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Fraction-Free Methods for Determinants

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FRACTION-FREE METHODS FOR DETERMINANTS

by

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ABSTRACT

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Given a matrix of integers, we wish to compute the determinant using a method that does not introduce fractions. Fraction-Free Triangularization, Bareiss' Algorithm (based on Sylvester's Identity) and Dodgson's Method (based on Jacobi's Theorem) are three such methods. However, both Bareiss' Algorithm and Dodgson's Method encounter division by zero for some matrices. Although there is a well-known workaround for the Bareiss Algorithm that works for all matrices, the workarounds that have been developed for Dodgson's method are somewhat difficult to apply and still fail to resolve the problem completely. After investigating new workarounds for Dodgson's Method, we give a modified version of the old method that relies on a well-known property of determinants to allow us to compute the determinant of any integer matrix.

DEDICATION

To the One from whom all things derive their order and beauty: *Soli Deo Gloria!* [1]

[1] For the Glory of God alone!

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So many people have been a part of this project that I shall never do them all justice in the space provided here.

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CHAPTER I

INTRODUCTION

Background Information

Before embarking on our study of fraction-free methods for computing determinants, we must have a thorough understanding of what is meant by the "determinant of a matrix." Thus we will begin with a discussion of relevant elementary definitions and properties of matrices and their determinants.

Geometrical Meaning of Determinant

> Algebra is nothing more than geometry in words; geometry is nothing more than algebra in pictures.
>
> — Sophie Germain.

The determinant of a square matrix can be defined using geometry. We will examine this definition by first developing the 2×2 case.

Notice that for any two vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$, the signed area of the parallelogram spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ is nonzero as long as the two vectors are linearly independent. Recall that the area of a parallelogram is the product of its base and its height. However, if we are given only the coordinates of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, the lengths of the base and height are unknown. If we choose the vector $\mathbf{v}_1$ to be the base, then the height is the length of the perpendicular line drawn from $\mathbf{v}_1$ to its parallel side, as shown in Figure 1.

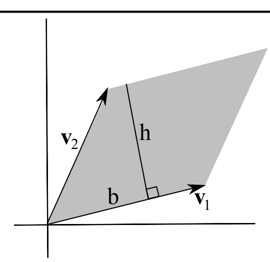

Figure 1. 2×2 determinant as area of a parallelogram.

Both the base, b, and the height, h, can be found using elementary geometry. Let us denote the two vectors as $\mathbf{v}_1 = (x_1, y_1)$ and $\mathbf{v}_2 = (x_2, y_2)$. Applying the distance formula, we can easily see that

$$\begin{aligned} b &= \sqrt{(x_1 - 0)^2 + (y_1 - 0)^2} \\ &= \sqrt{x_1^2 + y_1^2}. \end{aligned}$$

Finding h is slightly more involved. We begin by finding the equation of the line that passes through the points (x_2, y_2) and $(x_1 + x_2, y_1 + y_2)$:

$$y = \frac{y_1}{x_1}x + y_2 - \frac{y_1 x_2}{x_1}.$$

Next, we need the equation of the line that is perpendicular to the line above and passes through the origin:

$$y = -\frac{x_1}{y_1}x.$$

Now we find the intersection of the two lines, which we denote by (x_3, y_3):

$$\begin{aligned} x_3 &= \frac{y_1^2 x_2 - y_1 y_2 x_1}{x_1^2 + y_1^2}, \\ y_3 &= \frac{x_1^2 y_2 - y_1 x_2 x_1}{x_1^2 + y_1^2}. \end{aligned}$$

Finally, we find h using the distance formula:

$$\begin{aligned} h &= \sqrt{(x_3 - 0)^2 + (y_3 - 0)^2} \\ &= \sqrt{\left[\frac{y_1 y_2 x_1 - y_1^2 x_2}{x_1^2 + y_1^2}\right]^2 + \left[\frac{x_1^2 y_2 - y_1 x_2 x_1}{x_1^2 + y_1^2}\right]^2} \\ &= \sqrt{\frac{(x_1 y_2 - x_2 y_1)^2}{x_1^2 + y_1^2}} \end{aligned}$$

$$= \frac{x_1y_2 - x_2y_1}{\sqrt{x_1^2 + y_1^2}}.$$

Thus the formula for the area for the parallelogram is

$$\begin{aligned} Area &= bh \\ &= \sqrt{x_1^2 + y_1^2} \cdot \frac{x_1y_2 - x_2y_1}{\sqrt{x_1^2 + y_1^2}} \\ &= x_1y_2 - x_2y_1. \end{aligned}$$

Alternatively, this formula can be developed using vector analysis [11, pp. 273–274]. No matter which method is used, it turns out that the area of the parallelogram is equal to the determinant of the matrix A whose rows (or columns) are the vectors $\mathbf{v}_1 = (x_1, y_1)$ and $\mathbf{v}_2 = (x_2, y_2)$:

$$A = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}.$$

This equivalence becomes obvious if we recall the "shortcut" method for computing a 2×2 determinant that beginning linear algebra students are often taught:

$$\det(A) = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_1y_2 - x_2y_1 = Area.$$

We illustrate this relationship between the area of a parallelogram and the 2×2 determinant in Example 1.

Example 1. Find the area of the parallelogram in Figure 2.

The two vectors that define the parallelogram are $(1,1)$ and $(2,1)$. Thus the area of the parallelogram is

$$Area = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2 \cdot 1 - 1 \cdot 1 = 1.$$

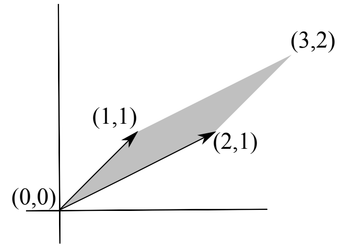

Figure 2. Area of the parallelogram corresponding to the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

It is believed to have been Lagrange who first used the determinant of a 3×3 matrix to calculate the volume of a three-dimensional solid [11, p. 301-302]. The volume definition of a determinant can be extended to the $n \times n$ case, as shown in the following theorem [11, p. 278].

Theorem 2. *For each $n \geq 1$, there is exactly one function D that associates to each ordered n-tuple of vectors $\mathbf{v}_1,...,\mathbf{v}_n \in \mathbb{R}^n$ a real number (the "signed volume" of the n-dimensional parallelepiped) and that has the following properties:*

1. *If any pair of the vectors $\mathbf{v}_1,...,\mathbf{v}_n$ are exchanged, D changes sign. That is,*

$$D\left(\mathbf{v}_1,...,\mathbf{v}_i,...,\mathbf{v}_j,...,\mathbf{v}_n\right) = -D\left(\mathbf{v}_1,...,\mathbf{v}_j,...,\mathbf{v}_i,...,\mathbf{v}_n\right)$$

   for any $1 \leq i < j \leq n$.

2. *For all $\mathbf{v}_1,...,\mathbf{v}_n \in \mathbb{R}^n$ and $c \in \mathbb{R}$, we have*

$$\begin{aligned} D(c\mathbf{v}_1, \mathbf{v}_2..., \mathbf{v}_n) &= D(\mathbf{v}_1, c\mathbf{v}_2..., \mathbf{v}_n) \\ &\vdots \\ &= D(\mathbf{v}_1, ..., \mathbf{v}_{n-1}, c\mathbf{v}_n) \\ &= cD(\mathbf{v}_1..., \mathbf{v}_n). \end{aligned}$$

3. *For any vectors* $\mathbf{v}_1,...,\mathbf{v}_n$ *and* $\mathbf{u}_i$ *we have*

$$D(\mathbf{v}_1,...,\mathbf{v}_{i-1},\mathbf{v}_i+\mathbf{u}_i,\mathbf{v}_{i+1},...,\mathbf{v}_n) = D(\mathbf{v}_1,...,\mathbf{v}_{i-1},\mathbf{v}_i,\mathbf{v}_{i+1},...,\mathbf{v}_n) + D(\mathbf{v}_1,...,\mathbf{v}_{i-1},\mathbf{u}_i,\mathbf{v}_{i+1},...,\mathbf{v}_n).$$

4. *If* $\{\mathbf{e}_1,...,\mathbf{e}_n\}$ *is the standard basis for* $\mathbb{R}^n$*, then we have*

$$D(\mathbf{e}_1,...,\mathbf{e}_n) = 1.$$

When it is necessary to write out the matrix entirely, the following notation is used:

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

In such cases, we use $A_{i...i+k,j...j+l}$ to refer to the $(k+1)\times(l+1)$ submatrix

$$A_{i...i+k,j...j+l} = \begin{pmatrix} a_{ij} & a_{i,j+1} & \dots & a_{i,j+l} \\ a_{i+1,j} & a_{i+1,j+1} & \dots & a_{i+1,j+l} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i+k,j} & a_{i+k,j+1} & \cdots & a_{i+k,j+l} \end{pmatrix}.$$

Motivation for the Study of Determinants

> Mathematics is the art of reducing problems to linear algebra.
>
> — William Hart, cited in [1]

Over the previous five centuries, many mathematicians have chosen to focus their studies on the determinant [10, pp. 1-3]. Consequently, we now have multiple uses for the determinant, some of which are more practical than others.

The famed Cramer's Rule uses determinants to solve systems of equations [2]. The determinant can also be used to determine the eigenvalues of a matrix A, which are the real roots of the characteristic polynomial of A:

$$p_A(t) = \det(A - tI).$$

Although these methods provide accurate results, they are rarely used in practice because of the large number of computations required. However, as the following theorems and definitions make clear, calculating the determinant of a square matrix is a convenient way to determine if the matrix is invertible.

Definition 3. An $n \times n$ matrix A is *singular* if and only if $\det(A) = 0$. Otherwise, A is *nonsingular*.

Nonsingular matrices have many useful properties, so it is often desirable to determine whether a matrix is nonsingular. Why?

Theorem 4. *Let A be an $n \times n$ matrix, and let $\mathbf{x}$ be a column vector of length n. Then A is nonsingular if and only if the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = 0$.*

Corollary 5. *Let A be an $n \times n$ matrix, and let $\mathbf{x}$ and $\mathbf{b}$ be column vectors of length n. Then A is nonsingular if and only if there is a unique solution to $A\mathbf{x} = \mathbf{b}$.*

We illustrate these ideas further in Example 6.

Example 6. Let

$$A = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Then

$$\begin{aligned} \det(A) &= 1 \begin{vmatrix} 4 & -6 \\ 8 & 9 \end{vmatrix} - (-2) \begin{vmatrix} -2 & 3 \\ 8 & 9 \end{vmatrix} + 7 \begin{vmatrix} -2 & 3 \\ 4 & -6 \end{vmatrix} \\ &= 84 - 84 + 0 \\ &= 0. \end{aligned}$$

Thus A is a singular matrix. We can illustrate Theorem 4 by the fact that $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions $\left(-\frac{45}{11}t, -\frac{6}{11}t, t\right)$. To illustrate Corollary 5, if $\mathbf{b} = (0,1,0)$, then there are no solutions to $A\mathbf{x} = \mathbf{b}$.

Common Methods for Computing Determinants

In introductory linear algebra courses, students are usually introduced to more than one method for computing determinants. The most fundamental method is the following. (For a definition of even and odd permutations, see [2, p. 113]).

Theorem 7. *Let A be an $n \times n$ matrix. Then*

$$\det(A) = \left[\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}\right],$$

where S_n is the set of all permutations of sequences of n elements, and $\operatorname{sgn}(\sigma)$ *is 1 if σ is even and -1 if σ is odd.*

However, this is impractical in general; there are several easier ways to compute the determinant. We discuss three of the most common methods here: special forms, cofactor expansion, and triangularization.

Special forms. This method is often introduced to students as a “short cut” for finding the determinant of two-by-two and three-by-three matrices. However, the method does not extend to higher dimensions. For a 2×2 matrix, students are taught to “cross-multiply” and subtract, as shown:

$$\begin{vmatrix} a & & b \\ & \times & \\ c & & d \end{vmatrix} = ad - bc.$$

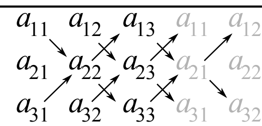

Figure 3. “Short cut” for 3×3 determinant.

For 3×3 matrices, the method is more involved. Students usually begin by recopying the first two columns on the right side of the original matrix:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \to \left.\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}\,\right|\,\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}\,.$$

They then multiply the entries of the diagonals as shown in Figure 3. This yields

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) \\ - (a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32} + a_{13}a_{22}a_{31})\,.$$

Example 8. Let A be the 3×3 matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & 3 \\ 2 & 4 & -1 \end{pmatrix}.$$

Then

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & 2 & 3 \\ -1 & 2 & 3 \\ 2 & 4 & -1 \end{vmatrix} \\ &= (1 \cdot 2 \cdot -1 + 2 \cdot 3 \cdot 2 + 3 \cdot (-1) \cdot 4) - (2 \cdot (-1) \cdot (-1) + 1 \cdot 3 \cdot 4 + 3 \cdot 2 \cdot 2) \\ &= (-2 + 12 - 12) - (2 + 12 + 12) \\ &= -28. \end{aligned}$$

Cofactor expansion. The method of cofactor expansion, discovered by Pierre-Simon Laplace [11, p. 301], is also sometimes referred to as “Laplace Expansion.” Unlike the “special forms” method, cofactor expansion can be used to find the determinant of square matrices of any dimension n. However, as n increases, the method becomes painfully cumbersome.

Definition 9. For a general $n \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

the *minor of entry* a_{ij}, denoted by M_{ij}, is the determinant of the matrix created by deleting the ith row and jth column of A. The *cofactor of entry* a_{ij} is

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

Example 10. For the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & 3 \\ 2 & 4 & -1 \end{pmatrix},$$

the cofactor of entry a_{13} is

$$\begin{aligned} C_{13} &= (-1)^{1+3} M_{13} \\ &= 1 \cdot \begin{vmatrix} -1 & 2 \\ 2 & 4 \end{vmatrix} \\ &= -8. \end{aligned}$$

Now that we have the necessary definitions, we are ready to look at the method itself, which is given in Theorem 11 [11, p. 292].

Theorem 11. *Let A be an $n \times n$ matrix. Then, for any fixed j, where $1 \leq j \leq n$, we have*

$$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij},$$

where C_{ij} is the submatrix of A created by deleting the i^{th} row and the j^{th} column of A.

The above proposition is also true if i is swapped with j. That is, if the row is fixed and the sum is over the columns instead of the rows. We illustrate this equivalence in Example 12.

Example 12. We use cofactor expansion to find the determinant of a 3×3 by expanding along the first row of the matrix:

$$\begin{aligned}\begin{vmatrix} 1 & 2 & 3 \\ -1 & 2 & 3 \\ 2 & 4 & -1 \end{vmatrix} &= (-1)^2(1)\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix} + (-1)^3(2)\begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} + (-1)^4(3)\begin{vmatrix} -1 & 2 \\ 2 & 4 \end{vmatrix} \\ &= (-2-12) - 2(1-6) + 3(-4-4) \\ &= -28.\end{aligned}$$

Triangularization. Like cofactor expansion, the triangularization method can also be used for any $n \times n$ matrix A. We begin our explanation with the definition of a *triangular matrix* [2, p. 69].

Definition 13. A *lower triangular matrix* L is an $n \times n$ matrix in which all entries above the main diagonal are zero. Likewise, an *upper triangular matrix* U is an $n \times n$ matrix in which all entries below the main diagonal are zero. Collectively, upper triangular and lower triangular matrices are referred to as simply *triangular matrices.*

The zero entries of triangular matrices are often denoted by blank spaces, as illustrated in Example 14.

Example 14. The matrices L and U are lower and upper triangular matrices, respectively:

$$L = \begin{pmatrix} 1 & & \\ 2 & 6 & \\ 3 & 4 & 5 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 2 & 3 \\ & 4 & 5 \\ & & 6 \end{pmatrix}.$$

Given a triangular matrix, we can use Theorem 15 to find its determinant.

Theorem 15. *Let A be a triangular $n \times n$ matrix. The determinant of A is the product of the diagonal entries of A. That is,*

$$\det(A) = a_{11}a_{22}\cdots a_{nn}.$$

Proof: First suppose A is an $n \times n$ upper triangular matrix. Applying Theorem 11, we choose to use cofactor expansion along the first column of A. Since all entries of this column are zero except a_{11}, all of the terms are zero except perhaps the one containing a_{11}:

$$\begin{aligned}\det(A) &= \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & a_{22} & \dots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{vmatrix} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} & \dots & a_{2n} \\ & a_{33} & \dots & a_{3n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{vmatrix}.\end{aligned}$$

Repeating this process for the determinant of the minor M_{11}, we have

$$\begin{aligned}a_{11} \begin{vmatrix} a_{22} & a_{23} & \dots & a_{2n} \\ & a_{33} & \dots & a_{3n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{vmatrix} &= a_{11}a_{22} \begin{vmatrix} a_{33} & a_{34} & \dots & a_{3n} \\ & a_{44} & \dots & a_{4n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{vmatrix} \\ &= a_{11}a_{22}a_{33} \begin{vmatrix} a_{44} & a_{45} & \dots & a_{4n} \\ & a_{55} & \dots & a_{5n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{vmatrix} \\ &\vdots \\ &= a_{11}a_{22}a_{33}\dots a_{nn}.\end{aligned}$$

If A is lower-triangular, expand along the first rows instead. □

However, what if our matrix A is not triangular? In such cases, we make use of the following theorem [11, p. 282]:

Theorem 16. *Let A be a square matrix.*

1. *Let A' be equal to A with two rows exchanged. Then* $\det(A') = -\det(A)$.
2. *Let A' be equal to A with one row multiplied by some scalar c. Then* $\det(A') = c\det(A)$.
3. *Let A' be equal to A after a multiple of one row has been added to another row. Then* $\det(A') = \det(A)$.

Each of the row operations listed above is called an elementary row operation.

Thus we can use elementary row operations to write A as a triangular matrix, a process we call *triangularization.* We then apply Theorem 16 to find the determinant of A in terms of the new triangular matrix, as illustrated in Example 17.

Example 17. We will use the triangularization method to find

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 2 & 3 \\ 2 & 4 & -1 \end{vmatrix}.$$

First, we add the first row to the second, which eliminates the first element in the second row:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 6 \\ 2 & 4 & -1 \end{vmatrix}.$$

Now we subtract twice the first row from the third row, which yields

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & -7 \end{vmatrix}.$$

Notice that the only operations used were the addition of a multiple of one row to another. Then, by part (3) of Theorem 16, the determinant of the triangular matrix is equal to that of the original matrix. Using Theorem 15, we multiply the diagonal elements to get $\det(A) = 1 \times 4 \times -7 = -28$. Note that this is the same answer as we found in Example 12.

Other Elementary Properties of the Determinant

The following propositions and examples illustrate other important properties of determinants. Notice that Theorem 18 is equivalent to statement (2) of Theorem 2.

Theorem 18. [2, p. 104] *Let A, B, and C be $n \times n$ matrices that differ only in the kth row, and assume that the kth row of C is equal to the sum of the kth row vectors of A and B. Then*

$$\det(C) = \det(A) + \det(B).$$

Example 19. Suppose

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & 3 \\ 2 & 4 & -1 \end{pmatrix}, \; B = \begin{pmatrix} 3 & 4 & 3 \\ -1 & 2 & 3 \\ 2 & 4 & -1 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 4 & 6 & 6 \\ -1 & 2 & 3 \\ 2 & 4 & -1 \end{pmatrix}.$$

Notice that matrices A, B, and C differ only in row one. In fact, the sum of row one of A and row one of B is equal to the first row of C, so we can use Theorem 18. Using cofactor expansion on A and B, we have

$$\det(A) = -28 \textit{ and } \det(B) = -46.$$

Applying Theorem 18 yields

$$\begin{aligned} \det(C) &= \det(A) + \det(B) \\ &= -28 + -46 \\ &= -74. \end{aligned}$$

Theorem 20. *[2, p. 287] If A is an $n \times n$ matrix, then the following are equivalent.*

(a) $\det(A) \neq 0$.

(b) *The column vectors of A are linearly independent.*

(c) The row vectors of A are linearly independent.

But what does it mean for vectors to be linearly independent?

Definition 21. [2, p. 241] If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a nonempty set of vectors, then the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n = \mathbf{0}.$$

That is, if each vector $\mathbf{v}_i$ can be written as a linear combination of the other vectors in the set, then the vectors in the set are said to be linearly dependent. Combining this definition with Theorem 20, we have $\det(A) = 0$ if and only if each of the row (or column) vectors of A can be written as a linear combination of the remaining row (or column) vectors.

The following proposition may be considered as a special case of Theorem 20.

Theorem 22. *[2, p. 104] If two rows of a square matrix are equal or are scalar multiples of each other, then* $\det(A) = 0$.

Example 23. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 4 & -1 \end{pmatrix}.$$

Since row two is equal to twice row one, $\det(A) = 0$ by Proposition 22.

Both Theorem 18 and Theorem 22 also hold for columns.

Theorem 24. *[8, p. 157] Let A be an $n \times n$ matrix of the form*

$$\begin{bmatrix} B & C \\ 0 & D \end{bmatrix},$$

where B is a $m \times m$ matrix, C is a $m \times (n-m)$ matrix, D is a $(n-m) \times (n-m)$ matrix, and 0 *is an $(n-m) \times m$ matrix of zeros. Then*

$$\det(A) = \det(B) \cdot \det(D).$$

Theorem 25. *[11, p. 284] Let A and B $n \times n$ matrices. Then*

$$\det(AB) = \det(A) \cdot \det(B).$$

Example 26. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & 3 \\ 2 & 4 & -1 \end{pmatrix} \quad and \quad B = \begin{pmatrix} 3 & 4 & 5 \\ 2 & -1 & 3 \\ 1 & 3 & 4 \end{pmatrix}.$$

Then the product of A and B is

$$AB = \begin{pmatrix} 10 & 11 & 23 \\ 4 & 3 & 13 \\ 13 & 1 & 18 \end{pmatrix},$$

and its determinant is

$$\det(AB) = \det(A) \cdot \det(B) = -28 \cdot -24 = 672.$$

Definition 27. Given an $n \times n$ matrix A, the *adjugate of A* is the transpose of the matrix of cofactors of A, as shown below:

$$\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \vdots & C_{nn} \end{bmatrix}.$$

Example 28. Suppose

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & 3 \\ 2 & 4 & -1 \end{pmatrix}.$$

Then the cofactors of A are

$$\begin{array}{lll} C_{11} = -14 & C_{12} = 5 & C_{13} = -3 \\ C_{21} = 14 & C_{22} = -7 & C_{23} = 0 \\ C_{31} = 0 & C_{32} = -6 & C_{33} = 4 \end{array}.$$

By substitution,

$$\mathrm{adj}(A) = \begin{bmatrix} -14 & 5 & -3 \\ 14 & -7 & 0 \\ 0 & -6 & 4 \end{bmatrix}^T$$

$$= \begin{bmatrix} -14 & 14 & 0 \\ 5 & -7 & -6 \\ -3 & 0 & 4 \end{bmatrix}.$$

It is important to note that the adjugate is sometimes confused with the *adjoint of A,* which is the complex conjugate of the adjugate of A. This confusion is understandable since the two are equivalent when A is defined over the field of real numbers.

Fraction-Free Determinant Computation

In this thesis, we will study efficient methods for the fraction-free computation of determinants of dense matrices. What does that mean? A dense matrix is a matrix whose entries are primarily non-zero, while a matrix containing a high percentage of zero entries is called a sparse matrix [5, p. 417].

Fraction-Free?

In some cases, the triangularization method for computing determinants introduces fractions into a matrix of integers.

Example 29. Suppose

$$A = \begin{pmatrix} 3 & 1 & 7 \\ -2 & 5 & 1 \\ 2 & 1 & 4 \end{pmatrix}.$$

Notice that all of the entries of A are integers. Theorem 7 implies that the determinant of A is also an integer. However, using the triangularization method to compute the determinant of A introduces fractions. We start by adding $\frac{2}{3}$ times row one to row two and $-\frac{2}{3}$ times row one to row three, then adding an appropriate multiple of the new row two to the new row three:

$$\begin{aligned}
\det(A) &= \det\begin{pmatrix} 3 & 1 & 7 \\ -2 & 5 & 1 \\ 2 & 1 & 4 \end{pmatrix} \\
&= \det\begin{pmatrix} 3 & 1 & 7 \\ 0 & \frac{17}{3} & \frac{17}{3} \\ 0 & \frac{1}{3} & -\frac{2}{3} \end{pmatrix} \\
&= \det\begin{pmatrix} 3 & 1 & 7 \\ 0 & \frac{17}{3} & \frac{17}{3} \\ 0 & 0 & -1 \end{pmatrix} \\
&= 3 \cdot \frac{17}{3} \cdot (-1) \\
&= -17.
\end{aligned}$$

Even though the final step yields an integer for the determinant of A, the fractions introduced along the way make the method somewhat inefficient by requiring additional time and space to compute the determinant. This is true both if the calculations are done by hand or if they are carried out by a computer program. Thus having a fraction-free method for computing determinants would be very useful:

Definition 30. *Fraction-free determinant computation* is a method of computing determinants such that any divisions that are introduced are exact [13, p. 262]. In contrast, *division-free determinant computation* is a method of computing determinants that requires no divisions.

Using these definitions, we see that cofactor expansion would be an example of a division-free method. However, because fractions introduced in the triangularization method are not necessarily eliminated at each step (see Example 29), triangularization is neither fraction-free nor division free.

Why do we not consider “floating point” instead of fractions and exact arithmetic?

We wish to avoid the rounding error that is introduced by the use of floating point arithmetic. Such error is especially problematic if the matrix contains entries that are approximately equal to zero, as divisions by zero can occur.

Although it might appear that using exact arithmetic would be painfully slow, it is actually still quite fast on finite fields. The Chinese Remainder Theorem allows to construct any answer from a finite field in the form of a rational number. The result is often fast and accurate [12, p. 101–105].

Why Not Cofactor Expansion?

The arguments above seem to lead us back to the elementary fraction-free method that we discussed earlier: cofactor expansion. However, as was also mentioned previously, this method becomes extremely cumbersome as n increases. In fact, for n as small as $n = 5$, the number of multiplications alone is a small multiple of 120. In general, the number of multiplications required to calculate the determinant of an $n \times n$ matrix using cofactor expansion is a small multiple of $n!$ [7, p. 295]. This is due to the recursive nature of the method, which is illustrated by the 5×5 case in Example 31.

Example 31. Suppose

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & -1 \\ 2 & -3 & 4 & 1 & -2 \\ -1 & 4 & -2 & 3 & 1 \\ 3 & -2 & 1 & -4 & 6 \\ 4 & 2 & -1 & 1 & 0 \end{pmatrix}.$$

We expand the determinant by cofactor expansion along the first row.

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 & 4 & -1 \\ 2 & -3 & 4 & 1 & -2 \\ -1 & 4 & -2 & 3 & 1 \\ 3 & -2 & 1 & -4 & 6 \\ 4 & 2 & -1 & 1 & 0 \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} -3 & 4 & 1 & -2 \\ 4 & -2 & 3 & 1 \\ -2 & 1 & -4 & 6 \\ 2 & -1 & 1 & 0 \end{vmatrix} - 2 \cdot \begin{vmatrix} 2 & 4 & 1 & -2 \\ -1 & -2 & 3 & 1 \\ 3 & 1 & -4 & 6 \\ 4 & -1 & 1 & 0 \end{vmatrix} + 3 \cdot \begin{vmatrix} 2 & -3 & 1 & -2 \\ -1 & 4 & 3 & 1 \\ 3 & -2 & -4 & 6 \\ 4 & 2 & 1 & 0 \end{vmatrix}$$

$$- 4 \cdot \begin{vmatrix} 2 & -3 & 4 & -2 \\ -1 & 4 & -2 & 1 \\ 3 & -2 & 1 & 6 \\ 4 & 2 & -1 & 0 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 2 & -3 & 4 & 1 \\ -1 & 4 & -2 & 3 \\ 3 & -2 & 1 & -4 \\ 4 & 2 & -1 & 1 \end{vmatrix}.$$

Now we must resolve five 4×4 determinants. If we wish to use expansion by cofactors on each of the five submatrices, we would have to compute the determinants of twenty 3×3 matrices. This means that, to find the determinant of A, we would have to find the determinants of sixty 2×2 determinants.

This is exceedingly tedious to do by hand. Thus we turn elsewhere in our search for an efficient, fraction-free method for computing determinants.

CHAPTER II

KNOWN FRACTION-FREE METHODS

In this chapter, we will discuss three fraction-free methods for computing determinants: fraction-free triangularization, the Bareiss Algorithm, and Dodgson's Method.

Fraction-Free Triangularization

The Method

Fraction-free triangularization is, as its name implies, a fraction-free version of the triangularization method discussed in Subsection 1. To make the triangularization method fraction-free, we must modify the way in which we use row operations to triangularize the matrix. Specifically, we will need to use Theorem 16, part (2) (multiplying a row by a scalar) in addition to part 3 of that proposition (adding a multiple of one row to another row). Thus, instead of using the following step to eliminate the first nonzero element in row $i+1$ of a matrix

$$[a^{(i)}_{i+1,i}, a^{(i)}_{i+1,i+1}, \ldots, a^{(i)}_{i+1,n}] = [a^{(i+1)}_{i+1,i}, a^{(i+1)}_{i+1,i+1}, \ldots, a^{(i+1)}_{i+1,n}] - \frac{a^{(i+1)}_{i+1,i}}{a^{(i+1)}_{i,i}}[a^{(i+1)}_{i,i}, a^{(i+1)}_{i,i+1}, \ldots, a^{(i+1)}_{i,n}],$$

we use $d = \gcd\left(a^{(i+1)}_{i,1}, a^{(i+1)}_{i+1,1}\right)$ to prevent the introduction of fractions into the matrix:

$$[a^{(i)}_{i+1,i}, a^{(i)}_{i+1,i+1}, \ldots, a^{(i)}_{i+1,n}] = \frac{a^{(i+1)}_{i,i}}{d}[a^{(i+1)}_{i+1,i}, a^{(i+1)}_{i+1,i+1}, \ldots, a^{(i+1)}_{i+1,n}] - \frac{a^{(i+1)}_{i+1,i}}{d}[a^{(i+1)}_{i,i}, a^{(i+1)}_{i,i+1}, \ldots, a^{(i+1)}_{i,n}].$$

To compensate for multiplying row $i+1$ by $\frac{a^{(i+1)}_{i,i}}{d}$, of Theorem 16(2) requires us to divide the determinant of the new matrix $A^{(i)}$ by $\frac{a^{(i+1)}_{i,i}}{d}$ in order to find $\det(A)$. Since there will be a factor $\frac{a^{(i+1)}_{i,i}}{d}$ for each row $i+1, i+2, \ldots, n$, for columns i through n, we store their product in the variable uf, meaning "unnecessary factor."

To better understand this method, let us reexamine Example 29:

$$A = \begin{pmatrix} 3 & 1 & 7 \\ -2 & 5 & 1 \\ 2 & 1 & 4 \end{pmatrix}.$$

In this case, the triangularization method introduces fractions, so we apply our fraction-free method. We begin by multiplying row one by -2 and subtracting it from 3 times row two:

$$\begin{aligned} row\,2 &= 3\begin{bmatrix} -2 & 5 & 1 \end{bmatrix} - (-2)\begin{bmatrix} 3 & 1 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 17 & 17 \end{bmatrix}. \end{aligned}$$

To finish the first column, we subtract 2 times row one from 3 times row three. This yields

$$A^{(2)} = \begin{pmatrix} 3 & 1 & 7 \\ 0 & 17 & 17 \\ 0 & 1 & -2 \end{pmatrix},$$

where $A^{(2)}$ is the matrix after the first column of A has been triangularized. Notice that no fractions were introduced in the matrix. Since we multiplied rows two and three by 3, we have

$$\begin{aligned} \det(A) &= \det\left(A^{(2)}\right) \div (3 \cdot 3) \\ &= \frac{\det\left(A^{(2)}\right)}{9} \end{aligned}$$

by Theorem 16(2). To completely triangularize the matrix, we repeat the process for column two to get

$$\det(A) = \frac{\det\left(A^{(1)}\right)}{153} = -17.$$

This fraction-free version of triangularization is outlined in Algorithm 1.

In Example 32, we apply the method to another 3×3 matrix.

Algorithm 1. fftd

algorithm *fftd*
  inputs
    $A \in \mathbb{Z}^{n \times n}$
  outputs
    $\det(A)$
  do
    — *uf is the unnecessary factor*
    Let $uf = 1$
    for $j \in \{1, \ldots, n-1\}$ **do**
      — *if 0 on diagonal, cannot eliminate, so swap*
      if $a_{jj} = 0$ **then**
        for $\ell \in \{j+1, \ldots, n\}$ **do**
          if $a_{\ell j} \neq 0$ **then**
            Swap rows ℓ and j
            Let $uf = -uf$
        — *unable to swap?* $\det(A) = 0$
        if $a_{jj} = 0$ **then**
          return 0
      for $i \in \{j+1, \ldots, n\}$ **do**
        if $a_{ij} \neq 0$ **then**
          Let $d = \gcd\left(a_{jj}, a_{ij}\right)$
          Let $b = \frac{a_{jj}}{d}$, $c = \frac{a_{ij}}{d}$
          — *account for multiplying row i by b*
          Let $uf = b \cdot uf$
          for $k \in \{j+1, \ldots, n\}$ **do**
            Let $a_{ik} = ba_{ik} - ca_{jk}$
    return $\frac{\prod_{i=1}^{n} a_{ii}}{uf}$

Example 32. Find $\det(A)$ using fraction-free triangularization, where

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 5 & 4 & 3 \\ 6 & 3 & 4 \end{pmatrix}.$$

Following the algorithm above, we begin with the first column ($j = 1$). First, we define $d = \gcd(3,5) = 1$. Then $a = 3$ and $b = 5$, which yields $uf = 3$ and

$$\begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 6 & 3 & 4 \end{pmatrix}.$$

Repeating this for the third row, we have $d = \gcd(3,6) = 3$, $a = 1$, $b = 2$, $uf = 3$, and

$$\begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 0 & -1 & 2 \end{pmatrix}.$$

Now we wish to work with the second column ($j = 2$). We define $d = \gcd(2,-1) = 1$.

This implies that $a = 2$ and $b = -1$, which yields $uf = 6$ and

$$\begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 8 \end{pmatrix}.$$

Thus

$$\det(A) = \frac{3 \cdot 2 \cdot 8}{6} = 8.$$

Proof of Fraction-Free Triangularization

We will now examine a proof of the fraction-free triangularization method (*fftd*).

In Chapter 1, we looked at a proof of the formula

$$\det(A) = a_{11}a_{22} \cdots a_{nn},$$

where A is an $n \times n$ triangular matrix (Theorem 15). Our proof of *fftd* will use this formula and Theorem 16.

Theorem 33. *Algorithm 1 terminates correctly in $O(n^3)$ ring operations.*

Proof: First, notice that since all loops in Algorithm 1 are over finite sets that are never modified, the algorithm terminates. To examine the correctness of the method, we will consider two cases.

Case (i): There were no zeros on the diagonal, or any zeros on the diagonal can be eliminated using row swaps.

Notice that all steps of the method can be justified using Proposition 15 and Proposition 16.

Case (*ii*): There are one or more zeros on the diagonal that cannot be eliminated by row swaps.

We use cofactor expansion. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

be such that, in the kth iteration, there is a zero in the diagonal that cannot be eliminated by row swaps. Since the previous $k-1$ iterations did not find a zero in the diagonal, the first k rows have been triangularized. Then, using cofactor expansion along the columns of A, we have

$$\begin{aligned} \det(A) &= a_{11} \begin{vmatrix} a_{22} & a_{23} & \dots & a_{2n} \\ a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} \\ &= a_{11}a_{22} \begin{vmatrix} a_{33} & a_{34} & \dots & a_{3n} \\ a_{43} & a_{44} & \dots & a_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n3} & a_{n4} & \dots & a_{nn} \end{vmatrix} \\ &\vdots \\ &= a_{11}a_{22}\dots a_{kk} \begin{vmatrix} a_{k+1,k+1} & a_{k+1,k+2} & \dots & a_{k+1,n} \\ a_{k+2,k+1} & a_{k+2,k+2} & \dots & a_{k+2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,k+1} & a_{n,k+2} & \dots & a_{nn} \end{vmatrix}. \end{aligned}$$

Since the zero in the $a_{k+1,k+1}$ entry could not be eliminated by row swaps, the other entries in the $k+1$ column must also be zero. This implies

$$\det(A) = a_{11}a_{22}\dots a_{kk} \begin{vmatrix} 0 & a_{k+1,k+2} & \dots & a_{k+1,n} \\ 0 & a_{k+2,k+2} & \dots & a_{k+2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n,k+2} & \dots & a_{nn} \end{vmatrix} = 0.$$

In general, Algorithm 1 runs three nested loops: one with $n-1$ iterations and two with

$n-2$ iterations. Thus fraction-free triangularization is $O\left(n^3\right)$. □

Bareiss' Algorithm

The Method

The Bareiss Algorithm is another fraction-free method for determinant computation. However, it can also be thought of as a sophisticated form of row reduction.

Suppose A is an $n \times n$ matrix. In the ith iteration of Bareiss' Algorithm, we multiply rows $i, \ldots, n$ of A by $a_{i-1,i-1}$. We then add an appropriate multiple of row $i-1$ to rows $i, \ldots, n$ and divide the new rows $i, \ldots, n$ by $a_{i-2,i-2}^{(i+2)}$. Entry $a_{nn}^{(1)}$ is the determinant of A.

Note that the divisions computed at any step are exact; thus Bareiss' Algorithm is indeed fraction-free. Algorithm 2 gives the method in detail.

Example 34. Find $\det(A)$ using the Bareiss Algorithm, where

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 5 & 4 & 3 \\ 6 & 3 & 4 \end{pmatrix}.$$

Following the algorithm above, we begin by defining $a_{0,0} = 1$. For the first iteration ($k = 1$) of the outside loop, we alter only a_{22}, a_{23}, a_{32}, and a_{33}. For $i = 2$, $j = 2$, we have

$$a_{22}^{(2)} = \frac{a_{22} \cdot a_{11} - a_{21} \cdot a_{12}}{a_{00}} = 2.$$

Continuing this process for the remaining entries gives us

$$A^{(2)} = \begin{pmatrix} 3 & 2 & 1 \\ 5 & 2 & 4 \\ 6 & -3 & 6 \end{pmatrix}.$$

For $k = 2$, the algorithm yields

$$A^{(1)} = \begin{pmatrix} 3 & 2 & 1 \\ 5 & 2 & 4 \\ 6 & -3 & 8 \end{pmatrix}.$$

Thus

$$\det(A) = A^{(1)}_{3,3} = 8.$$

Algorithm 2. Bareiss Algorithm

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algorithm *Bareiss Algorithm [13, p. 263]*
inputs
$A \in \mathbb{Z}^{n \times n}$, an $n \times n$ matrix whose principal minors $x^{(k)}_{kk}$ are all nonzero
outputs
$\det(A)$, which is stored in a_{nn}
do
— *Note that $a_{0,0}$ is a special variable*
Let $a_{0,0} = 1$.
for $k \in \{1, \ldots, n-1\}$ **do**
  for $i \in \{k+1, \ldots, n\}$ **do**
    for $j \in \{k+1, \ldots, n\}$ **do**
      Let $a_{ij} = \frac{a_{ij} \cdot a_{kk} - a_{ik} \cdot a_{kj}}{a_{k-1,k-1}}$.
return $a_{n,n}$

Proof of the Bareiss Algorithm

One of the most common proofs for the correctness of the Bareiss Algorithm uses Sylvester's Identity [4, 13]. Here, however, we will show that Algorithm 2 terminates correctly using the same principles of linear algebra as we used for our proof of *fftd* (Algorithm 1). To understand either the proof presented here or the proof that uses Sylvester's Identity, we need the following notation.

Let A be an $n \times n$ matrix. Then[2]

$$a^{|k|}_{r,s} = \left| \begin{array}{c|c} A^{(k-1)}_{1\ldots k,1\ldots k} & A^{(k-1)}_{1\ldots k,s} \\ \hline A^{(k-1)}_{r,1\ldots k} & a^{(k-1)}_{r,s} \end{array} \right|$$

is the r^{th} entry in column s of the k^{th} iteration matrix, $A^{(k)}$, where $A^{(k-1)}$ is the previous

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[2] If this notation appears too cumbersome, compare it to the notation in [4, 13].

iteration matrix. Note that

$$a_{r,s}^{(k-1)} \neq a_{r,s}^{|k-1|};$$

since $a_{r,s}^{(k-1)}$ is an entry in $A^{(k-1)}$ while $a_{r,s}^{|k-1|}$is a determinant of a submatrix of $A^{(k-2)}$.

Theorem 35 (Sylvester's Indentity). *[13, p. 262] Let A be an $n \times n$ matrix. Then*

$$\left(a_{k-1,k-1}^{|k-1|}\right)^{n-k} \det(A) = \begin{vmatrix} a_{k,k}^{|k|} & a_{k,k+1}^{|k|} & \cdots & a_{k,n}^{|k|} \\ a_{k+1,k}^{|k|} & a_{k+1,k+1}^{|k|} & \cdots & a_{k+1,n}^{|k|} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,k}^{|k|} & a_{n,k+1}^{|k|} & \cdots & a_{n,n}^{|k|} \end{vmatrix}.$$

Some of the ideas used in the proof below were also presented in *An Introduction to Numerical Linear Algebra* by L. Fox, which was cited by Bareiss in his 1968 paper on Sylvester's Identity (see [4, 6, pp. 84-85]).

Theorem 36. *If division by zero does not occur, Algorithm 2 terminates correctly in $O\left(n^3\right)$ ring operations.*

Proof: First, notice that since all loops in Algorithm 2 are over finite sets that are never modified, the algorithm terminates. To examine the correctness of the method, we will consider two cases.

Case (i): One or more principal minors $a_{kk}^{|k|}$ of A are zero. In this case, we cannot use Algorithm 2 to find the determinant of A.

Case (ii): All principal minors $a_{kk}^{|k|}$ of A are nonzero. Let A

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

be such that all principal minors of A are nonzero. Compute B from A by multiplying rows 2 through n by a_{11} and adding the necessary multiple of row 1 to eliminate entries

$a_{2,1}, a_{3,1}, \ldots, a_{n,1}$. We now have the matrix

$$B = \begin{pmatrix} a_{11} & * \; * \; * \\ & A' \end{pmatrix}.$$

Then

$$\det(A) = \frac{1}{a_{11}^{n-1}} \det(B) = \frac{1}{a_{11}^{n-1}} a_{11} \det\left(A'\right) = \frac{1}{a_{11}^{n-2}} \det\left(A'\right).$$

Compute C from A' by multiplying rows 2 through $n-1$ by $a_{11}^{'}$ and adding the necessary multiple of row 1 to eliminate entries $a_{2,1}^{'}, a_{3,1}^{'}, \ldots, a_{n-1,1}^{'}$. We now have the matrix

$$C = \begin{pmatrix} a_{11}' & * \; * \; * \\ & A'' \end{pmatrix}.$$

Then

$$\begin{aligned} \det(A) &= \tfrac{1}{a_{11}^{n-2}} \cdot \det(A') \\ &= \tfrac{1}{a_{11}^{n-2}} \left(\tfrac{1}{(a_{11}')^{n-2}} \cdot \det(C) \right) \\ &= \tfrac{1}{a_{11}^{n-2}} \left(\tfrac{1}{(a_{11}')^{n-2}} \cdot \det(C) \right) \\ &= \tfrac{1}{a_{11}^{n-2}} \left(\tfrac{1}{(a_{11}')^{n-2}} \cdot a_{11}' \cdot \det(A'') \right) \\ &= \tfrac{1}{a_{11}^{n-2}} \left(\tfrac{1}{(a_{11}')^{n-3}} \cdot \det(A'') \right) \\ &= \tfrac{1}{(a_{11}')^{n-3}} \left(\tfrac{1}{a_{11}^{n-2}} \cdot \det(A'') \right). \end{aligned} \tag{1}$$

Notice that every element in A' and A'' was obtained using the same 2×2 determinants as in the Bareiss Algorithm except that we have not yet divided each element by a_{11}. If we do so, we can rewrite Equation (1) as $\det(A) = \frac{1}{(a_{11}')^{n-3}} \cdot \det(D)$, where D is the matrix obtained from the Bareiss Algorithm. To show that D is fraction-free, we look at the j^{th} entry in the i^{th} column of A' $a_{ij}' = a_{11} \cdot a_{i+1,j+1} - a_{i+1,1} \cdot a_{1,j+1}$ and the corresponding entry of A''

$$a_{ij}'' = a_{11}' \cdot a_{i+1,j+1}' - a_{i+1,1}' \cdot a_{1,j+1}'$$

$$= (a_{11}a_{22} - a_{21}a_{12})\left(a_{11}a_{i+2,j+2} - a_{i+2,1}a_{1,j+2}\right)$$

$$- (a_{11}a_{i+2,2} - a_{i+2,1}a_{1,2})\left(a_{11}a_{2,j+2} - a_{2,1}a_{1,j+2}\right)$$

$$= a_{11}a_{22}a_{11}a_{i+2,j+2} - a_{11}a_{22}a_{i+2,1}a_{1,j+2} - a_{21}a_{12}a_{11}a_{i+2,j+2}$$

$$- a_{11}a_{i+2,2}a_{11}a_{2,j+2} + a_{11}a_{i+2,2}a_{2,1}a_{1,j+2} + a_{i+2,1}a_{1,2}a_{11}a_{2,j+2}.$$

Since the only two terms not containing a_{11} cancel, each a''_{ij} is divisible by a_{11}. Thus D is fraction-free.

As stated above, the remaining iterations are similar to the first. Thus the Bareiss Algorithm is a fraction-free method that can be used to find the determinant of an $n \times n$ matrix A.

Like Algorithm 1, Algorithm 2 uses three nested loops. Since each of these uses up to $n-1$ iterations, the Bareiss Algorithm takes $O\left(n^3\right)$ arithmetic steps [13, p. 264]. $\square$

Notice that the Bareiss Algorithm is performing the same operations as fraction-free triangularization with the exception that the divisions are performed during each iteration of Bareiss while *fftd* divides by the product of these factors (*uf*) in the final step.

Dodgson's Method

The Method

In each iteration of Dodgson's Method, we create a new matrix whose entries are contiguous two-by-two determinants made up of the entries of the matrix created in the previous iteration. We then divide each of the entries of the new matrix by the corresponding entry in the interior of the matrix created two iterations before. In this way, each iteration yields a matrix whose dimension is one less than that of the matrix from the previous iteration. It is for this reason that Dodgson's Method, like Bareiss' Algorithm, is often called a "condensation method."

The following example illustrates how Dodgson's Method is applied to a 4×4 matrix.

Example 37. Let

$$A = A^{(4)} = \begin{pmatrix} 1 & -2 & 1 & 2 \\ -1 & 4 & -2 & 1 \\ 3 & 3 & 3 & 4 \\ 2 & 5 & 2 & -1 \end{pmatrix}.$$

Then the first iteration of Dodgson's Method yields

$$A^{(3)} = \begin{pmatrix} \begin{vmatrix} 1 & -2 \\ -1 & 4 \end{vmatrix} & \begin{vmatrix} -2 & 1 \\ 4 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} \\ \\ \begin{vmatrix} -1 & 4 \\ 3 & 3 \end{vmatrix} & \begin{vmatrix} 4 & -2 \\ 3 & 3 \end{vmatrix} & \begin{vmatrix} -2 & 1 \\ 3 & 4 \end{vmatrix} \\ \\ \begin{vmatrix} 3 & 3 \\ 2 & 5 \end{vmatrix} & \begin{vmatrix} 3 & 3 \\ 5 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} \end{pmatrix}$$

$$\Longrightarrow A^{(3)} = \begin{pmatrix} 2 & 0 & 5 \\ -15 & 18 & -11 \\ 9 & -9 & -11 \end{pmatrix}.$$

For $i < n{-}1$, we divide elements of $A^{(i)}$ by the interior of $A^{(i+2)}$:

$$A^{(2)} = \begin{pmatrix} \dfrac{\begin{vmatrix} 2 & 0 \\ -15 & 18 \end{vmatrix}}{4} & \dfrac{\begin{vmatrix} 0 & 5 \\ 18 & -11 \end{vmatrix}}{-2} \\ \dfrac{\begin{vmatrix} -15 & 18 \\ 9 & -9 \end{vmatrix}}{3} & \dfrac{\begin{vmatrix} 18 & -11 \\ -9 & -11 \end{vmatrix}}{3} \end{pmatrix} = \begin{pmatrix} 9 & 45 \\ -9 & -99 \end{pmatrix}$$

$$\Longrightarrow A^{(1)} = \left(\frac{-486}{18} \right) = (-27).$$

Since $\det(A) = a_{1,1}^{(1)}$, we have $\det(A) = -27$.

Proof of Dodgson's Method

Dodgson's Method is based on the following theorem of Jacobi (as explained in [9]).

Theorem 38 (Jacobi's Theorem). *Let L be an $n \times n$ matrix and M an $m \times m$ minor of L chosen from rows $i_1, i_2, \ldots, i_m$ and columns $j_1, j_2, \ldots, j_m$. Let M' be the corresponding $m \times m$ minor of L', the matrix of cofactors of L, and let M^* be the complementary $(n-m) \times (n-m)$ minor of M in L. Then*

$$\det\left(M'\right) = \det\left(L\right)^{m-1} \cdot M^* \left(-1\right)^{\sum_{l=1}^{m} i_l + j_l}.$$

A sketch of a proof of Jacobi's Theorem can be found in [3]. To prove that Algorithm 3 terminates correctly, we first prove Dodgson's Condensation Theorem (Theorem [39]). The successful termination of Dodgson's Method follows directly from this theorem.

Algorithm 3. Dodgson's Method

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algorithm *Dodgson's Method*
  inputs
    $A \in \mathbb{Z}^{n \times n}$
  outputs
    $\det(A)$
  do

    Let $C = 0$
    while (number of columns in A)> 1 **do**
      Let m be the number of rows in A
      Let B be an $(m-1) \times (m-1)$ matrix of zeros
      for $i\varepsilon \in \{1, \ldots, m-1\}$ **do**
        for $j \in \{1, \ldots, m-1\}$ **do**
          Let $b_{ij} = a_{ij} \cdot a_{i+1,j+1} - a_{i+1,j} \cdot a_{i,j+1}$
      if $C \neq 0$ **then**
        for $i \in \{1, \ldots, m-1\}$ **do**
          for $j \in \{1, \ldots, m-1\}$ **do**
            Let $b_{ij} = \frac{b_{ij}}{c_{i+1,j+1}}$
      Let $C = A$
      Let $A = B$
    return A

Theorem 39 (Dodgson's Condensation Theorem). *Let A be an $n \times n$ matrix. After k successful condensations, Dodgson's method produces the matrix*

$$A^{(n-k)} = \begin{pmatrix} |A_{1\dots k+1,1\dots k+1}| & |A_{1\dots k+1,2\dots k+2}| & \cdots & |A_{1\dots k+1,n-k\dots n}| \\ |A_{2\dots k+2,1\dots k+1}| & |A_{2\dots k+2,2\dots k+2}| & \cdots & |A_{2\dots k+2,n-k\dots n}| \\ \vdots & \vdots & \ddots & \vdots \\ |A_{n-k\dots n,1\dots k+1}| & |A_{n-k\dots n,2\dots k+2}| & \cdots & |A_{n-k\dots n,n-k\dots n}| \end{pmatrix}$$

whose entries are the determinants of all $(k+1) \times (k+1)$ submatrices of A [9, p. 48].

Proof: (From [9, p. 48-49]) By "successful" condensations, we mean not encountering division by zero. The proof is by induction on k.

Base case: When $k = 1$, the theorem is trivial: the first condensation is

$$A^{(n-1)} = \begin{pmatrix} |A_{1\dots 2,1\dots 2}| & |A_{1\dots 2,2\dots 3}| & \cdots & |A_{1\dots 2,n-1\dots n}| \\ |A_{2\dots 3,1\dots 2}| & |A_{2\dots i+2,2\dots i+2}| & \cdots & |A_{2\dots 3,n-1\dots n}| \\ \vdots & \vdots & \ddots & \vdots \\ |A_{n-1\dots n,1\dots 2}| & |A_{n-i\dots n,2\dots i+2}| & \cdots & |A_{n-1\dots n,n-1\dots n}| \end{pmatrix}.$$

Inductive Hypothesis: Fix $k \geq 1$. Assume that for all $\ell = 1, \dots, k$, the ℓth condensation gives us $A^{(n-\ell)}$ where for all $1 \leq i, j \leq n$

$$a_{i,j}^{(n-\ell)} = \left|A_{i\dots i+(n-\ell),j\dots j+(n-\ell)}\right|.$$

Inductive Step: Let $i, j \in \{1, \dots, n-(k+1)\}$. The next condensation in Dodgson's method gives us

$$a_{i,j}^{(n-(k+1))} = \frac{a_{i,j}^{(n-k)} a_{i+1,j+1}^{(n-k)} - a_{i+1,j}^{(n-k)} a_{i,j+1}^{(n-k)}}{a_{i+1,j+1}^{(n-(k-1))}}.$$

From the inductive hypothesis, we know that

$$a_{i,j}^{(n-(k+1))} = \frac{\left(\begin{array}{l} |A_{i\dots i+k,j\dots j+k}|\,|A_{i+1\dots i+k+1,j+1\dots j+k+1}| \\ \qquad - |A_{i+1\dots i+k+1,j\dots j+k}|\,|A_{i\dots i+k,j+1\dots j+k+1}| \end{array} \right)}{|A_{i+1\dots i+k,j+1\dots j+k}|}.$$

Let $L = A_{i\dots i+k+1,j\dots j+k+1}$ and M be the 2×2 minor made up of the corners of L. Let M' be the corresponding 2×2 minor of L', and M^* be the interior of L (that is, the complementary $k \times k$ minor of M in L). By Jacobi's Theorem,

$$\det M' = (\det L)^{2-1} \cdot \det M^* \cdot (-1)^{i+(j+k+1)+(i+k+1)+j},$$

or

$$\det L = \frac{\det M'}{\det M^*} = \frac{\begin{pmatrix} \left|A_{i\ldots i+k,j\ldots j+k}\right| \left|A_{i+1\ldots i+k+1,j+1\ldots j+k+1}\right| \\ \quad - \left|A_{i+1\ldots i+k+1,j\ldots j+k}\right| \left|A_{i\ldots i+k,j+1\ldots j+k+1}\right| \end{pmatrix}}{\left|A_{i+1\ldots i+k,j+1\ldots j+k}\right|} = a_{i,j}^{(n-(k+1))},$$

so long as the denominator is not zero. Hence

$$a_{i,j}^{(n-(k+1))} = \det L = \left|A_{i\ldots i+k+1,j\ldots,j+k+1}\right|,$$

as claimed. $\square$

Theorem 40. *If division by zero does not occur, Algorithm 3 terminates correctly.*

Proof: For any $n \times n$ matrix A, the matrix produced at each iteration of Dodgson's Method is one less than that of the matrix from the previous iteration. Since n is finite, the dimension of the iteration matrix will eventually reach one, and the algorithm will terminate. From Dodgson's Condensation Theorem, it follows that

$$a_{11}^{(1)} = \left|A_{1\ldots n,1\ldots,n}\right| = \det(A).$$

$\square$

Problems With Bareiss and Dodgson Methods

Both Dodgson's Method and Bareiss' Algorithm sometimes encounter the same problem: division by zero. Here we will examine examples of cases where each method fails and discuss the workarounds that have been developed to resolve these problems.

Division by Zero in Bareiss' Algorithm

Recall that, for iteration matrices $k = n-2, n-3, \ldots, 2, 1$ of Dodgson's Method, each entry is the determinant of a 2×2 submatrix from the previous iteration divided by the corresponding element of the interior of the iteration matrix

$$a_{ij}^{(k)} = \frac{\left| \ a_{ij}^{(k-1)} \ \right|}{a^{(k-2)}}$$

where we begin with an example of a matrix for which Bareiss' Algorithm fails. Let A be the matrix

$$A = A^{(4)} = \begin{pmatrix} 1 & -4 & 1 & 2 \\ -1 & 4 & 4 & 1 \\ 3 & 3 & 3 & 4 \\ 2 & 5 & 2 & -1 \end{pmatrix}.$$

Applying one iteration yields

$$A^{(3)} = \begin{pmatrix} 1 & -4 & 1 & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 15 & 0 & -2 \\ 0 & 13 & 0 & -5 \end{pmatrix},$$

which has a zero at $a_{22}^{(3)}$. This means Bareiss' Algorithm will encounter division by zero at $A^{(1)}$. How do we work around this? The solution is somewhat simple: swap rows two and three of $A^{(3)}$ and continue with Bareiss' Algorithm; the resulting determinant will be $-\det(A)$, as shown below.

$$\begin{aligned} A^{(3)} = \begin{pmatrix} 1 & -4 & 1 & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 15 & 0 & -2 \\ 0 & 13 & 0 & -5 \end{pmatrix} &\Longrightarrow \widetilde{A}^{(3)} = \begin{pmatrix} 1 & -4 & 1 & 2 \\ 0 & 15 & 0 & -2 \\ 0 & 0 & 5 & 3 \\ 0 & 13 & 0 & -5 \end{pmatrix} \\ &\Longrightarrow \widetilde{A}^{(2)} = \begin{pmatrix} \ddots & & \\ & 75 & 45 \\ & 0 & -49 \end{pmatrix} \\ &\Longrightarrow \widetilde{A}^{(1)} = \begin{pmatrix} \ddots & \\ & -245 \end{pmatrix} \\ &\Longrightarrow \det(A) = 245. \end{aligned}$$

This method is the standard workaround for Bareiss' Algorithm and can be used to find the determinant of any integer matrix.

Division by Zero in Dodgson's Method

As with Bareiss' Algorithm, we begin with an example that illustrates how Dodgson's Method fails. Let A be the matrix

$$A = A^{(4)} = \begin{pmatrix} 1 & -4 & 1 & 2 \\ -1 & 4 & 4 & 1 \\ 3 & 3 & 3 & 4 \\ 2 & 5 & 2 & -1 \end{pmatrix}.$$

The first iteration of Dodgson's Method yields

$$A^{(3)} = \begin{pmatrix} 0 & -16 & -7 \\ -15 & 0 & 13 \\ 9 & -9 & -11 \end{pmatrix}.$$

Notice the zero in the interior of $A^{(3)}$. This implies Dodgson's Method will fail for $A^{(1)}$.

Following the workaround for Bareiss' Algorithm, we might be tempted to swap rows of an intermediate matrix and continue with Dodgson's Method. However, this yields the wrong answer for the determinant of the original matrix, as shown in Example 41.

Example 41. Swapping rows one and two in the matrix $A^{(3)}$ above yields

$$A'^{(3)} = \begin{pmatrix} -15 & 0 & 13 \\ 0 & -16 & -7 \\ 9 & -9 & -11 \end{pmatrix} \Longrightarrow A'^{(2)} = \begin{pmatrix} 60 & 52 \\ 48 & \frac{113}{3} \end{pmatrix}$$
$$\Longrightarrow A'^{(1)} = \left(\frac{-236}{-16} \right) = (14.75),$$

but $\det(A) = 245 \neq 14.75$.

Although we cannot swap rows of an intermediate matrix, we can swap rows of the original matrix. This is, in fact, the workaround used by Dodgson himself. Instead of simply guessing which rows to swap, he suggested that they be chosen by simply cycling through the remaining rows of the matrix. However, this method is somewhat inefficient; any method that uses row swapping requires us to recalculate many of the previous iterations of Dodgson's Method. In addition, there are many matrices for which this method does not remove the problem of division by zero. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}. \tag{2}$$

No combination of row swaps will allow us to have four non-zero entries in the interior of A. This leads us to ask "Are there other methods for correcting Dodgson's Method that are more efficient and allow us to compute the determinant of any integer matrix?" We examine the answer to this question in Chapter III.

CHAPTER III

FIXING DODGSON'S METHOD

As pointed out in Chapter II, Dodson's Method fails in many cases due to division by zero. In step k of each algorithm, the goal is to compute each determinant $\left|A_{i\ldots i+k,j\ldots j+k}\right|$, for all i, j such that $i+k,\ j+k \leq n$. Dodgson's method fails when the determinant of the interior of $A_{i\ldots i+k,j\ldots j+k}$ is zero. This means either the original matrix A has a zero in its interior or one of the intermediate matrices $A^{(n-k)}$ has a zero in its interior.

Thus far, the suggested algorithms for correcting Dodgson's Method have required us to make changes to the original matrix A and apply Dodgson's Method to the new matrix. However, we wish to find a workaround that enables us to use Dodgson's Method to find the determinant of any integer matrix without recalculating all previous steps of the original method. In our search for such a method, we examine three new methods, each of which is based on Dodgson's Condensation Theorem. In all three methods, we apply the original Dodgson's Method to a matrix A until we have division by zero. Only then do we apply the workaround.

Double-Crossing Method

Our first workaround is the Double-Crossing method, a method that works in many cases in which Dodgson's Method fails. However, it does not resolve all cases where Dodgson's Method fails and can be somewhat complicated to apply. The Double-Crossing Method applies Jacobi's Theorem in a slightly different fashion than that used in Dodgson's Method. This allows us to calculate $\det(A)$ whenever there exists at least one $(k-1)\times(k-1)$ minor of $A_{i\ldots i+k,j\ldots j+k}$ whose determinant is nonzero.

Theorem 42 (The Double-Crossing Theorem). [9, p. 49] *Let A be an $n \times n$ matrix. Suppose that while trying to evaluate $|A|$ using Dodgson's method, we encounter a zero in the interior of $A^{(n-k)}$, say in row i and column j. Let $r, s \in \{-1, 0, 1\}$. If the element α in row $i+r$ and column $j+s$ of $A^{(n-k)}$ is non-zero, then we compute $A^{(n-(k+1))}$ and $A^{(n-(k+2))}$ as usual, with the exception of the element in row $i-1$ and column $j-1$ of $A^{(n-(k+2))}$. Let $\ell = k+1$. Then*

(a) *identify the $(\ell+2) \times (\ell+2)$ submatrix L whose upper left corner is the element in row $i-1$ and column $j-1$ of $A^{(n)}$;*

(b) *identify the complementary minor M^* by crossing out the $\ell \times \ell$ submatrix M of L whose upper left corner is the element in row $r+2$ and column $s+2$ of L;*

(c) *compute the matrix M' of determinants of minors of M^* in L;*

(d) *compute the element in row $i-1$ and column $j-1$ of $A^{(n-(k+2))}$ by dividing the determinant of M' by α.*

The resulting matrix $A^{(n-(k+2))}$ has the form described by Dodgson's Condensation Theorem; that is, $a_{i,j}^{(n-(k+2))} = \left|A_{i\ldots i+(k+2), j\ldots j+(k+2)}\right|$, and the condensation can proceed as before.

The Double-Crossing Theorem applies whenever the non-zero element is immediately above, below, left, right, or catty-corner to the zero; that is, the non-zero element is *adjacent* to the zero. If the zero appears in a 3×3 block of zeroes, then the Double-Crossing method fails [9].

In Example 43, we use the Double-Crossing method to compute the determinant of matrix 2, whose determinant could not be found by row swapping.

Example 43. (From [9, p. 50-51]) Let A be the matrix

$$A^{(4)} = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}.$$

We have one zero element in the interior, at $(i, j) = (2,3)$. As set forth in the Double-Crossing Theorem, this affects element $(1,2)$ of $A^{(2)}$. The other three elements of $A^{(2)}$ can be computed as usual:

$$A^{(3)} = \begin{pmatrix} -1 & 3 & 3 \\ 1 & -2 & -2 \\ 2 & -4 & 2 \end{pmatrix} \quad \longrightarrow \quad A^{(2)} = \begin{pmatrix} 1 & ? \\ 0 & -6 \end{pmatrix},$$

but the Double-Crossing method is needed for the last element of $A^{(2)}$. The interior zero appears in the original matrix, so $\ell = 1$. We choose the 3×3 submatrix whose upper left corner is the element in row $i - 1 = 1$, column $j - 1 = 2$ of $A^{(4)}$,

$$L = \begin{pmatrix} 0 & 3 & 0 \\ -1 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}.$$

There is a non-zero element in row 1, column 2, so $r = -1$ and $s = 0$. Cross out the 1×1 submatrix in row 1, column 2 of L; identify the complementary matrix, and compute the corresponding matrix of minors in L,

$$M^* = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad M' = \begin{pmatrix} \begin{vmatrix} 3 & 0 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} \\ \\ \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ -1 & 0 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ 3 & 3 \end{pmatrix}.$$

The determinant of M' is 9; after dividing by the nonzero $a^{(4)}_{1,3} = 3$ we have $a^{(2)}_{2,1} = 3$ so

$$A^{(2)} = \begin{pmatrix} 1 & 3 \\ 0 & -6 \end{pmatrix}.$$

We conclude by computing

$$a_{1,1}^{(1)} = \frac{\begin{vmatrix} 1 & 3 \\ 0 & -6 \end{vmatrix}}{-2} = (3),$$

and, in fact, the determinant of A is 3.

New Variations of Dodgson's Method

Here we will present two methods that apply Theorem 18 to work around cases where the original Dodgson's Method fails due to division by zero. The first method, which we call "Dodgson+ε", is a perturbation method originally suggested by Dr. Jiu Ding. The second method, "Extended Dodgson's Method", focuses on recalculating the determinant of "problem" submatrices by first adding a multiple of one row of the submatrix to another row in the submatrix.

Method: Dodgson+ε

To use the workaround Dodgson+ε, we create a new matrix B by adding ε to one or more entries of the original matrix A in such a way as to remove the problem zero. If the offending zero is present in the original matrix, we simply replace it with ε. Otherwise, we add ε to strategic element(s) of the submatrix of A whose determinant is the zero entry of the intermediate matrix (see Theorem 39). Finally, we apply Dodgson's Method to B; the determinant of A is equal to $\lim_{\varepsilon\to 0} \det(B)$.

We demonstrate this method by applying it to two example matrices. For the first, we choose a 3×3 matrix whose interior element is one.

Example 44. Find $\det(A)$ using the modified Dodgson's Method, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 2 & -1 & -2 \end{pmatrix}.$$

Notice that since

$$A^{(2)} = \begin{pmatrix} -6 & 2 \\ -3 & 1 \end{pmatrix} \text{ and } A^{(1)} = \frac{-6-(-6)}{0} = \frac{0}{0},$$

the interior element $a_{22} = 0$ does not cause a problem for the original Dodgson's Method until the second iteration. However, upon detecting a zero in the interior of the matrix A, we can immediately add ε to a_{22}, which yields

$$B = B^{(3)} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & \varepsilon & 1 \\ 2 & -1 & -2 \end{pmatrix}.$$

We now apply Dodgson's Method to B.

$$B^{(2)} = \begin{pmatrix} \varepsilon - 6 & 2 - 3\varepsilon \\ -3 - 2\varepsilon & -2\varepsilon + 1 \end{pmatrix}$$

$$B^{(1)} = \frac{(\varepsilon-6)\cdot(-2\varepsilon+1) - (2-3\varepsilon)\cdot(-3-2\varepsilon)}{\varepsilon} = \frac{8\varepsilon - 8\varepsilon^2}{\varepsilon} = 8 - 8\varepsilon.$$

Then

$$\det(A) = \lim_{\varepsilon \to 0} \det(B) = \lim_{\varepsilon \to 0} (8 - 8\varepsilon) = 8,$$

which is the determinant of A.

We now turn to a 4×4 matrix which develops a zero in its interior when Dodgson's Method is applied.

Example 45. Find $\det(A)$ using the Dodgson+ε, where

$$A = \begin{pmatrix} 3 & -2 & 1 & 2 \\ -1 & 4 & 4 & 1 \\ 3 & 3 & 3 & 4 \\ 2 & 5 & 2 & -1 \end{pmatrix}.$$

We begin by computing $A^{(3)}$ using the original Dodgson's Method.

$$A^{(3)} = \begin{pmatrix} 10 & -12 & -7 \\ -15 & 0 & 13 \\ 9 & -9 & -11 \end{pmatrix}$$

Since the interior of $A^{(3)}$ contains a zero entry ($a_{22}^{(3)} = 0$), we add ε to an entry of the 2×2 submatrix of $A^{(4)}$ whose determinant was zero. This gives us

$$B^{(4)} = \begin{pmatrix} 3 & -2 & 1 & 2 \\ -1 & 4+\varepsilon & 4 & 1 \\ 3 & 3 & 3 & 4 \\ 2 & 5 & 2 & -1 \end{pmatrix}.$$

Now we use Dodson's Method to find

$$B^{(3)} = \begin{pmatrix} 10+3\varepsilon & -12-\varepsilon & -7 \\ -15-3\varepsilon & 3\varepsilon & 13 \\ 9 & -9 & -11 \end{pmatrix}.$$

Notice that the only entries of $B^{(3)}$ that differ from the corresponding entries of $A^{(3)}$ are those whose corresponding 2×2 submatrix in $B^{(4)}$ contains $b_{22}^{(4)} = 4+\varepsilon$. Thus we need only recalculate these 2×2 determinants. Continuing with Dodgson's Method, we have

$$B^{(2)} = \begin{pmatrix} 6\varepsilon - 45 & 2\varepsilon - 39 \\ 45 & -11\varepsilon + 39 \end{pmatrix}$$

and

$$B^{(1)} = \left(\frac{(6\varepsilon - 45) \cdot (-11\varepsilon + 39) - (6\varepsilon - 45) \cdot (-11\varepsilon + 39)}{3\varepsilon} \right) = (-22\varepsilon + 213).$$

Thus $\det(A) = \lim_{\varepsilon \to 0} \det(B) = \lim_{\varepsilon \to 0} (-22\varepsilon + 213) = 213$.

Notice that the method would be more efficient if we could add ε to the zero entry in the interior of intermediate matrix $A^{(n-k)}$ ($A^{(3)}$ in the previous example) instead of changing the original matrix. However, doing so does not yield the correct result, as shown below.

Example 46. Find $\det(A)$ using the Dodgson+ε, where

$$A = \begin{pmatrix} 3 & -2 & 1 & 2 \\ -1 & 4 & 4 & 1 \\ 3 & 3 & 3 & 4 \\ 2 & 5 & 2 & -1 \end{pmatrix}.$$

Recall from the previous example that the original Dodgson's Method yields

$$A^{(3)} = \begin{pmatrix} 10 & -12 & -7 \\ -15 & 0 & 13 \\ 9 & -9 & -11 \end{pmatrix},$$

whose interior element, a_{22}, is zero. Suppose we add ε to a_{22} to get the matrix $B^{(3)}$. Then

$$B^{(3)} = \begin{pmatrix} 10 & -12 & -7 \\ -15 & \varepsilon & 13 \\ 9 & -9 & -11 \end{pmatrix}.$$

Applying Dodgson's Method to $B^{(3)}$ yields

$$B^{(2)} = \begin{pmatrix} \dfrac{5\varepsilon - 90}{2} & \dfrac{-156 + 7\varepsilon}{4} \\ 45 - 3\varepsilon & -\dfrac{11\varepsilon + 117}{3} \end{pmatrix} \Longrightarrow B^{(1)} = \frac{47\varepsilon^2 - 1539\varepsilon}{\varepsilon} = (47\varepsilon - 1539).$$

Taking the limit as ε approaches zero, we have

$$\lim_{\varepsilon \to 0} \det(B) = \lim_{\varepsilon \to 0} (47\varepsilon - 1539) \neq 213 = \det(A),$$

as we predicted.

Theorem 47. *Algorithm 4 terminates correctly.*

Proof: Recall that we have already proven that Dodgson's Method terminates correctly. Thus we need only prove that the additional steps of *Dodgson's*+ε do not alter the determinant given by Dodgson's Method.

Suppose Dodgson's Method fails for a matrix A when calculating $a_{ij}^{(k)}$. This means there must have been a zero in the ith row, jth column of the interior of $A^{(k+2)}$; that is, $a_{i+1,j+1}^{(k+2)} = 0$. Let $n' = n - k + 2$. By Dodgson's Condensation Theorem,

$$a_{ij}^{(k)} = \begin{vmatrix} a_{ij} & a_{i,j+1} & \cdots & a_{i,j+n'} \\ a_{i+1,j} & a_{i+1,j+1} & \cdots & a_{i+1,j+n'} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i+n',j} & a_{i+n',j+1} & \cdots & a_{i+n',j+n'} \end{vmatrix} = \det(B),$$

and

$$\det(\text{int}(B)) = \begin{vmatrix} a_{i+1,j+1} & a_{i+1,j+2} & \cdots & a_{i+1,j+n'-1} \\ a_{i+2,j+1} & a_{i+2,j+2} & \cdots & a_{i+2,j+n'-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i+n'-1,j+1} & a_{i+n'-1,j+2} & \cdots & a_{i+n'-1,j+n'-1} \end{vmatrix} = a_{i+1,j+1}^{(k+2)} = 0.$$

Algorithm 4. Dodgson+ε

algorithm *Dodgson*+ε
  inputs
    $M \in \mathbb{Z}^{n\times n}$
  outputs
    $\det(M)$
  do
    Let $C = 0$
    Let $A = M$
    Let n be the number of columns in M
    while (number of columns in A)> 1 **do**
      Let m be the number of rows in A
      Let D be an $(m-1)\times(m-1)$ matrix of zeros
      for $i \in \{1,\ldots,m-1\}$ **do**
        for $j \in \{1,\ldots,m-1\}$ **do**
          Let $d_{ij} = a_{ij}\cdot a_{i+1,j+1} - a_{i+1,j}\cdot a_{i,j+1}$
        if $C \neq 0$ **then**
          for $i \in \{1,\ldots,m-1\}$ **do**
            for $j \in \{1,\ldots,m-1\}$ **do**
              if $c_{i+1,j+1} \neq 0$ **then**
                Let $d_{ij} = \frac{d_{ij}}{c_{i+1,j+1}}$
              else
                Let $l = n-(m-1)$
                Let $B = M_{i\ldots i+l,j\ldots j+l}$
                Let $b_{i+1,j+1} = b_{i+1.j+1} + \varepsilon$
                Let $d_{ij} = \lim_{\varepsilon\to 0}$det(B)
      Let $C = A$
      Let $A = D$
    return A

Thus in order to avoid having a zero at $a_{i+1,j+1}^{(k+2)}$, we need to alter the submatrix int(B) without changing the determinant of the submatrix B. To do this, we add some small number ε to one or more strategic elements in an interior row of B. Suppose we add this ε to m of the elements of row k of B (these elements need not be consecutive). This yields the matrix

$$B' = \begin{pmatrix} a_{ij} & a_{i,j+1} & & & \cdots & & & a_{i,j+n'} \\ a_{i+1,j} & a_{i+1,j+1} & & & \cdots & & & a_{i+1,j+n'} \\ \vdots & \vdots & & & \cdots & & & \vdots \\ a_{i+k-1,j} & a_{i+k-1,j+1} & \cdots & a_{i+k-1,l}+\varepsilon & \cdots & a_{i+k-1,l+m-1}+\varepsilon & \cdots & a_{i+k-1,j+n'} \\ \vdots & \vdots & & & \cdots & & & \vdots \\ a_{i+n',j} & a_{i+n',j+1} & & & \cdots & & & a_{i+n',j+n'} \end{pmatrix},$$

where entry l of row k is the first to have ε added to it. Notice that B, the new matrix B', and the matrix

$$E = \begin{pmatrix} a_{ij} & a_{i,j+1} & & & \cdots & & & a_{i,j+n'} \\ a_{i+1,j} & a_{i+1,j+1} & & & \cdots & & & a_{i+1,j+n'} \\ \vdots & \vdots & & & \cdots & & & \vdots \\ a_{i+k-2,j} & a_{i+k-2,j+1} & & & \cdots & & & a_{i+k-2,j+n'} \\ 0 & 0 & \cdots & \varepsilon & \cdots & \varepsilon & \cdots & 0 \\ a_{i+k,j} & a_{i+k,j+1} & & & \cdots & & & a_{i+k,j+n'} \\ \vdots & \vdots & & & \cdots & & & \vdots \\ a_{i+n',j} & a_{i+n',j+1} & & & \cdots & & & a_{i+n',j+n'} \end{pmatrix}$$

all differ only in row k. Also, row k of B' can be written as the sum of the kth rows of B and E. By Proposition 18,

$$\det\left(B'\right) = \det(B) + \det(E).$$

This implies that

$$\begin{aligned} \lim_{\varepsilon\to 0}\left[\det\left(B'\right)\right] &= \lim_{\varepsilon\to 0}\left[\det(B) + \det(E)\right] \\ &= \lim_{\varepsilon\to 0}\left[\det(B)\right] + \lim_{\varepsilon\to 0}\left[\det(E)\right]. \end{aligned}$$

We first examine $\lim_{\varepsilon\to 0}[\det(E)]$. Suppose we use cofactor expansion to find $\det(E)$. Since all of the entries of the kth row of E are either ε or 0, expanding along this row yields a sum numbers that each have a factor of ε. Since we previously assumed that

m entries of row k of E were ε, cofactor expansion along this row yields

$$\det(E) = \sum_{i=1}^{m} \varepsilon \cdot c_i = \varepsilon \cdot \sum_{i=1}^{m} c_i,$$

for some $c_1, c_2, \ldots, c_m \in \mathbb{Z}$. This implies that

$$\lim_{\varepsilon \to 0} [\det(E)] = \lim_{\varepsilon \to 0} \left[\varepsilon \cdot \sum_{i=1}^{m} c_i \right] = 0.$$

Recall that none of the entries of B contains ε. Then $\det(B)$ does not contain ε, so

$$\lim_{\varepsilon \to 0} [\det(B)] = \det(B).$$

By substitution, we have

$$\det(B) = \lim_{\varepsilon \to 0} \left[\det(B') \right].$$

Thus we need only use Dodgson's Method to find B', substitute this value for $a_{ij}^{(k)}$, and continue with Dodgson's Method (starting with $A^{(k)}$). By Dodgson's Condensation Theorem (Theorem 39), the result will be the determinant of the original matrix, A. Note that an argument similar to the one given above could be used to show that ε can be added to elements of more than one of the interior rows of B without the same result. Although this fact would allow us to compute the determinant of any submatrix of A, the method Dodson+ε becomes tedious for cases in which $\operatorname{int}(B)$ is sparse or when several of the rows of $\operatorname{int}(B)$ are linearly dependent. $\square$

Although this method works for any matrix of integers, it does have one important disadvantage: we often have to multiply and divide by polynomials in ε. These polynomials become more difficult to use the more elements to which we add ε.

Method: Extended Dodgson's Method

The Extended Dodgson's Method, like the previous two methods, uses the original

Dodgson's Method until division by zero occurs. As stated earlier, this can occur either when there is a zero in the interior of the original matrix or when Dodgson's Method introduces a zero in the interior of an intermediate matrix $A^{(n-k)}$. In either case, we determine the "problem submatrix" B by locating the submatrix of A whose interior has a determinant of zero. We then add a strategic multiple of row one of B to row two of B and use Dodgson's Method to find the determinant of B. If we encounter division by zero while calculating the determinant of B, then we add a strategic multiple of the last row of B to row two of B and recalculate the determinant of B. We will show that if Dodson's Method fails a third time, the determinant of B is zero. Whether the determinant of B is zero or not, we then continue with Dodgson's Method to find the determinant of A.

Thus this method can be used to find the determinant of any integer matrix. It also has two other major advantages: the reuse of calculations and the simplicity of the required calculations (unlike Dodgson+ε, no multiplication of polynomials).

Example 48. Find $\det(A)$ using the Extended Dodgson's Method, where

$$A = \begin{pmatrix} 1 & -4 & 1 & 2 & 1 \\ -1 & 4 & 4 & 1 & 0 \\ 3 & 3 & 3 & 4 & -2 \\ 2 & 5 & 2 & -1 & 4 \\ 4 & 1 & 3 & 2 & 1 \end{pmatrix}.$$

Since there are no zeros in the interior of $A^{(5)}$, we begin by applying Dodgson's Method as usual.

$$A^{(5)} = \begin{pmatrix} 1 & -4 & 1 & 2 & 1 \\ -1 & 4 & 4 & 1 & 0 \\ 3 & 3 & 3 & 4 & -2 \\ 2 & 5 & 2 & -1 & 4 \\ 4 & 1 & 3 & 2 & 1 \end{pmatrix} \Longrightarrow A^{(4)} = \begin{pmatrix} 0 & -20 & -7 & 1 \\ -15 & 0 & 13 & -2 \\ 9 & -9 & -11 & 14 \\ -18 & 13 & 7 & -9 \end{pmatrix}.$$

Notice the zero that is introduced in $a_{22}^{(4)}$. Since this is an interior element of $A^{(4)}$, we will have to apply Extended Dodgson's Method. As indicated in Algorithm 5, this means we will need to recalculate only the determinant of the submatrix whose interior has a determinant of zero. The remainder of the calculations will be carried out using the traditional Dodgson's Method,

$$A^{(3)} = \begin{pmatrix} \frac{300}{4} & \frac{-260}{4} & \frac{27}{1} \\ \frac{135}{3} & \frac{117}{3} & \frac{160}{4} \\ \frac{-45}{9} & \frac{80}{2} & \frac{1}{-1} \end{pmatrix} = \begin{pmatrix} 75 & -65 & 27 \\ 45 & 39 & 40 \\ -9 & 40 & -1 \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} \frac{0}{0} & \frac{-3653}{13} \\ \frac{2152}{-9} & \frac{-1639}{-11} \end{pmatrix} = \begin{pmatrix} ? & -281 \\ -239 & 149 \end{pmatrix}.$$

By Theorem 39, we know that the goal of $a_{11}^{(2)}$ is the determinant of the upper-right 4×4 submatrix of A. That is,

$$A^{(5)} = \begin{pmatrix} 1 & -4 & 1 & 2 & 1 \\ -1 & 4 & 4 & 1 & 0 \\ 3 & 3 & 3 & 4 & -2 \\ 2 & 5 & 2 & -1 & 4 \\ 4 & 1 & 3 & 2 & 1 \end{pmatrix} \Longrightarrow a_{11}^{(2)} = \begin{vmatrix} 1 & -4 & 1 & 2 \\ -1 & 4 & 4 & 1 \\ 3 & 3 & 3 & 4 \\ 2 & 5 & 2 & -1 \end{vmatrix}.$$

Thus to find $a_{11}^{(2)}$, we must find the determinant of this submatrix. To do this, we first subtract the first row of the submatrix from the second. We then use Dodgson's Method to calculate the determinant of the new submatrix:

$$a_{11}^{(2)} \begin{vmatrix} 1 & -4 & 1 & 2 \\ -1 & 4 & 4 & 1 \\ 3 & 3 & 3 & 4 \\ 2 & 5 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 1 & 2 \\ -2 & 8 & 3 & -1 \\ 3 & 3 & 3 & 4 \\ 2 & 5 & 2 & -1 \end{vmatrix} = 245.$$

By substituting this value into $A^{(2)}$, we have

$$A^{(2)} = \begin{pmatrix} 245 & -281 \\ -239 & 149 \end{pmatrix}$$
$$\Longrightarrow A^{(1)} = \left(\frac{(245) \cdot (149) - (-281) \cdot (-239)}{39} \right) = (-786).$$

Then $\det(A) = -786$.

To prove that Algorithm 5 terminates correctly, we first prove the following result.

Theorem 49. *Let A be an $n \times n$ matrix whose interior has a determinant of zero. Suppose adding a constant multiple of row 1 of A to row 2 of A still yields a zero for the determinant of the interior. Suppose the same is true when a constant multiple of row n of A is added to row 2 of A. Then* $\det(A) = 0$.

Proof: Let A be an $n \times n$ matrix whose interior has a determinant of zero, and let $\mathbf{r}_i = A_{i,2\ldots n-1}$, $i = 1,2,3,\ldots n$ be vectors. By Proposition 20, the row vectors of the interior of A (that is, $\mathbf{r}_2, \mathbf{r}_3, \ldots, \mathbf{r}_{n-1}$) are linearly dependent. Suppose that the determinant of the submatrix created by adding a constant multiple b of $\mathbf{r}_1$ to $\mathbf{r}_2$, $\begin{pmatrix} \mathbf{r}_2 + b\mathbf{r}_1 & \mathbf{r}_3 & \cdots & \mathbf{r}_{n-1} \end{pmatrix}^T$, is zero. This implies that $\mathbf{r}_2 + b\mathbf{r}_1, \mathbf{r}_3, \mathbf{r}_4, \ldots, \mathbf{r}_{n-1}$ are linearly dependent (Proposition 20). Then, by definition of linear dependence, there exist constants $c_3, c_4, \ldots, c_{n-1}$ such that

$$\mathbf{r}_2 + b\mathbf{r}_1 = c_3\mathbf{r}_3 + c_4\mathbf{r}_4 + \ldots + c_{n-1}\mathbf{r}_{n-1}.$$

Solving for $\mathbf{r}_1$ yields

$$\begin{aligned}\mathbf{r}_1 &= \frac{1}{b}\left(-\mathbf{r}_2 + c_3\mathbf{r}_3 + c_4\mathbf{r}_4 + \ldots + c_{n-1}\mathbf{r}_{n-1}\right)\\ &= -\frac{1}{b}\mathbf{r}_2 + \frac{c_3}{b}\mathbf{r}_3 + \frac{c_4}{b}\mathbf{r}_4 + \ldots + \frac{c_{n-1}}{b}\mathbf{r}_{n-1}.\end{aligned}$$

Since $\mathbf{r}_1$ can be rewritten as a sum of constant multiples of $\mathbf{r}_2, \mathbf{r}_3, \ldots, \mathbf{r}_{n-1}$, the vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{n-1}$ are linearly dependent. Suppose also that the submatrix $\begin{pmatrix} \mathbf{r}_2 + d\mathbf{r}_n & \mathbf{r}_3 & \cdots & \mathbf{r}_{n-1} \end{pmatrix}^T$, where d is a constant, has a determinant of zero. Then an argument similar to the one above can be used to show that

$$\mathbf{r}_n = -\frac{1}{d}\mathbf{r}_2 + \frac{c_3}{d}\mathbf{r}_3 + \frac{c_4}{d}\mathbf{r}_4 + \ldots + \frac{c_{n-1}}{d}\mathbf{r}_{n-1}.$$

Then $\mathbf{r}_2, \mathbf{r}_3, \ldots, \mathbf{r}_n$ are also linearly dependent.

Create a new matrix by swapping columns 1 and $n-1$ of A; now swap rows 1 and $n-1$ of the new matrix. Denote the resulting matrix by A', as shown:

$$A' = \begin{pmatrix} a_{n-1,n-1} & a_{n-1,2} & \ldots & a_{n-1n-2} & a_{n-1,1} & a_{n-1n} \\ a_{2,n-1} & a_{22} & \ldots & a_{2,n-2} & a_{21} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n-2,n-1} & a_{n-2,2} & \ldots & a_{n-2,n-2} & a_{n-2,1} & a_{n-2,n} \\ a_{1,n-1} & a_{12} & \ldots & a_{1,n-2} & a_{11} & a_{1n} \\ a_{n,n-1} & a_{n2} & \ldots & a_{n,n-2} & a_{n1} & a_{nn} \end{pmatrix}.$$

Since $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{n-1}$ and $\mathbf{r}_2, \mathbf{r}_3, \ldots, \mathbf{r}_n$ are linearly dependent, we can then use row reduction to create zero entries in the first $n-2$ columns of the last two rows of this matrix. This yields the matrix

$$A'' = \begin{pmatrix} a_{n-1,n-1} & a_{n-1,2} & \ldots & a_{n-1n-2} & a_{n-1,1} & a_{n-1n} \\ a_{2,n-1} & a_{22} & \ldots & a_{2,n-2} & a_{21} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n-2,n-1} & a_{n-2,2} & \ldots & a_{n-2,n-2} & a_{n-2,1} & a_{n-2,n} \\ 0 & 0 & \ldots & 0 & a''_{11} & a''_{1n} \\ 0 & 0 & \ldots & 0 & a''{}_{n1} & a''_{nn} \end{pmatrix},$$

where a''_{11}, a''_{1n}, a''_{nn}, and a''_{n1} are the corresponding entries of A' after row reduction has been applied.

Let D be the $(n-2) \times (n-2)$ matrix in the upper left corner of A''

$$D = \begin{pmatrix} a_{n-1,n-1} & a_{n-1,2} & \ldots & a_{n-1n-2} \\ a_{2,n-1} & a_{22} & \ldots & a_{2,n-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-2,n-1} & a_{n-2,2} & \ldots & a_{n-2,n-2} \end{pmatrix},$$

and let B be the 2×2 matrix in the bottom right corner of A''

$$\begin{pmatrix} a''_{11} & a''_{1n} \\ a''{}_{n1} & a''_{nn} \end{pmatrix}.$$

Then

$$A'' = \left(\begin{array}{cccc|cc} & & & & a_{n-1,1} & a_{n-1n} \\ & & D & & a_{21} & a_{2n} \\ & & & & \vdots & \vdots \\ & & & & a_{n-2,1} & a_{n-2,n} \\ 0 & 0 & \dots & 0 & & \\ 0 & 0 & \dots & 0 & \multicolumn{2}{c}{B} \end{array} \right) = \begin{pmatrix} D & * \\ 0 & B \end{pmatrix}.$$

From Proposition 24, we know that the determinant of matrices having this form is

$$\det(A) = \det(D) \cdot \det(B).$$

Notice that, after $n-3$ row swaps and $n-3$ column swaps, D is the interior of A. Then, by Proposition 16, the determinant of D is $(-1)^{2(n-3)}$ times the determinant of the interior of A. But the determinant of the interior of A is zero, so $\det(D) = 0$. Thus

$$\begin{aligned} \det(A) &= \det(D) \cdot \det(B) \\ &= 0 \cdot \det(B) \\ &= 0. \end{aligned}$$

Recall that the only operations used to create A'' from A were two row/column swaps and adding constant multiples of one row to another. Then, by Proposition16,

$$\begin{aligned} \det(A) &= (-1)^2 \det(A'') \\ &= 1 \cdot 0 \\ &= 0. \end{aligned}$$

□

Theorem 50. *Algorithm 5 (p. 53) terminates correctly.*

Proof: Recall that we have already proven that Dodgson's method terminates correctly.

Thus we need only show that the additional steps required by Extended Dodgson's Method do not change the determinant given by the original Dodgson's Method.

By the Condensation Theorem (Theorem 39), the jth entry of the ith row of $A^{(k)}$ is

$$a_{ij}^{(k)} = \left|A_{i\ldots i+n-k, j\ldots j+n-k}\right|.$$

Then, if we are encountering division by zero in Dodgson's Method when trying to calculate $a_{ij}^{(k)}$, we may instead calculate the determinant of the submatrix

$$B = A_{i\ldots i+n-k, j\ldots j+n-k}.$$

But, by the Condensation Theorem, if Dodgson's Method failed for $a_{ij}^{(k)}$, then the determinant of the interior of B is zero. Suppose we create a new matrix $\tilde{B}$ by adding to row two of B a strategic multiple of row one of B. By "strategic multiple" we mean a multiple of row one that does not introduce a zero in the interior of $\tilde{B}$. By Proposition 16, the determinant of $\tilde{B}$ is the same of B. Thus

$$a_{ij}^{(k)} = \det(B) = \det\left(\tilde{B}\right),$$

which allows us to calculate $a_{ij}^{(k)}$ and continue with Dodgson's Method to find $\det(A)$.

However, suppose that we also encounter division by zero when using Dodgson's Method to calculate $\det\left(\tilde{B}\right)$. Then we create a third matrix $\check{B}$ by adding a strategic multiple of the last row of B to the second row of B. By Proposition 16, the determinant of $\det\left(\check{B}\right) = \det(B) = a_{ij}^{(k)}$, so we may continue with Dodgson's Method to find $\det(A)$. Suppose that, instead of being able to calculate $\det\left(\check{B}\right)$, we again encounter division by zero. By Theorem 49, we have $\det(B) = 0$, so $a_{ij}^{(k)} = 0$.

Thus, in any case, we can find $a_{ij}^{(k)}$ and continue with Dodgson's Method to find det(A). Therefore, by Theorem 40, Extended Dodgson's Method terminates correctly. $\square$

Algorithm 5. Extended Dodgson's Method

---

algorithm *Extended Dodgson's Method*
  inputs
    $M \in \mathbb{Z}^{n \times n}$
  outputs
    $\det(M)$
  do
    Let $C = 0$
    Let $A = M$
    while (number of columns in A)> 1 **do**
      Let m be the number of rows in A
      Let D be an $(m-1) \times (m-1)$ matrix of zeros
      for $i \in \{1, \ldots, m-1\}$ **do**
        for $j \in \{1, \ldots, m-1\}$ **do**
          Let $b_{ij} = a_{ij} \cdot a_{i+1,j+1} - a_{i+1,j} \cdot a_{i,j+1}$
      if $C \neq 0$ **then**
        for $i \in \{1, \ldots, m-1\}$ **do**
          for $j \in \{1, \ldots, m-1\}$ **do**
            if $c_{i+1,j+1} \neq 0$ **then**
              Let $d_{ij} = \frac{d_{ij}}{c_{i+1,j+1}}$
            else
              Let $l = n - (m-1)$
              Let $B = M_{i \ldots i+l, j \ldots j+l}$
              Add to row 2 of B some multiple of row 1 of B that does not introduce a zero in the interior of B
              Use Dodgson's Method to find $\det(B)$
              if Dodgson's Method fails for B **then**
                Add to row 2 of B some multiple of row n of B
                if Dodgson's Method fails for B **then**
                  Let $\det(B) = 0$
              Let $d_{ij} = \det(B)$
      Let $C = A$
      Let $A = D$
    return A

CHAPTER IV

DIRECTION OF FUTURE WORK

Thus we now have workarounds for Dodgson's Method that can be used to find the determinant of any integer matrix (preferably a dense matrix) and do not require us to recalculate most of the previous successful iterations of Dodgson's Method. However, these new workarounds still use entries of the original matrix to recalculate selected entries of previous iterations. In the future, we wish to look for methods that, like the workaround for Bareiss' Algorithm, allow us to modify rows of intermediate matrices without going back to the original matrix A. For example, consider the following 4×4 matrix below.

$$
\begin{aligned}
A^{(4)} = \begin{pmatrix} 1 & -4 & 1 & 2 \\ -1 & 4 & 4 & 1 \\ 3 & 3 & 3 & 4 \\ 2 & 5 & 2 & -1 \end{pmatrix} &\Longrightarrow A^{(3)} = \begin{pmatrix} 0 & -16 & -7 \\ -15 & 0 & 13 \\ 9 & -9 & -11 \end{pmatrix} \\
&\Longrightarrow A^{(2)} = \begin{pmatrix} -60 & 52 \\ 45 & 39 \end{pmatrix} \\
&\Longrightarrow A^{(1)} = \left(\frac{-4680}{0} \right) = (?)\,.
\end{aligned}
$$

We wish to answer the question "Is there a way to modify $A^{(3)}$, $A^{(2)}$, or $A^{(1)}$ instead of $A^{(4)}$ such that Dodgson's Method will still yield the correct determinant of A?" Although we know that swapping rows of an intermediate matrix will alter the final result of Dodgson's Method (see Example 41), perhaps there is another workaround that would yield the correct result.

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