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Asymptotic Behavior of Finite-Time Ruin Probability in a By-Claim Risk Model with Constant Interest Rate

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ASYMPTOTIC BEHAVIOR OF FINITE TIME RUIN PROBABILITY IN A BY-CLAIM RISK MODEL WITH CONSTANT INTEREST RATE

by

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ABSTRACT

ASYMPTOTIC BEHAVIOR OF FINITE TIME RUIN PROBABILITY IN A BY-CLAIM RISK MODEL WITH CONSTANT INTEREST RATE

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Enlightened by the results of Li [8] and Wang [19], we study the ruin probability of a renewal risk model with constant interest rate and by-claim parts. We assume that the claim size and the inter-arrival time satisfy a certain dependent structure with some additional assumptions on their distribution functions. Furthermore, we give relevant preparation of theory and compare several existing risk models and dependent structures. In this way, we present our result and prove it.

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LIST OF ABBREVIATIONS

LND	Lower Negatively Dependent
UND	Upper Negatively Dependent
ND	Negatively Dependent
NQD	Negatively Quadrant Dependent
NLOD	Negatively Lower Quadrant Dependent
WUOD	Widely Upper Orthant Dependent
WLOD	Widely Lower Orthant Dependent
WOD	Widely Orthant Dependent
PQAI	Pairwise Quasi-asymptotically Independent
pSQAI	Pairwise Strong Quasi-asymptotically Independent

Chapter 1

Risk Model

1.1 Introduction

Risk theory plays an important role in financial mathematics and actuarial science. A variety of risk models have been investigated by many researchers. About a century ago, Lundberg [11] laid the foundation of actuarial risk model in his Uppsala thesis. Waters and Papatriandafylou [20] introduced delay in claims settlements in a discrete-time risk model and applied martingale technique to derive upper bounds for ruin probabilities. Yuen and Guo [23] applied the probability-generating functions to obtain ruin probabilities for the compound binomial model with delay by-claims. Tang [15] investigated a simple asymptotic formula for the ruin probability of the renewal risk model with constant interest force and regularly varying tailed claims. Recently, Weng et al. [21] studied in the tail probability of the Poisson shot noise process and established some asymptotic formulas for the finite and infinite ruin probabilities of a continuous time risk model. Related results can also be found in Chen and Ng [1], Zhu and Gao [24], Li et al. [9], and Yang and Wang [22].

1.2 The Renewal Risk Model

Consider the renewal risk model with the total capital reserve up to time t, denoted by $R_\delta(t,x)$, given by the following equilibrium equation

$$\begin{aligned} R_\delta(x,t) &= xe^{\delta t} + \int_{[0,t]} e^{\delta(t-s)} c ds - \sum_{k=1}^{\infty} X_k e^{\delta(t-\omega_k)} 1_{\{\omega_k \le t\}} \\ &\quad - \sum_{k=1}^{\infty} Z_k e^{\delta(t-\omega_k-T_k)} 1_{\{\omega_k+T_k \le t\}}, \end{aligned} \tag{1.1}$$

where x denotes the initial capital of the insurance company, $\delta > 0$ is the constant interest rate and c is the constant gross premium rate.

A basic formula for calculating annual compound interest is as follows

$$P = x\left(1+\frac{\delta}{m}\right)^{mt},$$

where P is the amount of money accumulated after t years, x is the principal amount, t is the number of years the amount is deposited or borrowed for, m is the number of times the interest is compounded per year and δ is the annual rate of interest. If the compounding period is infinitesimally small, i.e., m tends to infinity, we have the formula of continuous compound interest

$$P = x \lim_{m\to\infty}\left(1+\frac{\delta}{m}\right)^{mt} = xe^{\delta t}.$$

Therefore, $xe^{\delta t}$ denotes the total capital after time t generated by the initial capital reserve x.

In this renewal risk model, the deterministic linear function ct is the total amount of premiums accumulated up to time $t \geq 0$. Then $\tilde{c}(t) = \int_{[0,t]} e^{\delta(t-s)} cds = \frac{c}{\delta}(e^{\delta t} - 1)$ denotes total capital generated by the premiums by time t.

Consider the risk model in which the claim sizes and the arrival times of successive claims fulfill the following requirements:

1. The main claims sizes, $X_k, k \geq 1$, constitute a sequence of nonnegative random variables with common tail distribution

$$\overline{H}(x) = 1 - H(x) = P\{X_k \geq x\} > 0 \text{ for all } x > 0.$$

2. The arrival times of successive claims are $\omega_n = \sum_{k=1}^{n} Y_k, k \geq 1$. The inter-arrival time $\{Y_k; k \geq 1\}$ forms a sequence of random variables with common distribution function V but are not necessarily independent. The arrival times of successive claims can generate a renewal counting process

$$N(t) = \sum_{n=1}^{\infty} 1_{\{\omega_k \leq t\}},\ t \geq 0, \tag{1.2}$$

where 1_A is the indicator function of an event A. Then $N(t)$ describes the total number of claims occurred in finite interval $[0,t]$. Denote the renewal function by $\lambda(t) = EN(t), t \geq 0$, and assume that $\lambda(t) < \infty$ for all $0 < t < \infty$. As in Tang [14], define $\Lambda = \{t : \lambda(t) > 0\}$ with $\underline{t} = \inf\{t : \lambda(t) > 0\} = \inf\{t : P(Y_1 \leq t) > 0\}$, i.e.,

$$\Lambda = \begin{cases} [\underline{t}, \infty] & P(Y_1 = \underline{t}) > 0, \\ (\underline{t}, \infty] & P(Y_1 = \underline{t}) = 0. \end{cases}$$

3. We assume that $\{X_k; k \geq 1\}, \{Y_k; k \geq 1\}$ and $\{c(t); t \geq 0\}$ are mutually independent.

4. In our risk model, there are two parts of mutually independent claims, main claims and by-claims. We refer Z_n's as by-claims or delayed claims in the renewal risk model. They are identically distributed with common distribution F. They are usually induced by the main claim with some probability and the occurrence of a by-claim may be delayed depending on associated main claims amount. If the main claim occurs at the ω_k, then the by-claim occurs at the $T_k + \omega_k$. Let $T_k, k \geq 1$, be the corresponding delay times of the by-claim and they are identically distributed with common distribution function G and form a sequence of random variables, which are nonnegative, but possibly generated at 0. In this paper, we assume that the $\{X_n, Z_n; n \geq 1\}$, $\{\omega_n; n \geq 1\}$ and $\{T_n; n \geq 1\}$ are mutually independent.

   The claims can produce the dependent influence on each other and some additional damages and costs, such as a tornado, hurricane, heavy rain-storm, and so on. Hence, our renewal model with by-claim parts can better reflect the truth.

1.3 Ruin Probability

The ruin occurs in the finite time if the insurer's capital falls below zero in the finite time interval $[0,t]$, that is, the total claim exceeds the initial capital plus premium income. Once the capital is less than zero, the ruin occurs and the company will bankrupt. We investigate the asymptotic behavior of the ruin probability in the finite time in this thesis. The ruin probability in the finite time interval $[0,t]$ is given by

$$\Phi(x,t) = P(R_\delta(x,s) < 0, \text{for some } 0 \leq s \leq t). \tag{1.3}$$

We also introduce the ultimate ruin probability, which is defined as

$$\Phi(x) = P(R_\delta(x,s) < 0, \text{ for some } s \geq 0). \tag{1.4}$$

Chapter 2

Background

2.1 Notation

Throughout the paper, all limit relationships are for x tending to infinite unless otherwise stated. Define

$$a(x) = o(b(x)) \text{ if } \lim\ a(x)/b(x) = 0.$$

For two positive bivariate functions $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, the asymptotic relation $a(x,t) \sim b(x,t)$ holds uniformly for $t \in \Delta$ if

$$\lim_{x\to\infty} \sup_{t\in\Delta} \left| \frac{a(x,t)}{b(x,t)} - 1 \right| = 0 \tag{2.1}$$

or, equivalently,

$$\limsup_{x\to\infty} \sup_{t\in\Delta} \frac{a(x,t)}{b(x,t)} \leq 1, \tag{2.2}$$

and

$$\liminf_{x\to\infty} \inf_{t\in\Delta} \frac{a(x,t)}{b(x,t)} \geq 1. \tag{2.3}$$

2.2 Class of Heavy-tailed Distributions

In the insurance industry, practitioners usually choose heavy-tailed random variables to model large claims. We introduce some classes of heavy-tailed distributions with their basic properties.

For a distribution $H(x)$ on the $(-\infty, \infty)$, $\overline{H}(x) = 1 - H(x)$ is the tail of the distribution function H. We denote the upper and lower Matuszewska index of $H(x)$ distribution by

$$J_H^+ = -\lim_{x\to\infty} \frac{\log \overline{H_L}(x)}{\log x}, \qquad \overline{H}_L(x) = \lim_{x\to\infty} \inf \frac{\overline{H}(xy)}{\overline{H}(x)} \text{ for } y > 1\ ,$$

$$J_H^- = -\lim_{x\to\infty} \frac{\log \overline{H_U}(x)}{\log x}, \qquad \overline{H}_U(x) = \lim_{x\to\infty} \sup \frac{\overline{H}(xy)}{\overline{H}(x)} \text{ for } y > 1\ .$$

A distribution function H with support $(0, \infty)$ is subexponential, denoted by $\mathcal{S}$, if for all $n \geq 2$,

$$\lim_{x\to\infty} \frac{\overline{H^{*n}}(x)}{n\overline{H}(x)} = 1, \tag{2.4}$$

where the $\overline{H^{*n}}(x)$ denotes the n-fold convolution of H; see, e.g., Embrechts et al. [2].
If there is an integer $n \geq 2$, such that

$$\lim_{x\to\infty} \frac{\overline{H^{*n}}(x)}{\overline{H}(x)} \leq n \tag{2.5}$$

then $H \in S$. It provide us a sufficient condition for subexponentiality.

The class of dominated varying distribution is defined as

$$\mathcal{D} = \{H : \limsup_{x\to\infty} \frac{\overline{H}(xy)}{\overline{H}(x)} < \infty \text{ for any } y > 0\}.$$

If $H \in \mathcal{D}$, then for any $\eta > J_H^+$, there exists two positive constants c and d such that when $x \geq y \geq d$,

$$\frac{\overline{H}(y)}{\overline{H}(x)} \leq c \left(\frac{x}{y}\right)^{\eta}. \tag{2.6}$$

The class of long-tailed distribution is given by

$$\mathcal{L} = \{H : \limsup_{x\to\infty} \frac{\overline{H}(x+y)}{\overline{H}(x)} = 1, \text{for any y} > 0\}$$

It can be shown that a distribution $H \in \mathcal{L}$ if and only if there exists a function $l(\cdot) := l_H(x) : [0,\infty) \mapsto [0,\infty)$ such that $l(x) \to \infty$, $l(x) = o(x)$ and

$$\overline{H}(x+y) \sim \overline{H}(x) \tag{2.7}$$

holds uniformly for all $\mid y \mid \leq l(x)$. For any two long-tailed distribution H, F, let $l(x) = \min\{l_H(x), L_F(x)\}$. Clearly

$$\overline{H}(x+y) \sim \overline{H}(x) \quad \text{and} \quad \overline{F}(x+y) \sim \overline{F}(x) \tag{2.8}$$

hold uniformly for all $\mid y \mid \leq l(x)$. Without loss of generality, we can choose $l(x)$ satisfies (2.8) thoroughout the whole thesis.
We define a little smaller distribution class ERV. We say that $H \in ERV$ on $[0,\infty)$, if there are some $0 < \alpha \leq \beta < \infty$ such that

$$s^{-\beta} \leq \liminf \frac{\overline{H}(sx)}{\overline{H}(x)} \leq \limsup \frac{\overline{H}(sx)}{\overline{H}(x)} \leq s^{-\alpha}, \text{for all s} \geq 1. \tag{2.9}$$

Denoted by $H \in ERV(-\alpha, -\beta)$. If $\alpha = \beta$, we say that H belongs to the regular variation class and write $H \in R_{-\alpha}$.

It is well known that the following proper inclusion relationship holds

$$R_{-\alpha} \subset ERV(-\alpha, -\beta) \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{D}.$$

(See, e.g., Embrechts et al. [2])

2.3 Dependence Structure

From the study of many contexts and literatures, we easily found that the renewal risk model with constant interest rate mainly involves the independent structure between the claim sizes and arrival times of successive claims; this limits the usefulness of the obtained results to some extent. Introducing dependent structure to risk models has captured more and more researchers' attention in recent years, and it provides a new perspective for the ruin probabilities theory. Many researchers have already worked on this new topic, for example, Yang and Wang [22], Liu et al.[10], Wang et al. [19] and others. We summarize the current corresponding results.

We define random variables $\{\xi_i, i \geq 1\}$ as LND and UND if for each $i \geq 1$ and all $x_1,...x_n$

$$P(\cap_{i=1}^{n}\{\xi_i \leq x_i\}) \leq \prod_{i=1}^{n} P(\xi_i \leq x_i),$$

and

$$P(\cap_{i=1}^{n}\{\xi_i > x_i\}) \leq \prod_{i=1}^{n} P(\xi_i > x_i).$$

If the sequence can satisfy both the LND and UND, we can name it ND. When $n = 2$, the LND,UND and ND structures are equivalent; see, for example, Lehmann [7].

we say that two random variables $\{\xi_i, i \geq 1\}$ are NQD, if for all positive integers $i \neq j$,

$$P(\xi_1 \leq x_1, \xi_2 \leq x_2) \leq P(\xi_1 \leq x_1)P(\xi_2 \leq x_2),$$

or, equivalently,

$$P(\xi_1 > x_1, \xi_2 > x_2) \leq P(\xi_1 > x_1)P(\xi_2 > x_2).$$

Additionally, we also named the LND as the NLOD in Li et al. [9] with different notations and different formulas.

We define that $\{\xi_n, n \geq 1\}$ are WUOD. If there exists a finite real sequence $\{g_U(n), n \geq 1\}$ satisfying for each $n \geq 1$ and for all $x_i \in (-\infty, \infty), 1 \geq i \geq n$,

$$P(\cap_{i=1}^{n}\{\xi_i > x_i\}) \leq g_U(n) \prod_{i=1}^{n} P(\xi_i > x_i). \tag{2.10}$$

We can also define that $\{\xi_n, n \geq 1\}$ are WLOD. If there exists a finite real sequence $\{g_L(n), n \geq 1\}$ satisfying for each $n \geq 1$ and for all $x_i \in (-\infty, \infty), 1 \leq i \leq n$,

$$P(\cap_{i=1}^{n}\{\xi_i \leq x_i\}) \leq g_L(n) \prod_{i=1}^{n} P(\xi_i \leq x_i). \tag{2.11}$$

We would like remark that if $\{\xi_n, n \geq 1\}$ satisfies (2.10) and (2.11), it is also said to be WOD. $\big($See, e.g. Wang et al. [19] $\big)$

In the thesis, we will use the following assumptions in our main result, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} g_U(n)e^{-\varepsilon n} = 0, \tag{2.12}$$

and

$$\lim_{n\to\infty} g_L(n)e^{-\varepsilon n} = 0. \tag{2.13}$$

A sequence of random variables $\{\xi_n, n \geq 1\}$ are PQAI for any $i \neq j$,

$$\lim_{z\to\infty} P\big(\min\{|\xi_i|, \xi_j\} > z | \max\{\xi_i, \xi_j\} > z\big) = 0.$$

We also define the sequence of pSQAI random variables $\{\xi_n, n \geq 1\}$ if for any $i \neq j$,

$$\lim_{\min\{z_i, z_j\}\to\infty} P(|\ \xi_i\ | > z_i \mid \xi_j > z_j) = 0.$$

The dependent structure WUOD and WLOD can allow some negatively dependence and positively dependence. When the random variables are nonnegative, the two dependent structures pSQAI and PQAI are equivalent. The pSQAI structure is a more general dependent case than the WUOD structure.

2.4 Literature Review

We list several corresponding results and remark the methods in this section.

$[$*Result* 1$]$ Theorem 1 of Chen and Ng [1]. Consider the renewal risk model in Section 1, if the claim sizes $\{X_n; n \geq 1\}$ are pairwise ND with common distribution $H \in ERV$, the inter-arrival times Y_n are i.i.d random variables. And the $\{c(t), t \geq 0\}$ is a deterministic linear function, and then the asympototic for the ultimate ruin probability $\Phi(x)$

$$\Phi(x) \sim \int_0^\infty \big(\overline{H}\big(xe^{\delta t'}\big)\big) d\lambda(t').$$

$[$*Result* 2$]$ Theorem 1 of Li et al. [9]. Consider the renewal risk model in Section 1. If the claim sizes $\{X_n; n \geq 1\}$ are pairwise NQD with common distribution $H \in \mathcal{D}$, the inter-arrival times $\{Y_n; n \geq 1\}$ are NLOD, and the $\{c(t), t \geq 0\}$ is a deterministic linear function. In particular, if $H \in L$ and $J_H^- > 0$, we obtain the $\Phi(x,t)$

$$\Phi(x,t) \sim \int_0^t \big(\overline{H}\big(xe^{\delta t'}\big)\big) d\lambda(t'). \tag{2.14}$$

[*Result* 3] Theorem 1 of Kong and Zong [6]. Consider the renewal risk model in Section 1, if the claim sizes $\{X_n; n \geq 1\}$ are NOD random variables with common distribution $H \in \mathcal{L} \cap \mathcal{D}$, the inter-arrival times $\{Y_n; n \geq 1\}$ are i.i.d with common exponential distribution $\{N(t), t \geq 0\}$ is a homogeneous Poisson process.

$$\Phi(x,t) \sim \int_0^t \left(\overline{H}\left(xe^{\delta t'}\right)\right) d\lambda(t').$$

[*Result* 4] Theorem 1.1 of Wang et al. [19]. Consider the renewal risk model in Section 1. If the claim sizes $\{X_n; n \geq 1\}$ are WUOD with common distribution $H \in \mathcal{L} \cap \mathcal{D}$, the inter-arrival times $\{Y_n; n \geq 1\}$ are WLOD. Also holds the relations (2.12) and (2.13). Then for any finite T $\in \Lambda$, the relation (2.14) holds uniformly for t $\in \Lambda[0,T]$ and then we obtain the equivalent form for the $\Phi(x,t)$,

$$\Phi(x,t) \sim \int_0^t \left(\overline{H}\left(xe^{\delta t'}\right)\right) d\lambda(t').$$

[*Result* 5] Theorem 1.1 of Liu et al. [10]. Consider the renewal risk model in Section 1. If the claim sizes $\{X_n; n \geq 1\}$ are UTAI with common distribution $H \in \mathcal{L} \cap \mathcal{D}$, the inter-arrival times $\{Y_n; n \geq 1\}$ are WLOD such that the relation (2.13) holds. For any fixed T$\in \Lambda$, then

$$\Phi(x,t) \sim \int_0^t \left(\overline{H}\left(xe^{\delta t'}\right)\right) d\lambda(t').$$

[*Result* 6] Theorem 3.1 of Li [8]. Consider the by-claim model, assuming that $\{X_n, Y_n; n \geq 1\}$, $\{\theta_n; n \geq 1\}$ and $\{T_n; n \geq 1\}$ are mutually independent, X_1,Y_1,X_2,Y_2 are PQAI, and random pairs (X_1,Y_1),(X_2,Y_2)... are identically distributed. Let $\overline{H} \in ERV$ and $\overline{F}$ also $\in ERV$, then we obtain

$$\Phi(x) \sim \int_0^\infty \left(\overline{H}\left(xe^{\delta t}\right)\right) d\lambda(t) + \int_0^\infty \int_0^\infty \left(\overline{F}\left(xe^{\delta(s+t)}\right)\right) d\ G(s) d\lambda(t).$$

We remark these results in various aspects according to the motivation of research, such as the general risk model or renewal risk model, independent structure or dependent structure, some common heavy-tailed distribution classes, the constant interest rate or not. By analysis, we found that the claim sizes and the inter-arrival times in results of Li et al.[9], Wang et al. [19] and Li [8] satisfied the different dependent structures, it is a stronger restriction than the i.i.d condition in result of Chen and Ng [1]. But among the different dependent structures, we may have different choices in different risk models and then lead to different results, such as in Liu et al. [10], which required both the common distribution of claim sizes and inter-arrival times follow the intersection class, but in many cases, the author chose a more mild condition *ERV*. Furthermore, in terms of common distribution, some papers involve a more complicated case. In Li et al.[9] paper and Yang and Wang [22], the authors remarked the upper and lower Matuszewska index. We also consider the upper and lower Matuszewska index in the renewal risk model. But in Wang et al. [19], the authors canceled the condition J_H^-. In particular, in background section we introduced the relation (2.12) and (2.13). Wang et al. [19] considered them in $[Result\ 4]$, and we will discuss them in our renewal risk model. In addition, the $[Result\ 2]$ to $[Result\ 5]$ mainly investigate the asymptotic behavior of ruin probability in finite time. And then the $[Result\ 1]$ and $[Result\ 6]$ worked on the formula of ultimate ruin probability in risk model. Generally speaking, the premium function $c(t)$ is a general stochastic process, but in some papers, it is assumed that the $c(t)$ is a deterministic linear function, such as in Li et al.[9] and Chen and Ng [1]. Furthermore, we do not always require $\delta > 0$ and the inter-arrival times may not have an exponential distribution, but in most cases we define that the δ is constant interest rate, sometimes δ yield 0 and the inter-arrival times may follow a common exponential distribution. Finally, we consider the $N(t)$ factor. In the risk model section we define $N(t)$ to constitute a renewal counting process, but in the result of Kong and Zong [6], the $N(t)$ is a homogeneous Poisson process, which follows the Poisson distribution with associated parameter λ.

Chapter 3

Main Results

Our main results are for the approximation of finite ruin probability of the renewal risk model with constant interest rate and by-claim model.

Theorem 3.1 Consider the risk model in Section 1. Assume that $\{X_n, Y_n; n \geq 1\}$, $\{\omega_n; n \geq 1\}$ and $\{T_n; n \geq 1\}$ are mutually independent. Let the claims size $\{X_n, n \geq 1\}$ and $\{Z_n, n \geq 1\}$ be pSQAI with common distribution $H, F \in \mathcal{L} \cap \mathcal{D}$, $\overline{F} = O(\overline{H})$, and the inter-arrival claims $\{Y_n; n \geq 1\}$ be WLOD random variables with common distribution V satisfying the relation (2.12) and (2.13). Then for any fixed $T \in \Lambda$, it holds that uniformly for all t$\in \Lambda \cap [0, T]$

$$\Phi(x,t) \sim \int_0^t \left(\overline{H}\left(xe^{\delta t'}\right)\right) d\ \lambda(t') + \int_0^t \int_0^{t-t'} \left(\overline{F}\left(xe^{\delta(s'+t')}\right)\right) d\ G(s') d\lambda(t')$$

i.e.,

$$\lim_{x\to\infty} \sup_{t\in\Lambda\cap[0,T]} \left| \frac{\Phi(x,t)}{\int_0^t \left(\overline{H}\left(xe^{\delta t'}\right)\right) d\ \lambda(t') + \int_0^t \int_0^{t-t'} \left(\overline{F}\left(xe^{\delta(s'+t')}\right)\right) d\ G(s') d\lambda(t')} - 1 \right| = 0.$$

Chapter 4

Proofs

The proof of Theorem 3.1 is based on the following lemmas.

Lemma 4.1 For any positive integer m and events $A_1, \cdots, A_m$, it holds that

$$P\big(\cup_{i=1}^{m} A_i\big) \geq \sum_{i=1}^{m} P(A_i) - \sum_{1\leq i<j\leq m} P(A_iA_j). \tag{4.1}$$

Consequently,

$$P\Big(B \cap \big(\cup_{i=1}^{m} A_i\big)\Big) \geq \sum_{i=1}^{m} P(BA_i) - \sum_{1\leq i<j\leq m} P(A_iA_j). \tag{4.2}$$

Proof: We use mathematical induction to prove the relation (4.1). It is obviously true when $m=1$. Assume that it is true that when $m=k$, i.e.,

$$P\big(\cup_{i=1}^{k} A_i\big) \geq \sum_{i=1}^{k} P(A_i) - \sum_{1\leq i<j\leq k} P(A_iA_j). \tag{4.3}$$

When $m=k+1$, we can use the basic probability formula $P(A\cup B) = P(A)+P(B)-P(A\cap B)$ to get

$$P\big((\cup_{i=1}^{k} A_i)\cup(A_{k+1})\big) = P\big(\cup_{i=1}^{k} A_i\big) + P(A_{k+1}) - P((\cup_{i=1}^{k} A_i)\cap(A_{k+1})). \tag{4.4}$$

By induction assumption (4.3) and the equality (4.4), we obtain

$$P\big(\cup_{i=1}^{k+1} A_i\big) \geq \sum_{i=1}^{k} P(A_i) - \sum_{1\leq i<j\leq k} P(A_iA_j) + P(A_{k+1}) - P((\cup_{i=1}^{k} A_i)\cap(A_{k+1})). \tag{4.5}$$

Bonferroni inequality, $P(\cup_{i=1}^{k} E_i) \leq \sum_{i=1}^{k} P(E_i)$ for any events $E_1, \cdots, E_k$, yields

$$P((\cup_{i=1}^{k} A_i)\cap(A_{k+1})) = P(\cup_{i=1}^{k} A_iA_{k+1}) \leq \sum_{i=1}^{k} P(A_iA_{k+1}).$$

Combining with inequality (4.5), we get

$$P\big(\cup_{i=1}^{k+1} A_i\big) \geq \sum_{i=1}^{k+1} P(A_i) - \sum_{1\leq i<j\leq k+1} P(A_iA_j).$$

This ends the proof of the inequality (4.1). The inequality (4.2) can be easily proved by the inequality (4.1) with A_i replaced by BA_i. ■

The next lemma is integration by parts of Stieltjes Integral.

Lemma 4.2 (Integration by parts) Suppose that f and g are right continuous, nondecreasing and with left-hand limit functions on $[a,b]$, where $a<b\in\mathbf{R}$. Then

$$\int_{(a,b]} g(x)df(x) = g(b)f(b) - g(a)f(a) - \int_{(a,b]} f(x-)dg(x), \tag{4.6}$$

where $f(x-) = \lim_{t\to x-} f(t)$.

Proof: See Gut [3] or Shiryayev [13].

Lemma 4.3 Consider the renewal counting process $\{N(t), t\geq 0\}$ defined in (1.2). Suppose that Y_n, $n\geq 1$, satisfy the WLOD structure and the relation (2.13) holds. For any $T\in\Lambda$ and any $\gamma>0$, we obtain that

$$\lim_{x\to\infty} \sup_{t\in\Lambda\cap[0,T]} \lambda(t)^{-1} E(N(t))^{\gamma} 1_{\{N(t)>x\}} = 0. \tag{4.7}$$

Proof: See the proof of Lemma 2.1 in Wang et al. [19].

Let

$$A_{n,i}(x,t) = \{X_i e^{-\delta\omega_i} 1_{\{\omega_i\leq t\}} > x, N(t) = n\}$$

$$B_{n,i}(x,t) = \{Z_i e^{-\delta(\omega_i+T_i)} 1_{\{\omega_i+T_i\leq t\}} > x, N(t) = n\}.$$

Conditioning on $\{\omega_i\}, \{Y_i\}$, the property of class $\mathcal{D}$ yields the following result.

Lemma 4.4 Let $\{X_n, n\geq 1\}$ and $\{Z_n, n\geq 1\}$ have the common distribution function H and F, respectively, belonging to the class $\mathcal{L}\cap\mathcal{D}$. It holds that uniformly for $t\in\Lambda\cap[0,T]$, $1\leq i\leq n$, $k=1$ or 2,

$$P\big(A_{n,i}((x\pm 2l(x))/k,t)\big) \sim P\big(A_{n,i}(x/k,t)\big) = O(P(A_{n,i}(x,t))),$$

$$\text{and}\qquad P\big(B_{n,i}((x\pm 2l(x))/k,t)\big) \sim P\big(B_{n,i}(x/k,t)\big) = O(P(B_{n,i}(x,t))).$$

Lemma 4.5 Assume that $\{X_n, n\geq 1\}$ and $\{Z_n, n\geq 1\}$ be the mutual independent sequences of pSQAI random variables. Under the assumption of Lemma 4.4, for any $1\leq i\leq n$, it holds that

$$\liminf_{x\to\infty} \inf_{t\in\Lambda\cap[0,T]} \frac{P\big(X_i e^{-\delta\omega_i} 1_{\{\omega_i\leq t\}} + Z_i e^{-\delta(\omega_i+T_i)} 1_{\{\omega_i+T_i\leq t\}} > x + l(x), N(t) = n)\big)}{P(A_{n,i}(x,t)) + P(B_{n,i}(x,t))} \geq 1.$$

Proof: Since the claim size X_i and the by-claim size Z_i are nonnegative, we have

$$\begin{aligned}
& P\big(X_i e^{-\delta\omega_i} 1_{\{\omega_i \le t\}} + Z_i e^{-\delta(\omega_i+T_i)} 1_{\{\omega_i+T_i\le t\}} > x + l(x), N(t) = n\big) \\
\ge\ & P\big(X_i e^{-\delta\omega_i} 1_{\{\omega_i \le t\}} > x + l(x), N(t) = n) \\
& \cup (Z_i e^{-\delta(\omega_i+T_i)} 1_{\{\omega_i+T_i\le t\}}) > x + l(x), N(t) = n)) \\
=\ & P\big(X_i e^{-\delta\omega_i} 1_{\{\omega_i \le t\}} > x + l(x), N(t) = n) \\
& + P(Z_i e^{-\delta(\omega_i+T_i)} 1_{\{\omega_i+T_i\le t\}} > x + l(x), N(t) = n) \\
& - P\big(X_i e^{-\delta\omega_i} 1_{\{\omega_i \le t\}} > x + l(x), Z_i e^{-\delta(\omega_i+T_i)} 1_{\{\omega_i+T_i\le t\}} > x + l(x), N(t) = n\big) \\
=\ & P(A_{n,i}(x+l(x),t)) + P(B_{n,i}(x+l(x),t)) - P(A_{n,i}(x+l(x),t), B_{n,i}(x+l(x),t)).
\end{aligned}$$

By virtue of Lemma 4.4, it suffices to show that

$$\limsup_{x\to\infty} \sup_{t\in\Lambda\cap[0,T]} \frac{P(A_{n,i}(x+l(x),t), B_{n,i}(x+l(x),t))}{P(A_{n,i}(x,t)) + P(B_{n,i}(x,t))} = 0.$$

By the independence of $\{X_i, \omega_i\}$ and $\{Z_i\}$,

$$\begin{aligned}
P(A_{n,i}(x+l(x),t), B_{n,i}(x+l(x),t)) &\le P(A_{n,i}(x,t), Z_i > x + l(x)) \\
&= P(A_{n,i}(x,t)) P(Z_i > x + l(x)).
\end{aligned}$$

The desired result follows from Lemma 4.4 and the fact that $x + l(x) \to \infty$. ■

Lemma 4.6 Under the assumptions of Lemma 4.5, it holds that

$$\limsup_{x\to\infty} \sup_{t\in\Lambda\cap[0,T]} \frac{P(X_i e^{-\delta\omega_i} 1_{\{\omega_i \le t\}} + Z_i e^{-\delta(\omega_i+T_i)} 1_{\{\omega_i+T_i\le t\}} > x - l(x), N(t) = n)}{P(A_{n,i}(x,t)) + P(B_{n,i}(x,t))} \le 1.$$

Proof: To ease notation, define

$$C_{n,i}(x,t) = \{X_i e^{-\delta\omega_i} 1_{\{\omega_i \le t\}} + Z_i e^{-\delta(\omega_i+T_i)} 1_{\{\omega_i+T_i\le t\}} > x, N(t) = n\}.$$

By the simple formula that $P(A) = P(AB) + P(AB^C)$ and that $P(A \cap (B_1 \cup B_2)) \le P(B_1 \cup B_2) \le P(B_1) + P(B_2)$, we have

$$\begin{aligned}
P(C_{n,i}(x,t)) &= P\big(C_{n,i}(x,t), A_{n,i}(x-l(x),t) \cup B_{n,i}(x-l(x),t)\big) \\
&\quad + P\big(C_{n,i}(x,t), A_{n,i}^C(x-l(x),t) \cap B_{n,i}^C(x-l(x),t)\big) \\
&\le P(A_{n,i}(x-l(x),t)) + P(B_{n,i}(x-l(x),t)\big) \\
&\quad + P\big(C_{n,i}(x,t), A_{n,i}^C(x-l(x),t) \cap B_{n,i}^C(x-l(x),t)\big).
\end{aligned}$$

By virtue of Lemma 4.4, it suffices to show that

$$\limsup_{x\to\infty} \sup_{t\in\Lambda\cap[0,T]} \frac{P\big(C_{n,i}(x,t), A_{n,i}^C(x-l(x),t) \cap B_{n,i}^C(x-l(x),t)\big)}{P(A_{n,i}(x,t)) + P(B_{n,i}(x,t))} = 0. \tag{4.8}$$

It is easy to show that

$$\begin{aligned}&\{C_{n,i}(x,t),A_{n,i}^{C}(x-l(x),t)\}\subseteq\{B_{n,i}(l(x),t)\}\\ \text{and}\qquad&\{C_{n,i}(x,t),B_{n,i}^{C}(x-l(x),t)\}\subseteq\{A_{n,i}(l(x),t)\}.\end{aligned}$$

For any two random variables, X,Y, it is obviously true that $\{X+Y>x\}\subseteq\{X>x/2\}\cup\{Y>x/2\}$. Then,

$$\begin{aligned}&\{C_{n,i}(x,t),A_{n,i}^{C}(x-l(x),t)\cap B_{n,i}^{C}(x-l(x),t)\}\\ \subseteq\;&\big\{A_{n,i}(x/2,t),B_{n,i}(l(x),t)\big\}\cup\big\{B_{n,i}(x/2,t),A_{n,i}(l(x),t)\big\}\\ \subseteq\;&\big\{A_{n,i}(x/2,t),Z_i\geq l(x)\big\}\cup\big\{B_{n,i}(x/2,t),X_i\geq l(x)\big\}.\end{aligned}$$

By Boole's inequality and the independent of $\{X_i\},\{Z_i\}$, $\{\omega_i\},\{T_i\}$, we have

$$\begin{aligned}&P\big(C_{n,i}(x,t),A_{n,i}^{C}(x-l(x),t)\cap B_{n,i}^{C}(x-l(x),t)\big)\\ \leq\;&P\big(X_ie^{-\delta\omega_i}1_{\{\omega_i\leq t\}}>\frac{x}{2},N(t)=n)P(Z_i>l(x))\\ &+P\big(Z_ie^{-\delta(\omega_i+T_i)}1_{\{\omega_i+T_i\leq t\}}>\frac{x}{2},N(t)=n)P(X_i>l(x)).\end{aligned}$$

By the property of $\mathcal{D}$ and the fact that $l(x)\to\infty$, we can easily prove the relation (4.8). ■

Combining Lemma 4.5 and Lemma 4.6, the monotonicity of distribution functions yields the following result.

Lemma 4.7 Assume that $\{X_n,n\geq 1\}$ and $\{Z_n,n\geq 1\}$ be the mutual independent sequences of pSQAI random variables with common distribution function H and F, respectively, belonging to the class $\mathcal{L}\cap\mathcal{D}$. It holds that uniformly for $t\in\Lambda\cap[0,T]$, $1\leq i\leq n$,

$$\limsup_{x\to\infty}\sup_{t\in\Lambda\cap[0,T]}\left|\frac{P(X_ie^{-\delta\omega_i}1_{\{\omega_i\leq t\}}+Z_ie^{-\delta(\omega_i+T_i)}1_{\{\omega_i+T_i\leq t\}}>x\pm l(x),N(t)=n)}{P(A_{n,i}(x,t))+P(B_{n,i}(x,t))}-1\right|=0.$$

Lemma 4.8 For the renewal risk model introduced in Section 1, we have

$$\begin{aligned}&\sum_{k=1}^{\infty}P(X_ke^{-\delta\omega_k}1_{\{\omega_k\leq t\}}>x,N(t)\geq k)+P(Z_ke^{-\delta(\omega_k+T_k)}1_{\{\omega_k+T_k\leq t\}}>x,N(t)\geq k)\\ =\;&\int_{[0,t]}\overline{H}(xe^{\delta t'})d\lambda(t')+\int_{[0,t]}\int_{[0,t-t']}\overline{F}(xe^{\delta(t'+s')})d\,G(s')d\lambda(t').\end{aligned}$$

Proof: Define the distribution of ω_k as $V_{\omega_k}(t)$. Since $\{X_k\},\{\omega_k\},\{Z_k\},\{Y_k\}$ are independent, we have

$$
\begin{aligned}
& P\big(X_k e^{-\delta\omega_k} 1_{\{\omega_k \le t\}} > x, N(t) \ge k\big) + P\big(Z_k e^{-\delta(\omega_k+T_k)} 1_{\{\omega_k+T_k \le t\}} > x, N(t) \ge k\big) \\
= \;& \int_{[0,t]} \overline{H}(xe^{\delta t'}) dV_{\omega_k}(t') + \int_{[0,t]}\int_{[0,t-t']} \overline{F}(xe^{\delta(t'+s')}) dG(s') dV_{\omega_k}(t') \\
= \;& \int_{[0,t]} \overline{H}(xe^{\delta t'}) dV_{\omega_k}(t') + \int_{[0,t]}\int_{[0,t-s']} \overline{F}(xe^{\delta(t'+s')}) dV_{\omega_k}(t') dG(s') \\
= \;& \int_{[0,t]} \Big(1 - H(xe^{\delta t'})\Big) dV_{\omega_k}(t') + \int_{[0,t]}\int_{[0,t-s']} \Big(1 - F(xe^{\delta(t'+s')})\Big) dV_{\omega_k}(t') dG(s') \\
= \;& V_{\omega_k}(t) - \int_{[0,t]} H(xe^{\delta t'}) dV_{\omega_k}(t') + V_{\omega_k} * G(t) - \int_{[0,t]}\int_{[0,t-s']} F(xe^{\delta(t'+s')}) dV_{\omega_k}(t') dG(s').
\end{aligned}
$$

By the facts that $\int_{A\cup B} f(x)dg(x) = \int_A f(x)dg(x) + \int_B f(x)dg(x)$ and $\int_{\{a\}} f(x)dg(x) = f(a)(g(a) - g(a-))$, it is equal to

$$
\begin{aligned}
& V_{\omega_k}(t) - \int_{(0,t]} H(xe^{\delta t'}) dV_{\omega_k}(t') - H(x)\, V_{\omega_k}(0) + V_{\omega_k} * G(t) \\
& - \int_{[0,t]}\int_{(0,t-s']} F(xe^{\delta(t'+s')}) dV_{\omega_k}(t') dG(s') - \int_{[0,t]} F(xe^{\delta s'}) V_{\omega_k}(0) dG(s').
\end{aligned}
$$

Using Lemma 4.2, we know that it amounts to

$$
\begin{aligned}
& V_{\omega_k}(t) - H(xe^{\delta t}) V_{\omega_k}(t) + H(x)\, V_{\omega_k}(0) + \int_{(0,t]} V_{\omega_k}(t'-) dH(xe^{\delta t'}) \\
& - H(x)\, V_{\omega_k}(0) + V_{\omega_k} * G(t) \\
& - \int_{[0,t]} F(xe^{\delta t}) V_{\omega_k}(t - s') dG(s') + \int_{[0,t]} F(xe^{\delta s'}) V_{\omega_k}(0) dG(s') \\
& + \int_{[0,t]}\int_{(0,t-s']} V_{\omega_k}(t'-) dF(xe^{\delta(t'+s')}) dG(s') - \int_{[0,t]} F(xe^{\delta s'}) V_{\omega_k}(0) dG(s').
\end{aligned}
$$

Note that $V_{\omega_k}(t'-) = \mathrm{P}(\omega_k < t')$, $\{\omega_k \le t\} = \{N(t) \ge k\}$ and $\{\omega_k < t\} = \{N(t) < k\}$, $t \ge 0$. We can derive

$$
\sum_{k=1}^{\infty} V_{\omega_k}(t) = \sum_{k=1}^{\infty} P(N(t) \ge k) = EN(t) = \lambda(t) < \infty
$$

and

$$
\sum_{k=1}^{\infty} V_{\omega_k}(t-) = \sum_{k=1}^{\infty} P(N(t-) \le t) = EN(t-) = \lambda(t-).
$$

Therefore,

$$\begin{aligned}
&\sum_{k=1}^{\infty} P(X_k e^{-\delta\omega_k} 1_{\{\omega_k \le t\}} > x, N(t) \ge k) + P(Z_k e^{-\delta(\omega_k+T_k)} 1_{\{\omega_k+T_k \le t\}} > x, N(t) \ge k) \\
= \ & \lambda(t) - H(xe^{\delta t})\lambda(t) + H(x)\lambda(0) + \int_{(0,t]} \lambda(t'-) dH(xe^{\delta t'}) \\
&-H(x)\lambda(0) + \lambda * G(t) \\
&- \int_{[0,t]} F(xe^{\delta t})\lambda(t-s') dG(s') + \int_{[0,t]} F(xe^{\delta s'})\lambda(0) dG(s') \\
&+ \int_{[0,t]} \int_{(0,t-s']} \lambda(t'-) dF(xe^{\delta(t'+s')}) dG(s') - \int_{[0,t]} F(xe^{\delta s'})\lambda(0) dG(s').
\end{aligned}$$

Recall that $\lambda(t) = EN(t)$ is a nondecreasing and right continuous function. Using Lemma 4.2 again, it equals

$$\begin{aligned}
& \lambda(t) - \int_{(0,t]} H(xe^{\delta t'}) d\lambda(t') - H(x)\lambda(0) + \lambda * G(t) \\
& - \int_{[0,t]} \int_{(0,t-s']} F(xe^{\delta(t'+s')}) d\lambda(t') dG(s') - \int_{[0,t]} F(xe^{\delta s'})\lambda(0) dG(s') \\
= \ & \lambda(t) - \int_{[0,t]} H(xe^{\delta t'}) d\lambda(t') + \lambda * G(t) - \int_{[0,t]} \int_{[0,t-s']} F(xe^{\delta(t'+s')}) d\lambda(t') dG(s') \\
= \ & \int_{[0,t]} \overline{H}(xe^{\delta t'}) d\lambda(t') + \int_{[0,t]} \int_{[0,t-s']} \overline{F}(xe^{\delta(t'+s')}) d\lambda(t') dG(s') \\
= \ & \int_{[0,t]} \overline{H}(xe^{\delta t'}) d\lambda(t') + \int_{[0,t]} \int_{[0,t-t']} \overline{F}(xe^{\delta(t'+s')}) dG(s') d\lambda(t').
\end{aligned}$$

This ends the proof. ■

Lemma 4.9 Consider the risk model introduced in Section 1. Let $\{X_n\}$ be a sequence of pSQAI random variables with a common distribution function of H, $\{Z_n\}$ be a sequence of of pSQAI random variables with a common distribution function of F, independent of $\{X_n\}$. If $F, H \in \mathcal{L} \cap \mathcal{D}$, for any arbitrarily fixed n, it holds that uniformly for t $\in \Lambda \cap [0,T]$,

$$\begin{aligned}
& P\Bigg(\sum_{k=1}^{n} (X_k e^{-\delta\omega_k} 1_{\{\omega_k \le t\}} + Z_k e^{-\delta(\omega_k+T_k)} 1_{\{\omega_k+T_k \le t\}}) > x, N(t) = n \Bigg) \\
\sim \ & \sum_{k=1}^{n} P\Big(X_k e^{-\delta\omega_k} 1_{\{\omega_k \le t\}} > x, N(t) = n \Big) + \sum_{k=1}^{n} P\Big(Z_k e^{-\delta(\omega_k+T_k)} 1_{\{\omega_k+T_k \le t\}} > x, N(t) = n \Big).
\end{aligned}$$

Proof: It is equivalent to prove

$$\liminf_{x\to\infty} \inf_{t\in\Lambda\cap[0,T]} \frac{P\big(C_n(x,t)\big)}{\sum_{k=1}^{n} P(A_{n,k}(x,t)) + \sum_{k=1}^{n} P(B_{n,k}(x,t))} \ge 1 \tag{4.9}$$

and

$$\limsup_{x\to\infty} \sup_{t\in\Lambda\cap[0,T]} \frac{P\big(C_n(x,t)\big)}{\sum_{k=1}^n P(A_{n,k}(x,t)) + \sum_{k=1}^n P(B_{n,k}(x,t))} \leq 1, \tag{4.10}$$

where

$$C_n(x,t) = \Big\{ \sum_{k=1}^{n} (X_k e^{-\delta\omega_k} 1_{\{\omega_k\leq t\}} + Z_k e^{-\delta(\omega_k+T_k)} 1_{\{\omega_k+T_k\leq t\}}) > x, N(t) = n \Big\}.$$

We will prove the relation (4.10) first. By the inequality (4.2), we have

$$\begin{aligned} P\big(C_n(x,t)\big) \quad &\geq \quad \sum_{i=1}^{n} P\big(C_n(x,t), C_{n,i}(x+l(x),t)\big) \\ &\quad - \sum_{1\leq k<i\leq n} P\big(C_{n,i}(x+l(x),t), C_{n,k}(x+l(x),t)\big). \end{aligned} \tag{4.11}$$

It is easy to show that

$$\begin{aligned} &\big\{C_n(x,t), C_{n,i}(x+l(x),t)\big\} \\ \supseteq\ &\Big\{ \sum_{1\leq k\leq n, k\neq i} (X_k e^{-\delta\omega_k} 1_{\{\omega_k\leq t\}} + Z_k e^{-\delta(\omega_k+T_k)} 1_{\{\omega_k+T_k\leq t\}}) > -l(x), C_{n,i}(x+l(x),t) \Big\} \\ =\ &C_{n,i}(x+l(x),t) \\ &\quad \backslash \Big\{ \sum_{1\leq k\leq n, k\neq i} (X_k e^{-\delta\omega_k} 1_{\{\omega_k\leq t\}} + Z_k e^{-\delta(\omega_k+T_k)} 1_{\{\omega_k+T_k\leq t\}}) \leq -l(x), C_{n,i}(x+l(x),t) \Big\}. \end{aligned}$$

Since $\sum_{1\leq k\leq n, k\neq i} (X_k e^{-\delta\omega_k} 1_{\{\omega_k\leq t\}} + Z_k e^{-\delta(\omega_k+T_k)} 1_{\{\omega_k+T_k\leq t\}}) \leq -l(x)$ implies that one of $X_k e^{-\delta\omega_k} 1_{\{\omega_k\leq t\}} + Z_k e^{-\delta(\omega_k+T_k)} 1_{\{\omega_k+T_k\leq t\}}, 1 \leq k \neq i \leq n$, is at most $-\frac{l(x)}{n-1}$, we have

$$\begin{aligned} &P\big(C_n(x,t), C_{n,i}(x+l(x),t)\big) \\ \geq\ &P(C_{n,i}(x+l(x),t)) \\ &\quad -P\Big(\sum_{1\leq k\leq n, k\neq i} (X_k e^{-\delta\omega_k} 1_{\{\omega_k\leq t\}} + Z_k e^{-\delta(\omega_k+T_k)} 1_{\{\omega_k+T_k\leq t\}}) \leq -l(x), C_{n,i}(x+l(x),t) \Big) \\ \geq\ &P(C_{n,i}(x+l(x),t)) \\ &\quad - \sum_{1\leq k\leq n, k\neq i} P\Big(X_k e^{-\delta\omega_k} 1_{\{\omega_k\leq t\}} + Z_k e^{-\delta(\omega_k+T_k)} 1_{\{\omega_k+T_k\leq t\}} \leq -\frac{l(x)}{n-1}, C_{n,i}(x+l(x),t) \Big). \end{aligned}$$

In order to establish the relation (4.10), by the equality (4.11) and Lemma 4.5, it is sufficient to prove that for any $1 \leq k \neq i \leq n$,

$$\limsup_{x\to\infty} \sup_{t\in\Lambda\cap[0,T]} \frac{P\Big(X_k e^{-\delta\omega_k} 1_{\{\omega_k\leq t\}} + Z_k e^{-\delta(\omega_k+T_k)} 1_{\{\omega_k+T_k\leq t\}} \leq -\frac{l(x)}{n-1}, C_{n,i}(x+l(x),t) \Big)}{\sum_{k=1}^n P(A_{n,k}(x,t)) + \sum_{k=1}^n P(B_{n,k}(x,t))} = 0, \tag{4.12}$$

and

$$\limsup_{x\to\infty}\sup_{t\in\Lambda\cap[0,T]}\frac{P\big(C_{n,i}(x+l(x),t),C_{n,k}(x+l(x),t)\big)}{\sum_{k=1}^{n}P(A_{n,k}(x,t))+\sum_{k=1}^{n}P(B_{n,k}(x,t))}=0. \tag{4.13}$$

Note that

$$\begin{aligned}
&P\Big(X_ke^{-\delta\omega_k}1_{\{\omega_k\le t\}}+Z_ke^{-\delta(\omega_k+T_k)}1_{\{\omega_k+T_k\le t\}}\le-\frac{l(x)}{n-1},C_{n,i}(x+l(x),t)\Big)\\
\le\ & P\big(X_ke^{-\delta\omega_k}1_{\{\omega_k\le t\}}\le\frac{-l(x)}{2(n-1)},C_{n,i}(x+l(x),t)\big)\\
&+P\big(Z_ke^{-\delta(\omega_k+T_k)}1_{\{\omega_k+T_k\le t\}}\le\frac{-l(x)}{2(n-1)},,C_{n,i}(x+l(x),t)\big)\\
\le\ & P\big(X_ke^{-\delta\omega_k}1_{\{\omega_k\le t\}}\le\frac{-l(x)}{2(n-1)},X_ie^{-\delta\omega_i}1_{\{\omega_i\le t\}}>\frac{x+l(x)}{2},N(t)=n\big)\\
&+P\big(X_ke^{-\delta\omega_k}1_{\{\omega_k\le t\}}\le\frac{-l(x)}{2(n-1)},Z_ie^{-\delta(\omega_i+T_i)}1_{\{\omega_i+T_i\le t\}}>\frac{x+l(x)}{2},N(t)=n\big)\\
&+P\big(Z_ke^{-\delta(\omega_k+T_k)}1_{\{\omega_k+T_k\le t\}}\le\frac{-l(x)}{2(n-1)},X_ie^{-\delta\omega_i}1_{\{\omega_i\le t\}}>\frac{x+l(x)}{2},N(t)=n\big)\\
&+P\big(Z_ke^{-\delta(\omega_k+T_k)}1_{\{\omega_k+T_k\le t\}}\le\frac{-l(x)}{2(n-1)},Z_ie^{-\delta(\omega_k+T_k)}1_{\{\omega_k+T_k\le t\}}>\frac{x+l(x)}{2},N(t)=n\big).
\end{aligned}$$

Now the relation (4.12) follows from the pSQAI property of $\{X_i\},\{Z_i\}$ and Lemma 4.4. Similarly, uniformly for $t\in\Lambda\cap[0,T],1\le k\ne i\le n$,

$$\begin{aligned}
&P\big(C_{n,i}(x+l(x),t),C_{n,k}(x+l(x),t)\big)\\
\le\ & P\big(X_ke^{-\delta\omega_k}1_{\{\omega_k>t\}}>\frac{x+l(x)}{2},X_ie^{-\delta\omega_i}1_{\{\omega_i>t\}}>\frac{x+l(x)}{2},N(t)=n\big)\\
&+P\big(X_ke^{-\delta\omega_k}1_{\{\omega_k>t\}}>\frac{x+l(x)}{2},Z_ie^{-\delta(\omega_i+T_i)}1_{\{\omega_i+T_i>t\}}>\frac{x+l(x)}{2},N(t)=n\big)\\
&+P\big(Z_ke^{-\delta(\omega_k+T_k)}1_{\{\omega_k+T_k>t\}}>\frac{x+l(x)}{2},X_ie^{-\delta\omega_i}1_{\{\omega_i>t\}}>\frac{x+l(x)}{2},N(t)=n\big)\\
&+P\big(Z_ke^{-\delta(\omega_k+T_k)}1_{\{\omega_k+T_k>t\}}>\frac{x+l(x)}{2},Z_ie^{-\delta(\omega_k+T_k)}1_{\{\omega_k+T_k>t\}}>\frac{x+l(x)}{2},N(t)=n\big)\\
=\ & o\big(P(A_{n,k}(x,t))+P(A_{n,i}(x,t))+P(B_{n,k}(x,t))+P(B_{n,i}(x,t))\big),
\end{aligned}$$

i.e., the relation (4.13) follows from the pSQAI property of $\{X_i\},\{Z_i\}$ and Lemma 4.4. Next, we will prove the relation (4.10). By the basic equality $P(A)=P(AB)+P(AB^C)$, we have

$$\begin{aligned}
P\big(C_n(x,t)\big) &= P\big(C_n(x,t),\cup_{i=1}^{n}C_{n,i}(x-l(x),t)\big)\\
&\quad+P\big(C_n(x,t),\cap_{i=1}^{n}C_{n,i}^{C}(x-l(x),t)\big).
\end{aligned} \tag{4.14}$$

By the inequality (4.2) and Lemma 4.4, it is equivalent to prove

$$\limsup_{x\to\infty}\sup_{t\in\Lambda\cap[0,T]}\frac{P\big(C_n(x,t),\cap_{i=1}^{n}C_{n,i}^{C}(x-l(x),t)\big)}{\sum_{k=1}^{n}P(A_{n,k}(x,t))+\sum_{k=1}^{n}P(B_{n,k}(x,t))}=0.$$

Since $C_n(x,t) \subseteq \cup_{i=1}^{n} C_{n,i}(x/n,t)$ and

$$\begin{aligned}
& C_n(x,t) \cap C_{n,k}^{C}(x-l(x),t) \\
\subseteq\ & \{ \sum_{i=1, i\neq k}^{n} (X_k e^{-\delta\omega_k} 1_{\{\omega_k \le t\}} + Z_k e^{-\delta(\omega_k+T_k)} 1_{\{\omega_k+T_k \le t\}}) > l(x), N(t)=n \} \\
\subseteq\ & \cup_{i=1, i\neq k}^{n} \{ X_k e^{-\delta\omega_k} 1_{\{\omega_k \le t\}} + Z_k e^{-\delta(\omega_k+T_k)} 1_{\{\omega_k+T_k \le t\}} > l(x)/(n-1), N(t)=n \} \\
=\ & \cup_{i=1, i\neq k}^{n} C_{n,i}(l(x)/(n-1),
\end{aligned}$$

we know that

$$\begin{aligned}
P\big(C_n(x,t), \cap_{i=1}^{n} C_{n,i}^{C}(x-l(x),t)\big) &\le \sum_{k=1}^{n} P\big(C_{n,k}(x/n,t), C_n(x,t), \cap_{i=1}^{n} C_{n,i}^{C}(x-l(x),t)\big) \\
&\le \sum_{k=1}^{n} \sum_{i=1, i\neq k}^{n} P\big(C_{n,k}(x/n,t), C_{n,i}(l(x)/(n-1)\big) \\
&= o(\sum_{k=1}^{n} P(A_{n,k}(x,t)) + \sum_{k=1}^{n} P(B_{n,k}(x,t)))
\end{aligned}$$

holds uniformly for $t \in \Lambda \cap [0,T], 1 \le k \neq i \le n$ by the pSQAI property of $\{X_i\}, \{Z_i\}$ and Lemma 4.4. This ends the proof of the lemma. ■

Lemma 4.10 Assume that $\{X_n, Y_n, n \ge 1\}, \{\omega_n, n \ge 1\}$ and $\{T_n, n \ge 1\}$ are mutually independent. The claim sizes $\{X_n, n \ge 1\}$ and the by-claim parts $\{Z_n, n \ge 1\}$ follow the pSQAI structure with common distribution $H, F \in \mathcal{L} \cap D$ and $\overline{F} = O(\overline{H})$. The inter-arrival times $\{Y_n, n \ge 1\}$ are WLOD random variables. Suppose that the relation (2.12) and (2.13) hold. Then for any fixed finite T,

$$\begin{aligned}
& P\Big(R_\delta^\star(t) = \sum_{k=1}^{\infty} X_k e^{-\delta\omega_k} 1_{\{\omega_k \le t\}} + \sum_{k=1}^{\infty} Z_k e^{-\delta(\omega_k+T_k)} 1_{\{\omega_k+T_k \le t\}} > x\Big) \\
\sim\ & \int_0^t \overline{H}\big(xe^{\delta t'}\big) d\lambda(t') + \int_0^t \int_0^{t-t'} \overline{F}\big(xe^{\delta(s'+t')}\big) dG(s') d\lambda(t').
\end{aligned}$$

holds uniformly for $t \in \Lambda \cap [0,T]$.

Proof: Clearly, we have

$$\begin{aligned}
& P(R_\delta^\star(t) > x) \\
=\ & \Big(\sum_{n=1}^{m_0} + \sum_{n=m_0+1}^{\infty}\Big) P\Big(\sum_{k=1}^{n} X_k e^{-\delta\omega_k} 1_{\{\omega_k \le t\}} + Z_k e^{\delta(\omega_k+T_k)} 1_{\{\omega_k+T_k \le t\}} > x, N(t)=n\Big) \\
=\ & I_1 + I_2.
\end{aligned}$$

Firstly, we deal with the I_1. By Lemma 4.9, it holds that uniformly for $t \in \Lambda \cap [0,T]$

$$\begin{aligned}
I_1 &\sim \Big(\sum_{n=1}^{m_0}\sum_{k=1}^{n}\Big) P\big(X_k e^{-\delta\omega_k} 1_{\{\omega_k \le t\}} > x, N(t)=n\big) \\
&\quad + \sum_{n=1}^{m_0}\sum_{k=1}^{n} P\big(Z_k e^{-\delta(\omega_k+T_k)} 1_{\{\omega_k+T_k \le t\}} > x, N(t)=n\big) \\
&= \Big(\sum_{n=1}^{\infty} - \sum_{n=m_0+1}^{\infty}\Big) \sum_{k=1}^{n} \big(P(X_k e^{-\delta\omega_k} 1_{\{\omega_k \le t\}} > x, N(t)=n) \\
&\quad + P(Z_k e^{-\delta(\omega_k+T_k)} 1_{\{\omega_k+T_k \le t\}} > x, N(t)=n)\big) \\
&= I_3 - I_4.
\end{aligned}$$

Using Lemma 4.8, we have

$$\begin{aligned}
I_3 &= \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} \big(P(X_k e^{-\delta\omega_k} 1_{\{\omega_k \le t\}} > x, N(t)=n) \\
&\quad + P(Z_k e^{-\delta(\omega_k+T_k)} 1_{\{\omega_k+T_k \le t\}} > x, N(t)=n)\big) \\
&= \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} P(X_k e^{-\delta\omega_k} 1_{\{\omega_k \le t\}} > x, N(t) \ge k) \\
&\quad + \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} P(Z_k e^{-\delta(\omega_k+T_k)} 1_{\{\omega_k+T_k \le t\}} > x, N(t) \ge k) \\
&= \int_{[0,t]} \overline{H}(xe^{\delta t'}) d\lambda(t') + \int_{[0,t]}\int_{[0,t-t']} \overline{F}(xe^{\delta(t'+s')}) d\,G(s') d\lambda(t').
\end{aligned}$$

For I_4, because of the independent relationship among X_k, Z_k and ω_k,

$$\begin{aligned}
I_4 &\le \sum_{n=m_0+1}^{\infty}\sum_{k=1}^{n} \Big(P(X_k > x, N(t)=n) + P(Z_k > x, N(t)=n)\Big) \\
&= \sum_{n=m_0+1}^{\infty} n\big(P(X_k > x) + P(Z_k > x)\big) P(N(t)=n) \\
&= \overline{H}(x) \sum_{n=m_0+1}^{\infty} nP(N(t)=n) + \overline{F}(x) \sum_{n=m_0+1}^{\infty} nP(N(t)=n).
\end{aligned}$$

By the condition $F, H \in \mathcal{D}$ and Lemma 4.3, we can derive that

$$\lim_{m_0\to\infty} \limsup_{x\to\infty} \sup_{t\in\Lambda\cap[0,T]} \frac{I_4}{\int_{[0,t]} \overline{H}(xe^{\delta t'}) d\lambda(t') + \int_{[0,t]}\int_{[0,t-t']} \overline{F}(xe^{\delta(t'+s')}) d\,G(s') d\lambda(t')} = 0.$$

Note that

$$\begin{aligned}
&\Big\{\sum_{k=1}^{n} X_k e^{-\delta\omega_k} 1_{\{\omega_k \le t\}} + \sum_{k=1}^{n} Z_k e^{-\delta(\omega_k+T_k)} 1_{\{\omega_k+T_k \le t\}} > x\Big\} \\
\subseteq\ & \cup_{k=1}^{n}\Big\{X_k e^{-\delta\omega_k} 1_{\{\omega_k \le t\}} \ge \frac{x}{2n}\Big\} \cup \cup_{k=1}^{n}\Big\{Z_k e^{-\delta(\omega_k+T_k)} 1_{\{\omega_k+T_k \le t\}} \ge \frac{x}{2n}\Big\} \\
\subseteq\ & \cup_{k=1}^{n}\Big\{X_k \ge \frac{x}{2n}\Big\} \cup \cup_{k=1}^{n}\Big\{Z_k e^{-\delta(\omega_k+T_k)} 1_{\{\omega_k+T_k \le t\}} \ge \frac{x}{2n}\Big\}.
\end{aligned}$$

Then by the independence of $\{X_i, Z_i\}$ and $\{\omega_i, T_i\}$, we know

$$\begin{aligned}
& P\Big(\sum_{k=1}^{n} X_k e^{-\delta\omega_k} 1_{\{\omega_k\le t\}} + \sum_{k=1}^{n} Z_k e^{-\delta(\omega_k+T_k)} 1_{\{\omega_k+T_k\le t\}} > x, N(t)=n\Big) \\
\le\ & \sum_{k=1}^{n} P\Big(X_k \ge \frac{x}{2n}, N(t)=n\Big) + \sum_{k=1}^{n} P\Big(Z_k e^{-\delta(\omega_k+T_k)} 1_{\{\omega_k+T_k\le t\}} \ge \frac{x}{2n}, N(t)=n\Big) \\
\le\ & \sum_{k=1}^{n} P\Big(X_k \ge \frac{x}{2n}\Big) P(N(t)=n) + \sum_{k=1}^{n} P\Big(Z_k \ge \frac{x}{2n}\Big) P(\omega_k+T_k\le t, N(t)=n).
\end{aligned}$$

Since the probability of any event should not be greater than 1, it holds that

$$\begin{aligned}
I_2 \ \le\ & \sum_{m_0<n\le\frac{x}{d}} P\Big(\sum_{k=1}^{n} X_k e^{-\delta\omega_k} 1_{\{\omega_k\le t\}} + \sum_{k=1}^{n} Z_k e^{-\delta(\omega_k+T_k)} 1_{\{\omega_k+T_k\le t\}} > x, N(t)=n\Big) \\
& + \sum_{\frac{x}{d}<n<\infty} 1\cdot P(N(t)=n) \\
\le\ & \sum_{m_0<n<\frac{x}{d}} n\overline{H}\Big(\frac{x}{2n}\Big) P(N(t)=n) + \sum_{m_0<n<\frac{x}{d}} n\overline{F}\Big(\frac{x}{2n}\Big) \sum_{k=1}^{n} P(\omega_k+T_k\le t, N(t)=n) \\
& + P(N(t)\ge x/d).
\end{aligned}$$

Choose $\gamma > \max\{J_F^+, J_H^+\}$. Applying the Markov's inequality, we derive

$$P(N(t)\ge x/d) \le \frac{E(N(t))^{\gamma+1} 1_{\{N(t)\ge x/d\}}}{(x/d)^{\gamma+1}}.$$

By (2.6), we obtain that for some $c>0$ and large x

$$\frac{\overline{F}(x/2n)}{\overline{F}(x)} \le cn^{\gamma} \quad \text{and} \quad \frac{\overline{H}(x/2n)}{\overline{H}(x)} \le cn^{\gamma}.$$

Therefore,

$$\begin{aligned}
I_2 \ \le\ & c\overline{H}(x) \sum_{m_0<n<\frac{x}{d}} n^{\gamma+1} P(N(t)=n) + c\overline{F}(x) \sum_{m_0<n<\frac{x}{d}} n^{\gamma+1} \sum_{k=1}^{n} P(\omega_k+T_k\le t, N(t)=n) \\
& + \Big(\frac{x}{d}\Big)^{-\gamma-1} E(N(t))^{\gamma+1} 1_{\{N(t)\ge\frac{x}{d}\}} \\
\le\ & c\overline{H}(x) E(N(t))^{\gamma+1} 1_{\{N(t)>m_0\}} + c\overline{F}(x) E(N(t))^{\gamma+1} 1_{\{N(t)>m_0\}},
\end{aligned}$$

where the last inequality follows from the fact that $F, G \in \mathcal{D}$ and $\gamma > \max\{J_F^+, J_H^+\}$. Hence, by the condition of H and F belonging to the class $\mathcal{D}$, $\overline{F} = O(\overline{H})$ and applying

Lemma 4.3, we obtain

$$
\begin{aligned}
&\lim_{m_0\to\infty}\limsup_{x\to\infty}\sup_{t\in\Lambda\cap[0,T]}\frac{I_2}{\int_0^t(\overline{H}(xe^{\delta t'}))d\lambda(t')+\iint_{s',t'\geq 0,s'+t'\leq t}\overline{F}(xe^{\delta(s'+t')})dG(s')d\lambda(t')}\\
\leq\ &\lim_{m_0\to\infty}\limsup_{x\to\infty}\sup_{t\in\Lambda\cap[0,T]}\frac{c\overline{H}(x)E(N(t))^{\gamma+1}1_{\{N(t)>m_0\}}+c\overline{F}(x)E(N(t))^{\gamma+1}1_{\{N(t)>m_0\}}}{\overline{H}(xe^{\delta T})\lambda(t)}\\
\leq\ &c\limsup_{x\to\infty}\frac{\overline{H}(x)+\overline{F}(x)}{\overline{H}(xe^{\delta T})}\lim_{m_0\to\infty}\sup_{t\in\Lambda\cap[0,T]}\lambda(t)^{-1}E(N(t))^{\gamma+1}1_{\{N(t)>m_0\}}\\
=\ &0.
\end{aligned}
$$

Thus, we complete the proof of the lemma. ■

Proof of Theorem 3.1 Recall that

$$
\begin{aligned}
R_\delta(x,t) &= xe^{\delta t}+\int_{[0,t]}e^{\delta(t-s)}cds-\sum_{k=1}^{\infty}X_ke^{\delta(t-\omega_k)}1_{\{\omega_k\leq t\}}\\
&\quad -\sum_{k=1}^{\infty}Z_ke^{\delta(t-\omega_k-T_k)}1_{\{\omega_k+T_k\leq t\}}.
\end{aligned}
$$

Define

$$\tilde{R}_\delta(x,t)=x+\tilde{c}(t)-R_\delta^\star(t),$$

where $\tilde{c}(t)=\int_{[0,t]}e^{\delta(t-s)}cds=\frac{c}{\delta}(e^{\delta t}-1)$. Then the finite time ruin probability is

$$
\begin{aligned}
\Phi(x,t) &= P(R_\delta(x,s)<0,\text{for some }0\leq s\leq t)\\
&= P(e^{-\delta t}R_\delta(x,s)<0,\text{for some }0\leq s\leq t)\\
&= P(R_\delta^\star(s)-\tilde{c}(s)>x,\text{for some }0\leq s\leq t)\\
&= P(\sup_{0\leq s\leq t}(R_\delta^\star(s)-\tilde{c}(s))>x).
\end{aligned}
$$

Lemma 4.10 and the fact $H,F\in\mathcal{D}$ lead to

$$
\begin{aligned}
\Phi(x,t) &\geq P(R_\delta^\star(x)>x+\tilde{c}(t)=x+c/\delta(e^{\delta t}-1))\\
&\sim \int_0^t\overline{H}\big((x+c/\delta(e^{\delta t}-1))e^{\delta t'}\big)d\lambda(t')\\
&\qquad +\int_0^t\int_0^{t-t'}\overline{F}\big((x+c/\delta(e^{\delta t}-1))e^{\delta(s'+t')}\big)dG(s')d\lambda(t')\\
&\sim \int_0^t\overline{H}\big(xe^{\delta t'}\big)d\lambda(t')+\int_0^t\int_0^{t-t'}\overline{F}\big(xe^{\delta(s'+t')}\big)dG(s')d\lambda(t'),
\end{aligned}
$$

which holds uniformly for $t\in\Lambda\cap[0,T]$.

Since $\sup_{0\leq s\leq t}(R_\delta^\star(s)-\tilde{c}(s))\leq\sup_{0\leq s\leq t}R_\delta^\star(s)\leq R_\delta^\star(t)$, it follows from Lemma 4.10 again that

$$\Phi(x)\leq P(R_\delta^\star(t)>x)\sim\int_0^t\overline{H}\big(xe^{\delta t'}\big)d\lambda(t')+\int_0^t\int_0^{t-t'}\overline{F}\big(xe^{\delta(s'+t')}\big)dG(s')d\lambda(t').$$

This completes the proof of Theorem 3.1. ■

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