1, 2, \dots$$

In this paper, we direct our attention to the Chebyshev polynomials of first kind and we use them for approximating a function and a particular solution for second order ODEs.

### 2.1 Method of Chebyshev Polynomial Approximation

In this section, we give an introduction to the Chebyshev polynomials and their basic properties. See the references [1, 9, 5, 2] for more details.

Chebyshev polynomials of the first kind are denoted by  $T_n$  and the first several polynomials are listed below:

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_2(x) &= 2x^2 - 1, \\ T_3(x) &= 4x^3 - 3x, \\ T_4(x) &= 8x^4 - 8x^2 + 1, \\ T_5(x) &= 16x^5 - 20x^3 + 5x, \\ T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1, \\ &\dots \end{aligned}$$

The first six Chebyshev polynomials  $T_j$ ,  $j = 0, 1, \dots, 6$  are shown in Figure 1.

A Chebyshev polynomial can be found using the previous two polynomials by the recursive formula

$$T_{n+1} = 2xT_n(x) - T_{n-1}(x), \quad \text{for } n \geq 1.$$

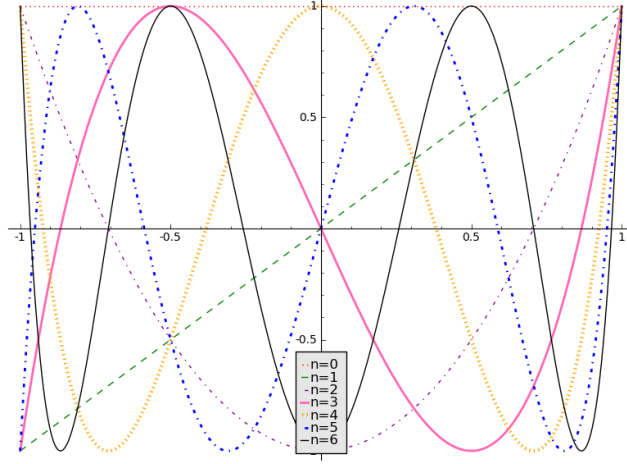


Figure 1: Plot of Chebyshev polynomials  $T_n$ ,  $n = 1, 2, \dots, 6$

[1] The leading coefficient of the  $T_n(x)$  is  $2^{n-1}$  for  $n \geq 1$  as we can see from the recursive formula shown above. These polynomials are orthogonal with respect to the weight function  $\frac{1}{\sqrt{1-x^2}}$  on the interval  $[-1, 1]$ ,

$$\int_{-1}^1 T_i(x) T_j(x) \frac{1}{\sqrt{1-x^2}} dx = \begin{cases} 0, & i \neq j, \\ \pi, & i = j = 0, \\ \frac{\pi}{2}, & i = j \neq 0. \end{cases}$$

As we have mentioned,  $T_n(x)$  denotes the Chebyshev polynomial of degree  $n$ . It has  $n$  roots, which are also known as Chebyshev nodes. These nodes [5] can be calculated by the formula

$$x_i = \cos\left(\frac{i - \frac{1}{2}}{n}\right) \pi, \text{ for } i = 1, 2, \dots, n.$$

A function  $f(x)$  can be approximated [4] by an  $n$ -th degree polynomial  $P_n(x)$  expressed in terms of  $T_0, \dots, T_n$ ,

$$P_n(x) = C_0 T_0(x) + C_1 T_1(x) + \dots + C_n T_n(x) - \frac{1}{2} C_0 \quad (7)$$

where

$$C_j = \frac{2}{n} \sum_{k=1}^{n+1} f(x_k) T_j(x_k), \quad j = 0, 1, \dots, n. \quad (8)$$

and  $x_k, k = 1, \dots, n+1$  are zeros of  $T_{n+1}$ .

Since

$$T_j(x) = \cos(j \arccos x),$$

we have

$$\begin{aligned} T_j(x_k) &= \cos(j \arccos x_k) \\ &= \cos\left(\frac{j(k - \frac{1}{2})}{n + 1}\pi\right). \end{aligned}$$

Let  $f(x)$  be a continuous function defined on the interval  $[-1, 1]$ , that we want to approximate by a polynomial  $P_n$  defined by (7) and (8). We can measure how good an approximation is of  $f(x)$  by the uniform norm,

$$\|f - P_n\| = \max_{-1 \leq x \leq 1} |f(x) - P_n(x)|.$$

This means that the measure of the error of the approximation is given by the greatest distance between  $f(x)$  and  $P_n(x)$  with  $x$  going through the interval  $[-1, 1]$ [5].

For  $m$  that is much less than  $n$ ,  $f(x) \approx C_0T_0(x) + C_1T_1(x) + \dots + C_mT_m(x) - \frac{1}{2}C_0$ , is a truncated approximation with  $C_{m+1}T_{m+1}(x) + \dots + C_nT_n(x)$  being the truncated part of the sum. The error in this approximation is dominated by the leading term of the truncated part  $C_{m+1}T_{m+1}(x)$  since typically the coefficients  $C_k$  are rapidly decreasing. We know that the  $m + 2$  equal extrema of  $T_{m+1}(x)$  spread out smoothly over the interval  $[-1, 1]$ , thus the error spreads out smoothly over the interval  $[-1, 1]$ .

The equations (7) and (8) are used to approximate a function  $f(x)$  defined on the interval  $[-1, 1]$ . For a function  $f(x)$  that is defined on an interval  $[a, b]$ , we can obtain  $f(y)$  with  $y \in [-1, 1]$  by a change of variable

$$y \equiv \frac{x - \frac{1}{2}(b + a)}{\frac{1}{2}(b - a)}.$$

## 2.2 Illustrations of Chebyshev Polynomial Approximation

Let  $P_n$  be the  $n^{\text{th}}$  degree polynomial approximation to the function  $f(x)$ . Example 2.2.1. Let  $f(x) = \frac{3}{125}x^3 - \frac{x}{80} + 1$ . The function  $f(x)$  and its Chebyshev polynomial approximation  $P_3$  are shown in Figure 2. The approximation polynomial is identical to the function  $f(x)$  as we expected.

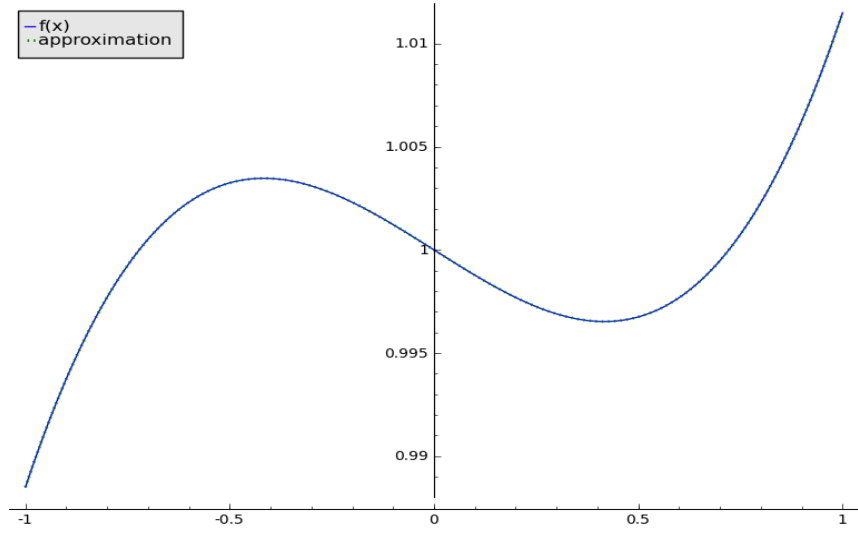
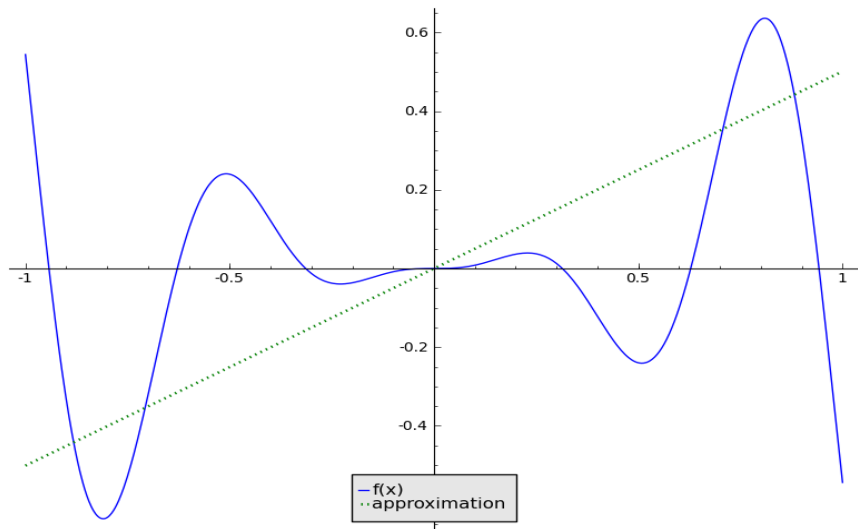
Example 2.2.2. Let  $f(x) = x^2 \sin(10x)$ . Its Chebyshev polynomial approximation  $P_n$  with  $n = 1, 4, 9, 16$  are shown in Figures 3-6. As  $n$  increases, the error of the approximation polynomial decreases.

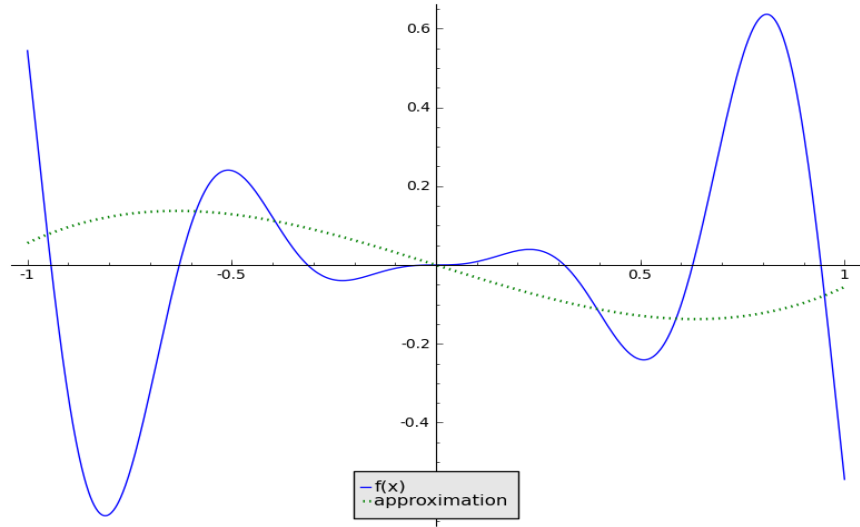
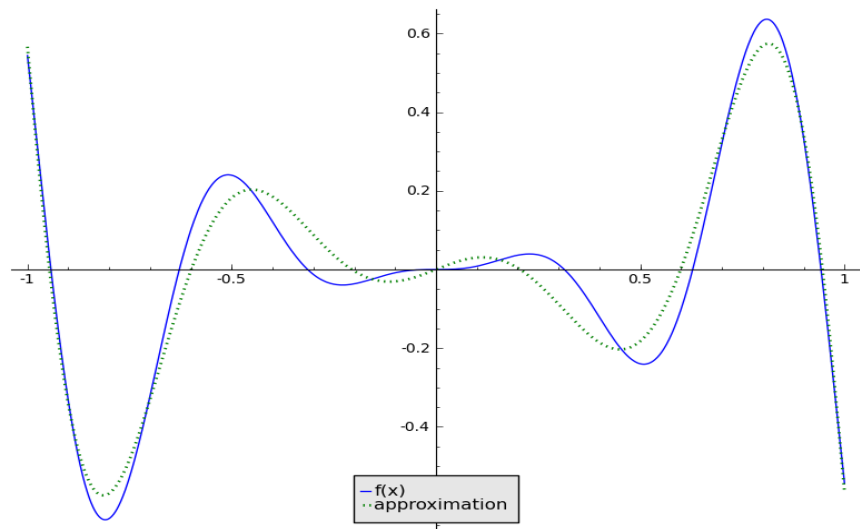
## 3 An Approximate Particular Solution

### 3.1 Method of Reduction of Order

We recall that we can reduce an initial value differential equation problem to an algebraic equation through the Laplace transform [12, 6]. Here we propose



Figure 2: Plot of  $P_3$ Figure 3: Plot of  $f(x)$  with  $T_1$

Figure 4: the plot of  $f(x)$  with  $T_4$ Figure 5: the plot of  $f(x)$  with  $T_9$

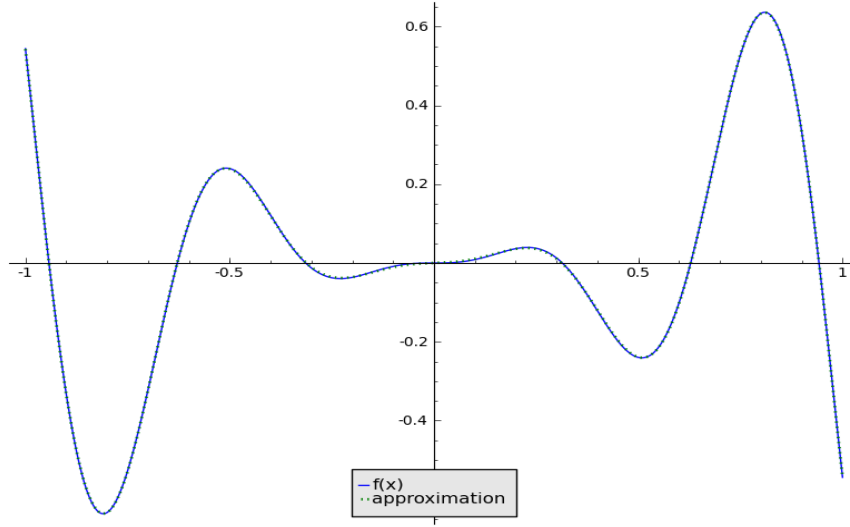


Figure 6: the plot of  $f(x)$  with  $T_{16}$

a method that achieves the similar goal using Chebyshev polynomial approximation and order reduction techniques.

By defining the Laplace transform to  $f(t)$ ,

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt = F(s),$$

we get a new function  $F(s)$ . Assume that the functions  $f, f', \dots, f^{(n-1)}$  are continuous and that  $f^{(n)}$  is piecewise continuous on any interval  $0 \leq t \leq A$ . Suppose further that there exist constants  $K, a$ , and  $M$  such that  $|f(t)| \leq Ke^{at}$ , and  $|f'(t)| \leq Ke^{at}, \dots, |f^{(n-1)}(t)| \leq Ke^{at}$  for  $t \geq M$ . Then  $\mathcal{L}\{f^{(n)}(t)\}$  exists for  $s > a$  and is given by,

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0).$$

If the solution  $y(t)$  of the equation (1), when substituted for  $f$ , satisfies the above condition for  $n = 2$ , we can apply the Laplace transform to (1) to get the following algebraic equation,

$$a[s^2 Y(s) - sy(0) - y'(0)] + b[sY(s) - y(0)] + cY(s) = F(s)$$

where  $Y(s)$  and  $F(s)$  are respectively the Laplace transforms of  $y(t)$  and  $f(t)$ . By solving the above equation for  $Y(s)$ , we find that

$$Y(s) = \frac{(as + b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}.$$

With the given initial conditions  $y(0)$  and  $y'(0)$ , we can find a unique solution  $Y(s)$ . The inverse Laplace transform  $\mathcal{L}^{-1}\{Y(s)\}$  will give the solution to the original initial value problem.

When employing the Laplace transform for an initial value problem, we first transform a differential equation into an algebraic one by taking into account the given initial data. Then we obtain the solution by inverse Laplace transform. In our proposed MPS, we express a particular solution and the derivatives of the particular solution by Chebyshev polynomials. Then using identities to obtain a system of algebraic equations the coefficients should satisfy. Thus we find a particular solution by solving algebraic equations. To get the solution of an IVP, we need to get the solution of the corresponding homogenous problem.

We first approximate  $f(x)$  in Eq. (1) by  $P_n(x)$  using Chebyshev polynomials. Then we find a particular solution  $y_p$  of the equation

$$ay''(x) + by'(x) + cy(x) = P_n(x). \quad (9)$$

We let the particular solution be in the form of

$$y_p = \sum_{j=0}^m q_j T_j(x). \quad (10)$$

The coefficients  $q_j$  for (10) are to be determined. We let  $m = n$  if  $c \neq 0$  in Equ. (9);  $m = n + 1$  if  $c = 0$ ,  $b \neq 0$ ;  $m = n + 2$  if  $c = 0$ ,  $b = 0$ .

Substituting (10) into (9),

$$a \sum_{j=0}^m q_j T_j''(x) + b \sum_{j=0}^m q_j T_j'(x) + c \sum_{j=0}^m q_j T_j(x) = P_n(x).$$

Our next goal is to use linear combinations of Chebyshev polynomials to represent  $T_j'(x)$  and  $T_j''(x)$  for each  $j$ . That is, the first and second order derivatives of Chebyshev polynomials  $T_j'(x)$  and  $T_j''(x)$  are represented in terms of  $T_k(x)$ ,  $k = 0, \dots, j$ . Then we arrive at a system of algebraic equations of  $q_j$  by equating coefficients to find the solution  $y_p$  given by (10).

According to [8] and its tables for representation coefficients, we reduce the first and second order derivatives as follows

$$T_j'(x) = \sum_{k=0}^{j-1} b_k T_k(x), \quad (11)$$

where

$$b_{2l} = 0, \text{ for } l = 0, 1, \dots, \frac{j}{2} - 1,$$

$$b_{2l+1} = 2j, \text{ for } l = 0, 1, \dots, \frac{j}{2} - 1,$$

for even  $j$ , and

$$\begin{aligned} b_0 &= j, \\ b_{2l} &= 2j, \text{ for } l = 1, \dots, \frac{j-1}{2}, \\ b_{2l+1} &= 0, \text{ for } l = 0, 1, \dots, \frac{j-3}{2}, \end{aligned}$$

for odd  $j$ .

Next, we expand the second order derivative  $T_j''(x)$  in terms of Chebyshev polynomials  $T_i(x)$ ,  $i = 0, \dots, j-2$ . With  $c_i$  being the representation coefficients for  $T_j''(x)$ , that is,

$$T_j''(x) = \sum_{i=0}^{j-2} c_i T_i(x). \quad (12)$$

We use the tables provided in [8] to calculate  $c_i$ , for  $i = 0, \dots, j-2$ . By Eq. (11),

$$\begin{aligned} T_j''(x) &= (T_j'(x))' \\ &= \left( \sum_{k=0}^{j-1} b_k T_k(x) \right)' \\ &= \sum_{k=0}^{j-1} b_k T_k'(x) \\ &= \sum_{k=0}^{j-1} b_k \left( \sum_{i=0}^{k-1} b_i T_i(x) \right) \\ &= \sum_{k=0}^{j-1} \sum_{i=0}^{k-1} b_k b_i T_i(x) \end{aligned} \quad (13)$$

### 3.2 Examples of Approximation

**Example 3.1.1.** We consider the equation

$$y''(x) + 2y(x) = P_4(x)$$

with  $P_4(x) = 2T_1(x) + 5T_4(x)$ . We look for a particular solution  $y_p = \sum_{j=0}^4 q_j T_j(x)$ .

By (11)-(13), we list the coefficients for  $T_j'(x)$ ,  $j = 0, 1, \dots, 4$  in Table 1, which is equivalent to the following,

$$\begin{aligned} (T_0)' &= 0, \\ (T_1)' &= T_0(x), \\ (T_2)' &= 4T_1(x), \\ (T_3)' &= 3T_0(x) + 6T_2(x), \\ (T_4)' &= 8T_1(x) + 8T_3(x). \end{aligned}$$

j=0				
j=1	$b_0 = 1$			
j=2	$b_0 = 0$	$b_1 = 4$		
j=3	$b_0 = 3$	$b_1 = 0$	$b_2 = 6$	
j=4	$b_0 = 0$	$b_1 = 8$	$b_2 = 0$	$b_3 = 8$

Table 1: Coefficients for  $T_j'(x)$ ,  $j = 0, 1, \dots, 4$ 

$j = 0$	
$j = 1$	$b_0 = 1,$
$j = 2$	$b_0 = 0$ $b_1 = 4, b_1 \cdot b_0 = 4 \cdot 1$ $b_0 = 3,$
$j = 3$	$b_1 = 0, b_1 \cdot b_0 = 0 \cdot 1$ $b_2 = 6, b_2 \cdot b_0 = 6 \cdot 0, b_2 \cdot b_1 = 6 \cdot 4$ $b_0 = 0,$
$j = 4$	$b_1 = 8, b_1 \cdot b_0 = 8 \cdot 1$ $b_2 = 0, b_2 \cdot b_0 = 0 \cdot 0, b_2 \cdot b_1 = 0 \cdot 4$ $b_3 = 8, b_3 \cdot b_0 = 8 \cdot 3, b_3 \cdot b_1 = 8 \cdot 0, b_3 \cdot b_2 = 8 \cdot 6$

Table 2: Coefficients for  $T_j''(x)$ ,  $j = 0, 1, \dots, 4$ 

Thus,

$$y_p' = (q_1 + 3q_3)T_0(x) + (4q_2 + 8q_4)T_1(x) + 6q_3T_2(x) + 8q_4T_3(x). \quad (14)$$

The coefficients for second order derivatives  $T_j''(x)$ ,  $j = 0, 1, \dots, 4$  are listed in Table 2 below,

Now we obtain the second order derivatives for  $T_j$ ,  $j = 0, 1, \dots, 4$ . We list the results as follows:

$$\begin{aligned} (T_0)'' &= 0, \\ (T_1)'' &= 0, \\ (T_2)'' &= 4T_0(x), \\ (T_3)'' &= 24T_1(x), \\ (T_4)'' &= 32T_0(x) + 48T_2(x). \end{aligned}$$

We notice, for example,  $c_0 = 32$ ,  $c_1 = 0$ , and  $c_2 = 48$  for (12) when  $j = 4$ .

Therefore,

$$\begin{aligned} y_p'' &= 4q_2T_0(x) + 24q_3T_1(x) + 32q_4T_0(x) + 48q_4T_2(x) \\ &= (4q_2 + 32q_4)T_0(x) + 24q_3T_1(x) + 48q_4T_2(x). \end{aligned}$$

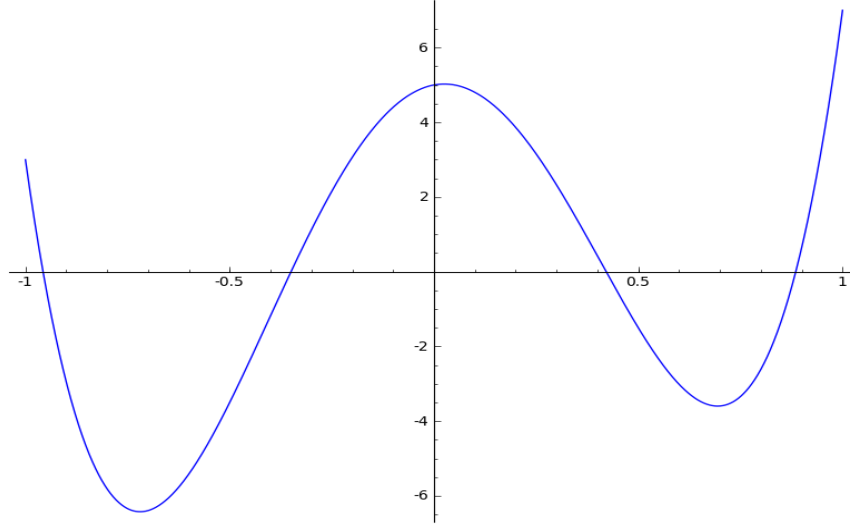


Figure 7: Plot of the right hand side function in Example 3.1.1

We obtain the following linear system of equations by comparing coefficients of  $T_j$ ,  $j = 0, 1, \dots, 4$ .

$$\begin{cases} 2q_0 + 4q_2 + 32q_4 = 0, \\ 2q_1 + 24q_3 = 2, \\ 2q_2 + 48q_4 = 0, \\ 2q_3 = 0, \\ 2q_4 = 5. \end{cases} \quad (15)$$

The solution for (15) is  $q_4 = 5/2$ ,  $q_3 = 0$ ,  $q_2 = -60$ ,  $q_1 = 1$ ,  $q_0 = 80$ . The plots of the right hand side function and the particular solution are shown in Figures 7-8.

Assume we use  $P_n$  defined by (7)-(8) with  $n = 4$  to approximate the function  $f(x)$  in Eq. (1). Without loss of generality, we assume  $c \neq 0$  in (1). Using our method of reduction of order, the coefficients  $q_j$ ,  $j = 0, 1, \dots, 4$  of the particular solution satisfy the following system of equations,

$$\begin{aligned} cq_0 + bq_1 + 4aq_2 + 3bq_3 + 32aq_4 &= \frac{1}{2}C_0, \\ cq_1 + 4bq_2 + 24aq_3 + 8bq_4 &= C_1, \\ cq_2 + 6bq_3 + 48aq_4 &= C_2, \\ cq_3 + 8bq_4 &= C_3, \\ cq_4 &= C_4. \end{aligned} \quad (16)$$

**Example 3.1.2.** In this example, we consider

$$y''(x) + y'(x) + 1.25y(x) = 3x^3 + x^2 + 2x + 7.$$

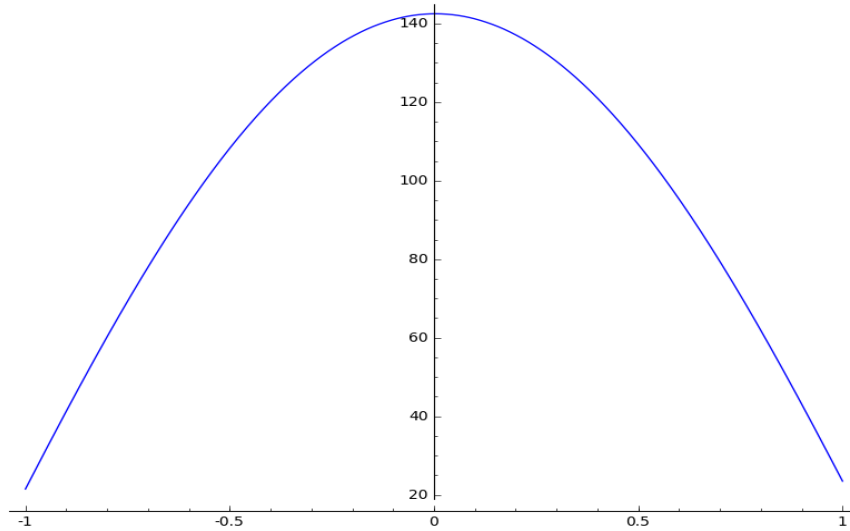


Figure 8: Plot of the particular solution for Example 3.1.1

We notice here the function  $f(x) = 3x^3 + x^2 + 2x + 7$ . We can still look for a particular solution in the form of  $y_p = \sum_{j=0}^4 q_j T_j(x)$ , but  $q_4$  will be 0 as expected. The Chebyshev polynomial representation for  $f(x)$  is

$$f(x) = \sum_{j=0}^3 p_j T_j(x)$$

with  $p_0 = 15/2$ ,  $p_1 = 17/4$ ,  $p_2 = 1/2$ ,  $p_3 = 3/4$ . According to the system of equations (16), we obtain the following system for  $q_j$ ,  $j = 0, \dots, 4$ ,

$$\begin{aligned} 1.25q_0 + q_1 + 4q_2 + 3q_3 + 32q_4 &= \frac{15}{2}, \\ 1.25q_1 + 4q_2 + 24q_3 + 8q_4 &= \frac{17}{4}, \\ 1.25q_2 + 6q_3 + 48q_4 &= \frac{1}{2}, \\ 1.25q_3 + 8q_4 &= \frac{3}{4}, \\ 1.25q_4 &= 0. \end{aligned}$$

Using backward substitution, we get  $q_4 = 0$ ,  $q_3 = 3/5$ ,  $q_2 = -62/25$ ,  $q_1 = -23/125$ ,  $q_0 = 7902/625$ .

**Example 3.1.3.** In this example, we consider the Cauchy-Euler equations



that are expressible in the form of

$$ax^2y''(x) + bxy'(x) + cy(x) = h(x),$$

where  $a$ ,  $b$  and  $c$  are constants. This important class of variable coefficient differential equations can be solved by the particular solutions method of reduction of order. As an illustration, we solve a specific Cauchy-Euler equation as follows,

$$x^2y''(x) - 2xy'(x) + 2y(x) = x^3. \quad (17)$$

By a change of variable  $x = e^t$ , the equation (17) can be transformed into the constant coefficient equation in the new independent variable  $t$ ,

$$y''(t) - 3y'(t) + 2y(t) = e^{3t}.$$

Using the method described above, we approximate the right hand side function by Chebyshev polynomial approximation, then we use the reduction of order method to solve for the approximate particular solution.

The approximation to  $f(t) = e^{3t}$  by  $P_4$  is

$$f(t) = \frac{C_0}{2}T_0(x) + C_1T_1(x) + C_2T_2(x) + C_3T_3(x) + C_4T_4(x),$$

where the numerical values of  $C_0, \dots, C_4$  are given as follows,

$$\frac{C_0}{2} = 4.88075365707809,$$

$$C_1 = 7.90647046809621,$$

$$C_2 = 4.48879683378407,$$

$$C_3 = 1.9105629481,$$

$$C_4 = 0.608043176705983.$$

Solving the resulting system of equations,

$$2q_0 - 3q_1 + 4q_2 - 9q_3 + 32q_4 = \frac{1}{2}C_0,$$

$$2q_1 - 12q_2 + 24q_3 - 24q_4 = C_1,$$

$$2q_2 - 18q_3 + 48q_4 = C_2,$$

$$2q_3 - 24q_4 = C_3,$$

$$2q_4 = C_4,$$

we get

$$q_0 = 344.481628331765,$$

$$q_1 = 170.637487061598,$$

$$q_2 = 36.3797465297907,$$

$$q_3 = 4.60354053428590,$$

$$q_4 = 0.304021588352991,$$

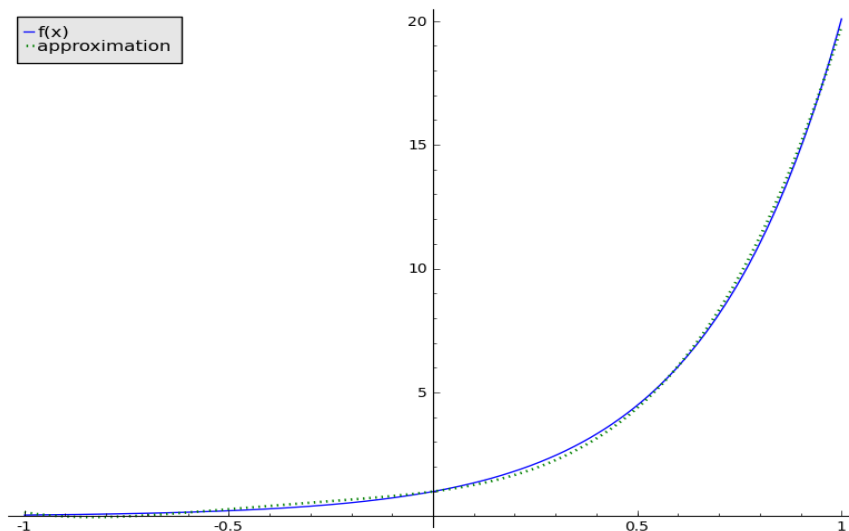


Figure 9: Plot of  $f(t)$  and  $P_4$  of Example 3.1.3

which are coefficients of  $T_j(x)$  for an approximate particular solution  $y_p(t)$  of the constant coefficient equation. The plots of the right hand side function and the particular solution are shown in Figures 9-10. An approximate particular solution of the Cauchy-Euler equation is  $y_p(\ln x)$ .

**Example 3.1.4.** As another example we consider the initial value problem,

$$y'' + 3y' - 4y = \sin(2x),$$

with initial data,

$$y(0) = 1,$$

$$y'(0) = 2.$$

First, the Chebyshev approximation of  $\sin(2x)$  by  $P_4$  is,

$$\sin(2x) \approx \frac{C_0}{2}T_0(x) + C_1T_1(x) + C_2T_2(x) + C_3T_3(x) + C_4T_4(x),$$

where the coefficients  $C_0, \dots, C_4$  are

$$\frac{C_0}{2} = 5.55111512312578e^{-17},$$

$$C_1 = 1.15344467691257,$$

$$C_2 = 0,$$

$$C_3 = 0.257536611098051,$$

$$C_4 = 2.22044604925031e^{-16}.$$

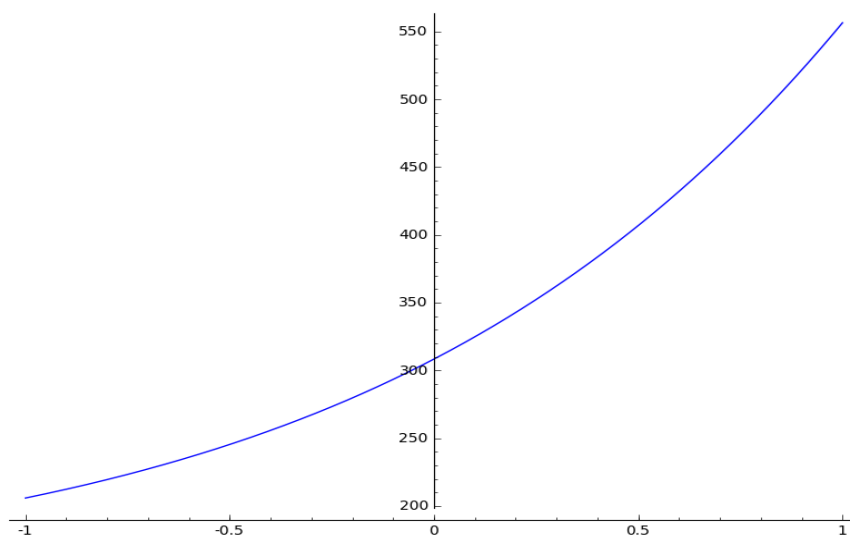


Figure 10: Plot of an approximate particular solution of Example 3.1.3

The resulting system by the reduction of order method with  $n = 4$  gives

$$\begin{aligned} -4q_0 + 3q_1 + 4q_2 + 9q_3 + 32q_4 &= \frac{1}{2}C_0, \\ -4q_1 + 12q_2 + 24q_3 + 24q_4 &= C_1, \\ -4q_2 + 18q_3 + 48q_4 &= C_2, \\ -4q_3 + 24q_4 &= C_3, \\ -4q_4 &= C_4. \end{aligned}$$

The coefficients  $q_0, \dots, q_4$  for the particular solution of the IVP is given by

$$\begin{aligned} q_0 &= 1.15994038863409 \\ q_1 &= 0.9671298098748495 \\ q_2 &= 0.289728687485306 \\ q_3 &= 0.0643841527745125 \\ q_4 &= -5.55111512312578e^{-17} \end{aligned}$$

So we arrive at the particular solution,

$$y_p(x) = q_0T_0(x) + q_1T_1(x) + q_2T_2(x) + q_3T_3(x) + q_4T_4(x).$$

The derivative of  $y_p$  is given by (14) and thus

$$\begin{aligned} y_p'(0) &= (q_1 + 3q_3)T_0(0) + (4q_2 + 8q_4)T_1(0) + 6q_3T_2(0) + 8q_4T_3(0) \\ &= (q_1 + 3q_3) * 1 + (4q_2 + 8q_4) * 0 + 6q_3 * (-1) + 8q_4 * 0 \\ &= q_1 - 3q_3. \end{aligned}$$

This together with

$$\begin{aligned} y_p(0) &= q_0T_0(0) + q_1T_1(0) + q_2T_2(0) + q_3T_3(0) + q_4T_4(0) \\ &= q_0 * 1 + q_1 * 0 + q_2 * (-1) + q_3 * 0 + q_4 * 1 \\ &= q_0 - q_2 + q_4 \end{aligned}$$

gives the initial data for the corresponding homogeneous problem,

$$y'' + 3y' - 4y = 0,$$

that is subject to the initial data,

$$y(0) = 1 - y_p(0) = 1 - q_0 + q_2 - q_4 =,$$

$$y'(0) = 2 - y_p'(0) = 2 - q_1 + 3q_3 = .$$

We need to find the solution  $y_h$  to the above homogeneous problem so that we can obtain the numerical solution to the original IVP. The homogenous solution is,

$$y_h = c_1e^{-4x} + c_2e^x$$

where  $c_1 = -0.219246869919495$  and  $c_2 = 0.349035168770711$ .

Now we obtain the numerical solution of the IVP as follows,

$$\tilde{y}(x) = y_p - 0.219246869919495e^{-4x} + 0.349035168770711e^x$$

The exact solution of the problem is

$$y(x) = \frac{1}{20}\cos(2x) - \frac{1}{20}\sin(2x) - \frac{9}{40}e^{-4x} + \frac{6}{5}e^x.$$

Figure 11 shows the right hand side function and its approximation, and Figure 12 shows an approximate particular solution together with the exact solution of the IVP.

## 4 An Alternative Method

### 4.1 Method of Superposition

In this section, we use the idea of paper [13] to solve our 1-D problem. We want to be able to find a particular solution for Eq. (1). Assume that  $f(x)$  has the Chebyshev polynomial approximation  $P_n(x)$  given by (7) and (8), i.e.,

$$Ly = ay'' + by' + cy = P_n(x). \quad (18)$$

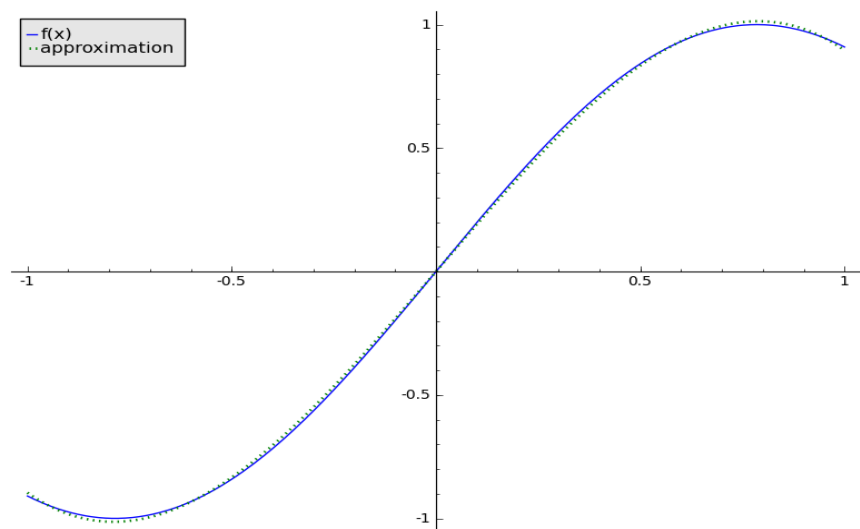
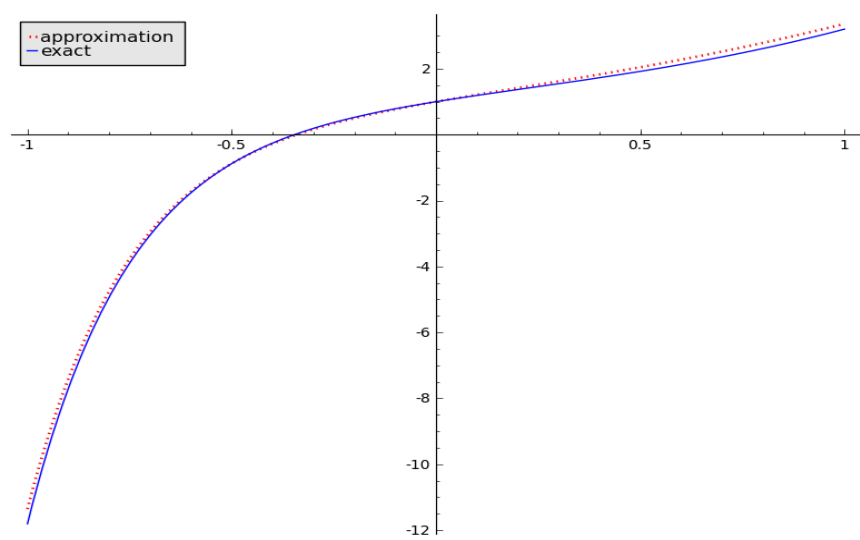
Figure 11: Plot of  $f(t)$  and  $P_4$  for Example 3.1.4.

Figure 12: Plot of the numerical solution and the exact solution for Example 3.1.4.

We first look for a particular solution  $y^k$  for

$$ay'' + by' + cy = T_k(x), \quad (19)$$

with  $k = 0, 1, \dots, n$ .

To find a particular solution for (19), we expand the polynomial  $T_k(x)$  in terms of the polynomial basis  $\{1, x, x^2, \dots, x^k\}$ . That is,  $T_k(x) = d_k x^k + d_{k-1} x^{k-1} + \dots + d_1 x + d_0$ .

For the equation (19), we consider three cases:

Case 1) If  $c \neq 0$ , we look for a particular solution that is in the form of

$$y_p = e_k x^k + e_{k-1} x^{k-1} + \dots + e_1 x + e_0; \quad (20)$$

Case 2) If  $c = 0$  and  $b \neq 0$ , we look for a particular solution that is in the form of

$$y_p = e_{k+1} x^{k+1} + e_k x^k + \dots + e_1 x + e_0;$$

Case 3) If  $c = b = 0$ , the particular solution is

$$y_p = e_{k+2} x^{k+2} + e_{k+1} x^{k+1} + \dots + e_1 x + e_0.$$

We consider case 1) with the assumption  $c \neq 0$  in the following since other cases can be handled similarly. We The particular solution (20) is substituted into (19),

$$a\left(\sum_{j=0}^k e_j x^j\right)'' + b\left(\sum_{j=0}^k e_j x^j\right)' + c\left(\sum_{j=0}^k e_j x^j\right) = \sum_{j=0}^k d_j x^j.$$

By comparing coefficients, the  $e_j$  should satisfy the following system of equations

$$\begin{aligned} ce_k &= d_k, \\ ce_{k-1} + bke_k &= d_{k-1}, \\ ce_i + b(i+1)e_{i+1} + a(i+2)(i+1)e_{i+2} &= d_i, \text{ for } i = k-2, \dots, 0. \end{aligned}$$

Using backward substitution, we obtain the coefficients

$$\begin{aligned} e_k &= \frac{d_k}{c}, \\ e_{k-1} &= \frac{d_{k-1} - bke_k}{c}, \\ e_i &= \frac{d_i - b(i+1)e_{i+1} - a(i+2)(i+1)e_{i+2}}{c}, \text{ for } i = k-2, \dots, 0. \end{aligned} \quad (21)$$

Since  $y^k$  is the particular solution corresponding to the Chebyshev polynomial  $T_k$  and  $L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c$  is a linear differential operator,

$$L\left(\sum_{k=0}^n c_k y^k(x)\right) = \sum_{k=0}^n c_k L(y^k(x))$$

which is,

$$L\left(\sum_{k=0}^n c_k y^k(x)\right) = \sum_{k=0}^n c_k T_k(x), \quad (22)$$

The equation (22) means that

$$y_p = \sum_{k=0}^n c_k y^k(x)$$

is a particular solution of  $Ly = P_n(x)$ , with  $P_n(x) = \sum_{k=0}^n c_k T_k(x)$ .

## 4.2 Examples of Approximation

**Example 4.2.1** We consider the differential equation

$$y''(x) + y'(x) + 1.25y(x) = 3x^3 + x^2 + 2x + 7.$$

Due to the polynomial form of the right hand side function, the coefficients for particular solution can be directly determined by (21), that is,

$$\begin{aligned} e_3 &= \frac{d_3}{c} = \frac{12}{5}, \\ e_2 &= \frac{d_2 - 3be_3}{c} = -\frac{124}{25}, \\ e_1 &= \frac{d_1 - 2be_2 - 6ae_3}{c} = -\frac{248}{125}, \\ e_0 &= \frac{d_0 - be_1 - 2ae_2}{c} = \frac{9452}{625} \end{aligned}$$

Thus, a particular solution for the equation is  $y_p = e_0 + e_1x + e_2x^2 + e_3x^3$ . This verifies the result we obtained in Example 3.1.2.

**Example 4.2.2** We consider

$$y''(x) + y'(x) + 1.25y(x) = f(x)$$

where  $f(x)$  has Chebyshev polynomial approximation  $P_3(x) = \frac{15}{2}T_0 + \frac{17}{4}T_1 + \frac{1}{2}T_2 + \frac{3}{4}T_3$ . We must find particular solutions  $y^{(0)}$ ,  $y^{(1)}$ ,  $y^{(2)}$ ,  $y^{(3)}$  with respect to the corresponding Chebyshev polynomials  $T_0$ ,  $T_1$ ,  $T_2$ ,  $T_3$ . The coefficients for  $y^{(3)}$  is

$$\begin{aligned} e_3 &= \frac{d_3}{c} = \frac{4}{1.25} = \frac{16}{5}, \\ e_2 &= \frac{d_2 - 3be_3}{c} = \frac{0 - 3e_3}{1.25} = -\frac{192}{25}, \\ e_1 &= \frac{d_1 - 2be_2 - 6ae_3}{c} = \frac{-3 - 2e_2 - 6e_3}{1.25} = -\frac{684}{125}, \\ e_0 &= \frac{d_0 - be_1 - 2ae_2}{c} = \frac{0 - e_1 - 2e_2}{1.25} = \frac{10416}{625}. \end{aligned}$$

The coefficients for  $y^{(2)}$  is

$$\begin{aligned} e_2 &= \frac{d_2}{c} = \frac{2}{1.25} = \frac{8}{5}, \\ e_1 &= \frac{d_1 - b2e_2}{c} = \frac{0 - 2e_2}{1.25} = -\frac{64}{25}, \\ e_0 &= \frac{d_0 - be_1 - 2ae_2}{c} = \frac{-1 - e_1 - 2e_2}{1.25} = -\frac{164}{125}. \end{aligned}$$

The coefficients for  $y^{(1)}$  is

$$\begin{aligned} e_1 &= \frac{d_1}{c} = \frac{1}{1.25} = \frac{4}{5}, \\ e_0 &= \frac{d_0 - be_1}{c} = \frac{0 - e_1}{1.25} = -\frac{16}{25}. \end{aligned}$$

The coefficients for  $y^0$  is

$$e_0 = \frac{1}{1.25} = \frac{4}{5}.$$

We list these particular solutions as follows:

$$\begin{aligned} y^0 &= \frac{4}{5} \\ y^1 &= \frac{4}{5} - \frac{16}{25}x \\ y^2 &= \frac{-164}{125} - \frac{64}{25}x + \frac{8}{5}x^2 \\ y^3 &= \frac{10416}{625} - \frac{685}{125}x - \frac{192}{25}x^2 + \frac{16}{5}x^3 \end{aligned}$$

Hence we find an approximate particular solution for the equation as

$$\begin{aligned} y_p &= \frac{15}{2}y^0 + \frac{17}{4}y^1 + \frac{1}{2}y^2 + \frac{3}{4}y^3 \\ &= \frac{9452}{625} - \frac{248}{125}x - \frac{124}{25}x^2 + \frac{12}{5}x^3. \end{aligned}$$

The alternative approach further verifies the result obtained by the reduction of order method in Example 3.1.2.

## 5 Conclusions

The Chebyshev polynomials have been in existence for over a hundred years and they have been used for solving many different problems. In this thesis, we have a new strategy using chebyshev polynomials for a common problem. We obtain satisfactory results because of the excellent convergence rate of Chebyshev approximation, which is very close to the minimax polynomial which minimizes the maximum error in approximation. We find approximate particular solutions by Chebyshev polynomial approximation and the reduction of order for



the derivatives of Chebyshev polynomials. For comparison purpose, we solve same problems using an existing approach as an alternative approach. The first approach does not need the expansion of Chebyshev polynomials. We end up with an algebraic system of equations by expressing the derivatives of Chebyshev polynomials in terms of these polynomials. We can compare the coefficients of the Chebyshev polynomials in the resulting system of equations so that we can determine the coefficients of the particular solution. The second approach uses the approximation of a right hand side function, which we need to do expansion for each Chebyshev polynomial basis function. We use the particular solutions corresponding to each Chebyshev polynomial to construct a particular solution of the differential equation by superposition principle. Since we are approximating a particular solution with polynomials in both cases, a particular solution by either approach will be exact if the right hand function is already a polynomial.

## References

- [1] L. Fox and I. B. Parker, Chebyshev Polynomials in Numerical Analysis, Oxford University Press (C) 1968.
- [2] William Karush, The Crescent Dictionary of Mathematics, 7th Edition, MacMillan Publishing Co., Inc., 1974.
- [3] Philip Hartman, Ordinary Differential Equations, Birkhauser Boston, 1982.
- [4] M. C. Seiler and F. A. Seiler, Numerical Recipes in C: The art of scientific computing, Cambridge University Press. Programs Copyright (C) 1988-1992 by Numerical Recipes Software.
- [5] Theodore J. Rivlin, Chebyshev Polynomials: From Approximation Theory to Algebra and Number Theory, Wiley and Sons Copyright (C) 1990 by Wiley-Interscience.
- [6] R. K. Nagle, E. B. Saff and A. D. Snider, Fundamentals of Differential Equations, 6th Edition, Pearson, 2004.
- [7] S. Reutskiy and C.S. Chen, Approximation of multivariate functions and evaluation of particular solutions using Chebyshev polynomial and trigonometric basis functions International Journal for Numerical Methods in Engineering 2006, 67: 1811-1829
- [8] H.Y. Tian, Reducing order of derivatives and derivation of coefficients for Chebyshev polynomial approximation, preprint, 2006.
- [9] Timothy Sauer, Numerical Analysis, Pearson Education Inc. (C) 2006 - 2012.
- [10] David J. Benson, Music, a mathematical offering, Cambridge, 2007.

- [11] S. Reutskiy, C.S. Chen and H.Y. Tian, A boundary meshless method using Chebyshev interpolation and trigonometric basis function for solving heat conduction problems, *International Journal for Numerical Methods in Engineering*, 2008, 74: 1621-1644.
- [12] W. E. Boyce and R. C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, John Wiley & Sons, Inc, 2009.
- [13] J. Ding, H.Y. Tian and C.S. Chen, The recursive formulation of particular solutions for inhomogeneous elliptic PDEs with Chebyshev basis functions, *Communications in Computational Physics*, 2009, 5, 942-958.