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The Structure and Properties of Clique Graphs of Regular Graphs

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THE STRUCTURE AND PROPERTIES

OF CLIQUE GRAPHS OF REGULAR GRAPHS

by

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ABSTRACT

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In the following thesis, the structure and properties of G and its clique graph $cl_t(G)$ are analyzed for graphs G that are non-complete, regular with degree δ, and where every edge of G is contained in a t-clique. In a clique graph $cl_t(G)$, all cliques of order t of the original graph G become the clique graph's vertices, and the vertices of the clique graph are adjacent if and only if the corresponding cliques in the original graph have at least 1 vertex in common. This thesis mainly investigates if properties of regular graphs are carried over to clique graphs of regular graphs. In particular, the first question considered is whether the clique graph of a regular graph must also be regular. It is shown that while line graphs, $cl_2(G)$, of regular graphs are regular, the degree difference of the clique graph $cl_3(R)$ can be arbitrarily large using δ-regular graphs R with $\delta \geq 3$. Next, the question of whether a clique graph can have a large independent set is considered (independent sets in regular graphs can be composed of half the vertices in the graph at the most). In particular, the relation between the degree difference and the independence number of $cl_t(G)$ will be analyzed. Lastly, we close with some further questions regarding clique graphs.

ACKNOWLEDGMENTS

First of all, I would like to thank The University of Southern Mississippi and my graduate committee members for giving me the opportunity to pursue my master studies in mathematics (in spite of my bachelor's business background which certainly did not make me the strongest applicant). Thank you for trusting in my capabilities and giving me a chance.

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Thirdly, I would like to thank my thesis committee members – Dr. Harris, Dr. Lambers and Dr. Lyle – for allowing me to defend my prospectus from the other end of the country, so I could finish in December 2014. And lastly, of course, I would like to thank you, Dr. Lyle, for your guidance, patience, and time as my thesis adviser. Although it was quite a bit of work, I thoroughly enjoyed the process of meeting with you on a weekly basis and of drafting this thesis. Thank you for guiding me through this experience. I really appreciate it.

Now, enjoy reading and learning about what I have been working on for the last couple of months.

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INTRODUCTION

1.1 Definitions

This thesis is concerned with problems from an area of mathematics called graph theory. To begin, essential terms and definitions that will be used throughout the entire thesis are provided. In notation, we will follow West [10].

Let $G = (V,E)$ be a *graph* with vertex set $V(G)$ and edge set $E(G)$. The *order* of G is $|G| = |V(G)| = n$, that is, the number of vertices in G. Let $u \in V(G)$ be a vertex of G. Then $N_G(u)$ stands for the *neighborhood* of G (vertices in the neighborhood are called the neighbors of u), that is, the set of vertices of G adjacent to u excluding u itself. The *degree* of a vertex u of a graph G is the number of edges adjacent to u and is denoted by $d(u)$. A *path* in a graph G on n vertices, denoted by P_n, is a sequence of edges which connect a sequence of vertices which are all distinct from one another. A *cycle* of a graph G on n vertices, denoted by C_n, is a subset of the edge set of G that forms a path such that the first vertex of the path corresponds to the last. A *regular graph* is a graph where each vertex has the same number of neighbors. A regular graph with vertices of degree δ is called a δ-regular graph. The *minimum degree* of G is denoted by $\delta(G)$. The *maximum degree* $\Delta(G)$ of G is the maximum number of neighbors of any vertex in that graph.

A graph $H = G^c$ on the same vertices as graph G, such that 2 distinct vertices of H are adjacent if and only if they are not adjacent in G, is called the *complement* of the graph G. A *complete graph* is a graph in which every pair of distinct vertices is connected by an edge. A complete graph on n-vertices is denoted by K_n. A set S with property P is maximal (with respect to P) if no set S' exists with S properly contained in S' such that S' has property P. A set S with property P is maximum (with respect to P) if no set S' exists with $|S| < |S'|$ such that S' has property P.

A *subgraph* is a graph G' whose vertices and edges form subsets of the vertices and edges of a given graph G. A set of subgraphs of G is *vertex-disjoint* if no 2 of them have any common vertex in G. An edge-induced subgraph is a subset of the edges of a graph G together with any vertices that are their endpoints. An *induced subgraph* of a graph (or vertex-induced subgraph) is a subset of vertices, with all the edges between those vertices that are present in the larger graph. An induced subgraph that is a complete graph is called a *clique*. A clique of order t is called t-clique. A maximal clique is a clique that cannot be extended by including 1 more adjacent vertex, meaning it is not a subset of a larger clique. A maximum clique is a clique of the largest possible size in a given graph. The *clique number* $\omega(G)$ of a graph G is the order of the largest clique in that graph.

In a clique graph $cl(G)$, all maximal cliques of the original graph G become vertices of $cl(G)$. In this thesis, a slightly different definition for the clique graph will be used. Namely, in a *clique graph* $cl_t(G)$, all cliques of order t of the original graph G become vertices of the clique graph. In either case, the vertices of the clique graph are adjacent if and only if the corresponding cliques in the original graph have at least 1 vertex in common. For example, in Figure 1.1, the maximal clique of G is of order 3, which is a triangle. Therefore, all triangles in G become vertices in $cl(G)$. In contrast, for $cl_t(G)$, if $t = 2$ is selected, then all cliques of order 2 become vertices in the clique graph. Neighboring cliques are t-cliques that share at least 1 vertex in the original graph G and whose corresponding vertices in $cl_t(G)$ will therefore be adjacent.

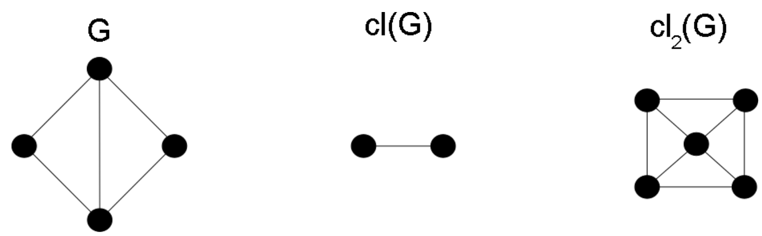


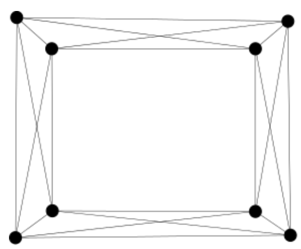
Figure 1.1: Comparison between $cl(G)$ and $cl_t(G)$

An *independent set* is a set of vertices in a graph G of which no pair is adjacent. An independent set $S \subset V(G)$ such that $S + v$ is no longer an independent set for any choice of $v \in V(G) - S$ is called a maximal independent set. The *independence number* $\alpha(G)$ of a graph G is the order of the largest independent set of G. A dominating set $S \subseteq V(G)$ in a graph G is a subset of the vertices, such that for every vertex $v \in V(G)$, either $v \in S$, or $N_G(v) \cap S \neq \emptyset$ (there is a neighbor of v contained in S). A *dominating vertex* is a vertex in G that has degree $n-1$. An independent dominating set $S \subseteq V(G)$ in a graph G is a dominating set that is additionally independent, i.e., for any vertices $u, v \in S$, $(u, v) \notin E(G)$. The order of the smallest independent dominating set is called the *independent domination number*, and is denoted by $i(G)$. Note that any maximal independent set is also an independent dominating set.

The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors needed to assign colors to the vertices in G, so adjacent vertices have different colors. The *ceiling* of the number x, denoted by $\lceil x \rceil$, is the smallest integer greater than or equal to x. Similarly, the *floor* of the number x, denoted by $\lfloor x \rfloor$, is the greatest integer less than or equal to x.

1.2 Main topic

In particular, this thesis is concerned with the structure and properties of $cl_t(G)$ of δ-regular graphs G where every edge of G is contained in a t-clique.



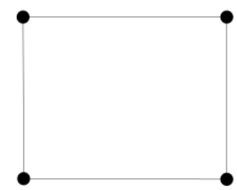
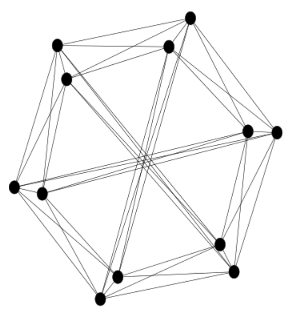


Figure 1.2: Graph H with $\delta = 5$ and its corresponding clique graph $cl_4(H)$



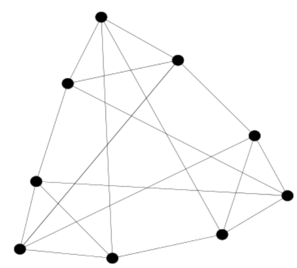


Figure 1.3: Graph J with $\delta = 7$ and its corresponding clique graph $cl_4(J)$

In order to clarify the topic and the definitions, 2 graphs H (Figure 1.2) and J (Figure 1.3) that meet the following conditions and their respective clique graphs are now provided:

- G is δ-regular,
- G is non-complete,
- Every edge of G is in a clique of order t,
- t is close to δ.

In contrast, the graphs and their respective clique graphs in Figures 1.4, 1.5, and 1.6 that do not meet 1 of the criteria are presented. Although the focus of this thesis' analysis is not on these graphs, they are now added in order to clarify the topic and the stated conditions.

First of all, the following graph L (Figure 1.4) is not regular and therefore does not meet the first criterion.

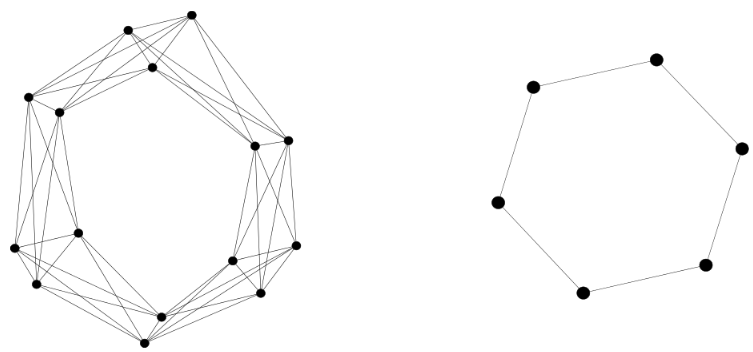


Figure 1.4: Non-regular graph L with degrees 6 and 7 and its corresponding clique graph $cl_4(L)$

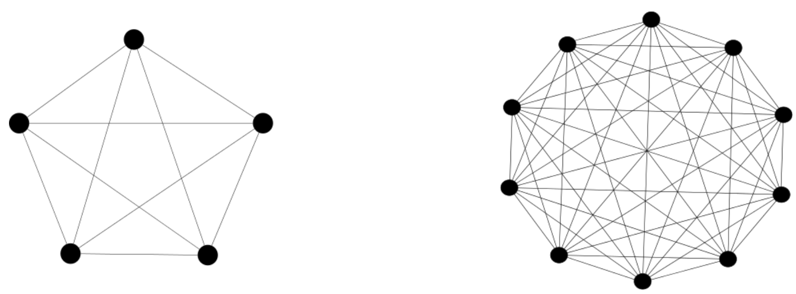


Figure 1.5: Complete graph $M = K_5$ and its corresponding clique graph $cl_3(M) = K_{10}$

Second, the graph M (Figure 1.5) is the complete graph on 5 vertices, K_5, and therefore does not meet the second criterion.

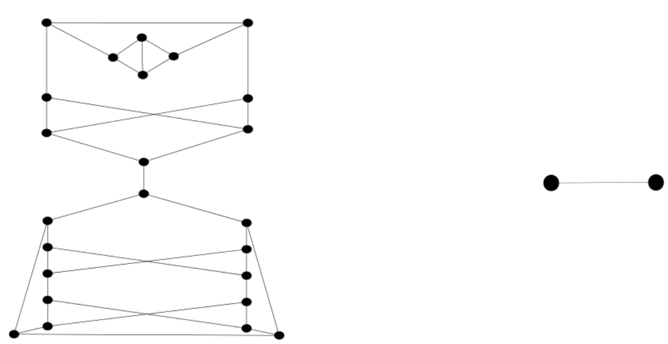


Figure 1.6: Graph N where not every edge is in a clique of size $t = 3$ and its corresponding clique graph $cl_3(N)$

Third, in the graph N (Figure 1.6), not every edge is in a clique of order $t = 3$. Therefore, N does not meet the third criterion.

The analysis focuses on graphs that meet the outlined conditions due to the following reasons: Whereas complete graphs are too easy to analyze, non-regular graphs may not have much structure in their clique graphs. Therefore, the analysis focuses on non-complete and regular graphs. Furthermore, as seen in Figure 1.6, if not every edge is contained in a t-clique, a lot of information about all the edges of G that are not in a t-clique is lost in $cl_t(G)$. Lastly, if t is close to δ, the results, namely the clique graph $cl_t(G)$, reveal a lot of information about the structure about the original graph G.

Remark: While the analysis of the clique graph $cl_3(N)$ (Figure 1.6) would not be considered (since not every edge in N is in a 3-clique), the clique graph $cl_2(N)$ is since every edge of N is in a 2-clique.

As you might have observed, all of the clique graphs in Figures 1.2, 1.3, 1.4, 1.5, and 1.6 are regular. This raises some of the following questions: Will all clique graphs be regular? What are the implications of the regularity condition on the independence number of the clique graph? What are the consequences on the results of the first 2 questions if t is changed? If the clique graph is a clique, does the original graph G have to be a clique?

Such questions concerning the structure and properties of clique graphs (of graphs that meet the stated criteria) and their discussions' implications on the original graphs will be addressed in the following chapters.

1.3 Literature review

1.3.1 Characteristics of regular graphs and their corresponding clique graphs

Lyle [7] considered upper bounds on the independent domination number of G, $i(G)$, for regular graphs with large degrees. First of all, regarding the independent domination of regular graphs, Lyle's main result is the following theorem: Let G be a δ-regular graph such that G^c is connected, and k be any integer. Then,

$$i(G) \leq \begin{cases} \left(\frac{3}{5}\right)(n-\delta), & \text{if } \frac{1}{6}n < \delta < \frac{1}{4}n, \\ \left(\frac{5}{8}\right)(n-\delta), & \text{if } \frac{1}{4}n < \delta < (n-8), \text{ and } \delta \neq \frac{k-3}{k}n \text{ (for } k \geq 4), \\ \left(\frac{2}{k}\right)n, & \text{if } \delta = \frac{k-3}{k}n \text{ (for } k \geq 4) \end{cases} \tag{1.1}$$

In the following, all 3 cases are illustrated. Recall that the order of the smallest independent dominating set is called the independent domination number and is denoted by $i(G)$.

First of all, let $n = 10$. Then, if $\frac{1}{6}n < \delta < \frac{1}{4}n$, let $\delta = 2$. We see that C_{10}'s (Figure 1.7) $i(G) = 4$ does satisfy $i(G) \leq \frac{3}{5}(n-\delta) = \frac{3}{5}(10-2) = 4.8$.

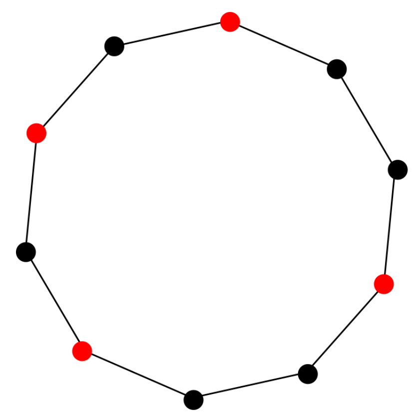


Figure 1.7: Graph $G = C_{10}$ with $i(G) = 4$

For the second case, let $n = 13$. Since $\frac{1}{4}n < \delta < (n-8)$, let $\delta = 4$. Then, we have to ensure that, for any $k \geq 4$, $\delta = 4 \neq \frac{k-3}{k}n$. If $k = 4$, then $\frac{k-3}{k}n = 3.25 \neq 4$ and if $k = 5$,then $\frac{k-3}{k}n = 5.2 \neq 4$. Since for any $k > 5$, $\frac{k-3}{k}n > 5.2$, for any $k \geq 4$, $\delta = 4 \neq \frac{k-3}{k}n$ holds true. Therefore, let $n = 13$ and $\delta = 4$ (Figure 1.8) in order to illustrate the second case. Lyle's theorem states that $i(G) \leq \frac{5}{8}(n-\delta) = \frac{5}{8}(13-4) = 5.625$. In this case, $i(G) = 3$ illustrates the theorem.

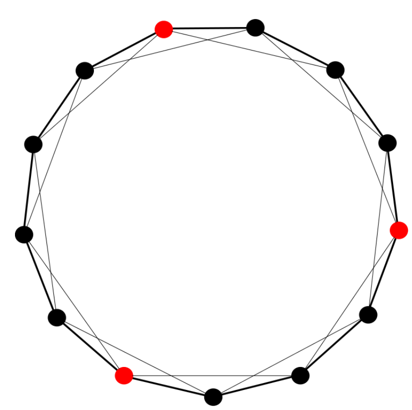


Figure 1.8: Graph G with $n = 13$, $\delta = 4$ and $i(G) = 3$

For the third case, let $n = k = 10$. Then, $\delta = \frac{k-3}{k}n = \frac{10-3}{10} \cdot 10 = 7$. This resulting graph's independent domination number is bounded above by $i(G) \leq \frac{2}{k}n = \frac{2}{10} \cdot 10 = 2$. The graph in Figure 1.9 illustrates this as $i(G) = 2$.

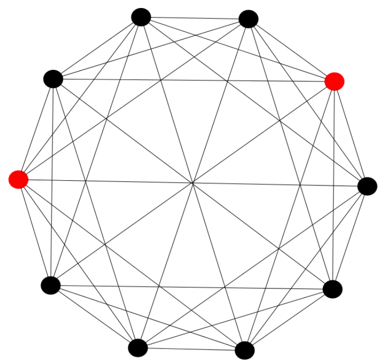


Figure 1.9: Graph G with $n = k = 10$, $\delta = 7$ and $i(G) = 2$

Second, Lyle proved the following: Let G be a δ-regular graph such that every edge is in a t-clique and G^c is connected and $cl_t(G)$ has a dominating vertex. Then, $t \leq \frac{3}{5}(\delta + 1)$.

Third, Lyle proved that if G is a connected, δ-regular graph with $t > \frac{3}{5}(\delta+1)$, then there is some $k \geq 4$, such that $cl_t(G)$ contains an induced copy of C_k. The following graph G with $\delta = 8$ and its clique graph $cl_6(G)$ (Figure 1.10) serve as an example. In this example, $t = 6 > \frac{3}{5}(\delta+1) = \frac{3}{5}(8+1) = 5.4$ and $cl_6(G)$ does contain an induced copy of C_7. So, the conditions hold.

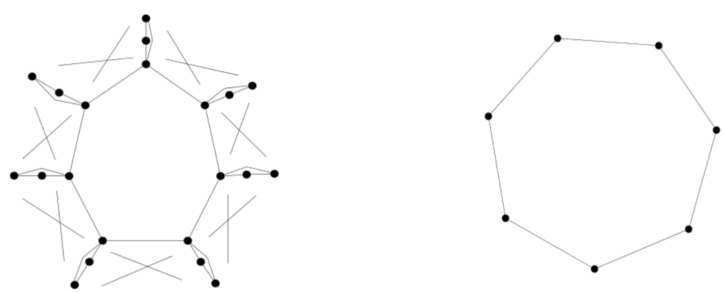


Figure 1.10: Graph G with $\delta = 8$ and its corresponding clique graph $cl_6(G) = C_7$

1.3.2 Clique partition problems

The next topic considered is clique partitions, specifically with disjoint copies of K_3 (triangles).

Corradi and Hajnal [2] proved that if G is a graph of order $n = 3k$ with $\delta(G) \geq 2k$, then G contains k disjoint triangles. For instance, graph G (Figure 1.11) with $n = 3k = 3 \cdot 2 = 6$ and $\delta(G) \geq 2k = 2 \cdot 2 = 4$ has $k = 2$ disjoint triangles. Furthermore, Dirac [3] proved that if G is a graph of order $n \geq 3k$ with $\delta(G) \geq \frac{n+k}{2}$, then G contains k disjoint triangles. For instance, graph G (Figure 1.12) with $n \geq 3k = 3 \cdot 3 = 9$ and $\delta(G) \geq \frac{n+k}{2} = \frac{9+3}{2} = 6$ has $k = 3$ disjoint triangles.

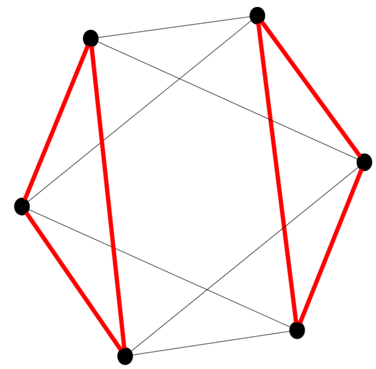


Figure 1.11: Graph G with $n = 6$ and $\delta = 4$

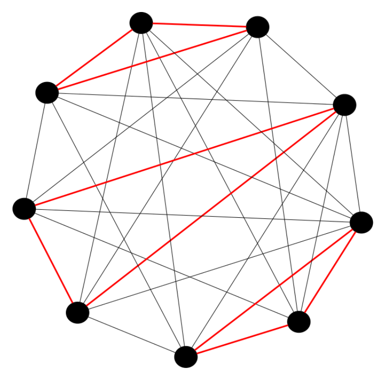


Figure 1.12: Graph G with $n = 9$ and $\delta = 6$

Results exist for other disjoint small cliques, namely K_3s and K_4s, in a graph. Specifically, let s and k be 2 integers with $0 \leq s \leq k$. Yan, Gao, and Zhang [12] proved that if $n \geq 3s + 4(k-s)$ and $d(u) + d(v) \geq 3\left(\frac{n-s}{2}\right) + k - 1$ for any pair of non-adjacent vertices u,v of G, then G contains s vertex-disjoint K_3s and $(k-s)$ vertex-disjoint K_4s, such that all of them are vertex-disjoint. For instance, graph G (Figure 1.13) with $n \geq 3s + 4(k-s) = 3 \cdot 2 + 4 \cdot (3-2) = 10$ and $d(u) + d(v) \geq 3\left(\frac{n-s}{2}\right) + k - 1 = 3\left(\frac{10-2}{2}\right) + 3 - 1 = 14$ has $s = 2$ vertex-disjoint K_3s and $(k-s) = (3-2) = 1$ vertex-disjoint K_4, such that all of them are vertex-disjoint.

Furthermore, the following example shows that the degree sum condition $d(u) + d(v) \geq 3\left(\frac{n-s}{2}\right) + k - 1$ is sharp, that is the right side of the inequality cannot be further decreased. For example, let G (Figure 1.14) be a 4-cycle. While, for $k = s = 1$, $n \geq 3s + 4(k-s)$ ($\Leftrightarrow 4 \geq 3$) holds, the second inequality $d(u) + d(v) \geq 3\left(\frac{n-s}{2}\right) + k - 1$ ($\Leftrightarrow 4 \geq 4.5$) does not

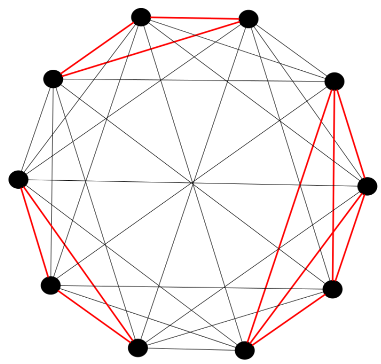


Figure 1.13: Graph G with $n = 10$ and $\delta = 7$

hold. If both inequalities had held, graph G would have had to have $s = 1$ K_3 which is not true. Therefore, the degree condition is indeed sharp.

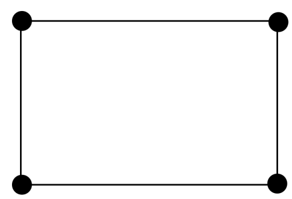


Figure 1.14: Graph G with $n = 4$ and $\delta = 2$

Lastly, for the discussion of the next paper, 2 additional definitions are needed. Since these definitions are very specific to this particular paper, they are just now introduced. Namely, let $B^c(G)$ be the maximum number of disjoint maximal cliques in G, and let $b^c(G)$ be the minimum number of disjoint maximal cliques in G, that is no additional cliques can be added. Figure 1.15 illustrates an example of the definition. The main point of the paper is to find lower bounds for the maximum and minimum numbers of disjoint maximal cliques in G. These bounds help to discover lower bounds for the maximum and minimum numbers of pairwise disjoint maximal independent sets in the graph's complement. Erdos, Hobbs, and Payan [4] specifically proved that if graph G has maximum degree k, then $b^c(G) \geq \frac{4n}{(k+2)^2}$ and $B^c(G) \geq \frac{6n}{(k+3)^2}$. Furthermore, if the graph G is regular of degree δ, then $b^c(G) \geq \frac{8n}{(\delta+3)^2}$.

In the example (Figure 1.15), we have the regular graph G on $n = 8$ vertices with $\delta = 3$. Therefore, we get $b^c(G) \geq \frac{8n}{(\delta+3)^2} = 1.78$. Therefore, we should see at least 2 pairwise disjoint maximal cliques in the graph. In this case, $b^c(G) = 3$ illustrating the formula.

Furthermore, the authors proved the following: Let G be a graph on n vertices and minimum degree $\delta(G) = n - k$. If $k < -1 + 2\sqrt{n}$, then G includes 2 disjoint maximal

Figure 1.15: $B^c(G) = 4$ and $b^c(G) = 3$

independent sets of vertices. Further, if G is regular of degree $\delta(G) = n - k$ and if $k < -2 + 2\sqrt{2n}$, then G includes 2 disjoint maximal independent sets of vertices.

For instance, let $G = C_5$. Then, $n = 5$ and $\delta(G) = 2$. If $\delta(G) = n - k$, then $k = 3$. Since $k < -1 + 2\sqrt{n} = -1 + 2\sqrt{5} = 3.47$ holds, G should have 2 disjoint maximal independent sets. Further, since G is regular with degree $\delta(G) = 2$, we can use the same example to test the other conditions. Again, since $k < -2 + 2\sqrt{2n} = -2 + 2\sqrt{10} = 4.32$ holds, G should have 2 disjoint maximal independent sets, according to the second part of the theorem. The following illustration (Figure 1.16) confirms the analysis;

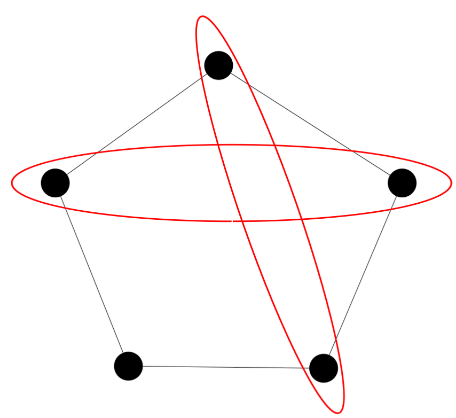


Figure 1.16: Graph $G = C_5$ with 2 maximal independent sets

These results concern independent sets in $cl(G)$. We will consider independent sets in $cl_t(G)$ in Chapter 3.

1.3.3 Independent set in G

Rabern [8] proved that if G is a graph with $\omega(G) \geq \frac{3}{4}(\Delta(G) + 1)$, then G has an independent set I, such that $\omega(G - I) < \omega(G)$. The basic idea is that for graphs that meet the condition, there exists an independent set that contains 1 vertex from each maximum clique. Then, in $G-I$, every maximum clique is reduced by 1, and therefore $\omega(G-I) < \omega(G)$ holds. In order to illustrate this theorem, let $\omega(G) = \Delta(G) = 3$ (Figure 1.17 and Figure 1.18). In both illustrations, we have $\omega(G) = 3$ and $\omega(G-I) = 2$. Therefore, $\omega(G-I) < \omega(G)$ holds. In sum, using the title of the paper, all maximum cliques will be hit by an independent set.

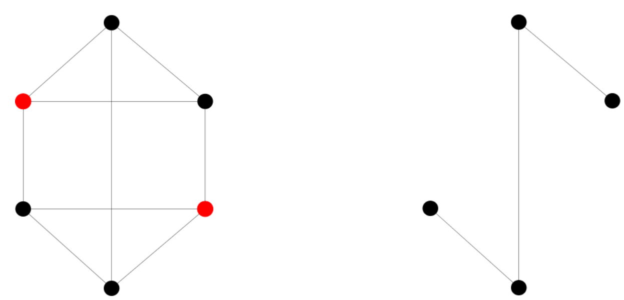


Figure 1.17: Graph G with $\omega(G) = \Delta = 3$ and graph $G-I$

Figure 1.18: Graph G with $\omega(G) = \Delta = 3$ and graph $G-I$

The results of Rabern can be used to discuss Reed's conjecture. Reed [9] proved that for every graph G we have $\chi(G) \leq \left\lceil \frac{\omega(G)+\Delta(G)+1}{2} \right\rceil$. If we let the independent set in $(G-I)$ be maximal, then Reed's conjecture holds true for graphs with $\omega(G) \geq \frac{3}{4}(\Delta(G)+1)$. Therefore, a minimum counterexample to the conjecture satisfies $\omega(G) < \frac{3}{4}(\Delta(G)+1)$ and also $\chi(G) > \left\lceil \frac{7}{6}\omega(G) \right\rceil$. The second solution is obtained by substituting Δ from $\chi(G) \leq \left\lceil \frac{\omega(G)+\Delta(G)+1}{2} \right\rceil$ into $\omega(G) < \frac{3}{4}(\Delta(G)+1)$ and solving for $\chi(G)$.

1.3.4 Line graphs

The graphs, $cl_2(G)$, for any G are a special kind of graphs, which are called *line graphs*. Therefore, the line graph of a graph G is another graph $L(G)$ that represents the adjacencies between the edges of G. Beineke [1] showed that a graph is a line graph if and only if no subset of its vertices induces 1 of the 9 subgraphs in Figure 1.19.

For example, the top left forbidden subgraph in Figure 1.19 is a claw, which is a complete bipartite graph $K_{1,3}$. A line graph $L(G)$ cannot contain a claw, because if 3 edges e_1, e_2,

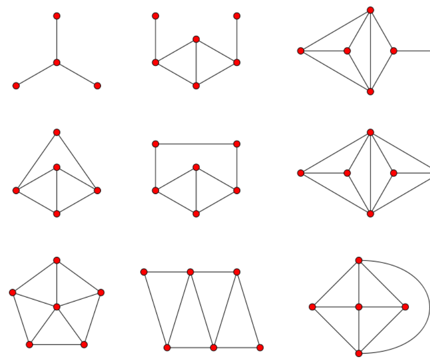


Figure 1.19: 9 forbidden subgraphs for line graphs

and e_3 in G all share endpoints with another edge e_4 (Figure 1.20) then by the pigeonhole principle (which states that if n items are put into m containers, with $n > m$, then at least 1 container must contain more than 1 item [6]) at least 2 of e_1, e_2, and e_3 must share 1 of those endpoints with each other.

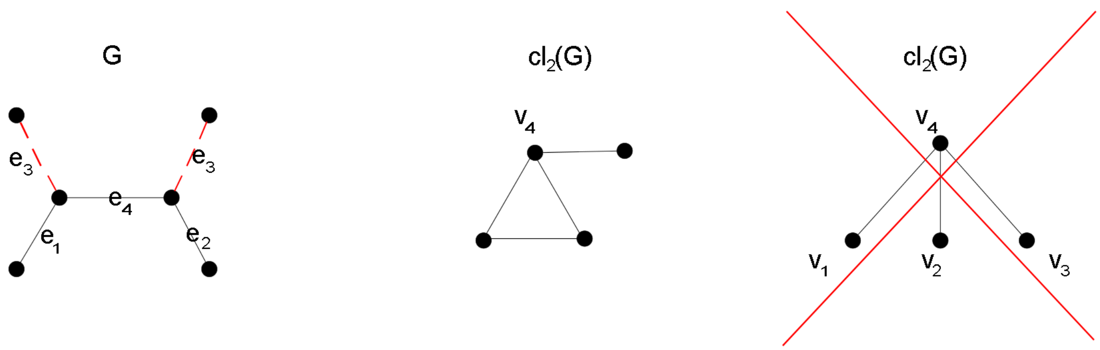


Figure 1.20: Why the line graph $cl_2(G)$ cannot contain the claw $K_{1,3}$

Furthermore, for the line graph $L(G)$ of the graph G to exist, a collection of cliques in the graph $L(G)$ (allowing some of the cliques to be single vertices) exists that partitions the edges of the graph $L(G)$, such that each vertex of $L(G)$ belongs to exactly 2 of the cliques. Additionally, for G not to be a multigraph, no 2 vertices of $L(G)$ are both in the same 2 cliques [5]. For instance, the vertex in the center of a claw is in 3 2-cliques, violating the requirement that each vertex appears in exactly 2 cliques. Thus, any graph that has a claw as an induced subgraph is not a line graph.

Chapter 2

REGULARITY OF THE CLIQUE GRAPH

In this section, the variations of the following question will be discussed: Is every clique graph of a regular graph regular?

2.1 Regularity of line graphs, $cl_2(G)$

To begin, the line graphs, $cl_2(G)$, of regular graphs G will be analyzed.

Theorem 2.1.1. *The line graph $L(G)$ of a regular graph G is regular.*

Proof. Let G be a δ-regular graph. Let $L(G) = cl_2(G)$ be the line graph of G. Let $u \in E(G)$. Let $v_1(u)$ and $v_2(u)$ be the endpoints of the edge u. Let $s \in V(cl_2(G))$ be the vertex of the line graph corresponding to the edge u in G (Figure 2.1). The edge u has the vertex $v_1(u)$ in common with $\delta - 1$ edges and the vertex $v_2(u)$ in common with $\delta - 1$ edges in G. Therefore, the vertex s of the line graph has degree $\delta - 1 + \delta - 1 = 2\delta - 2$. Therefore, the line graph $cl_2(G)$ is regular.

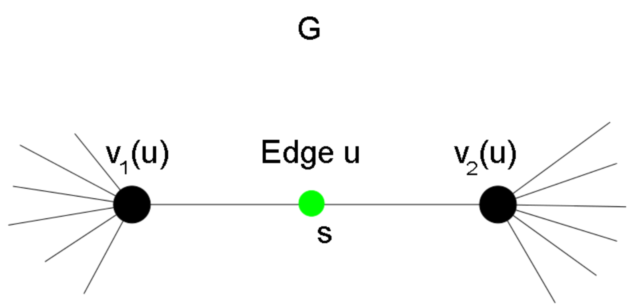


Figure 2.1: The line graph of a regular graph is regular

2.2 Degree difference of $cl_3(G)$ - Preliminary discussion

After having shown that every line graph $L(G)$ of a regular graph G is regular, the clique graphs $cl_3(G)$ of regular graphs G will be analyzed. First of all, the graphs G and their corresponding clique graphs $cl_3(G)$ in Figure 2.2 and Figure 2.3 are examples of non-regular clique graphs.

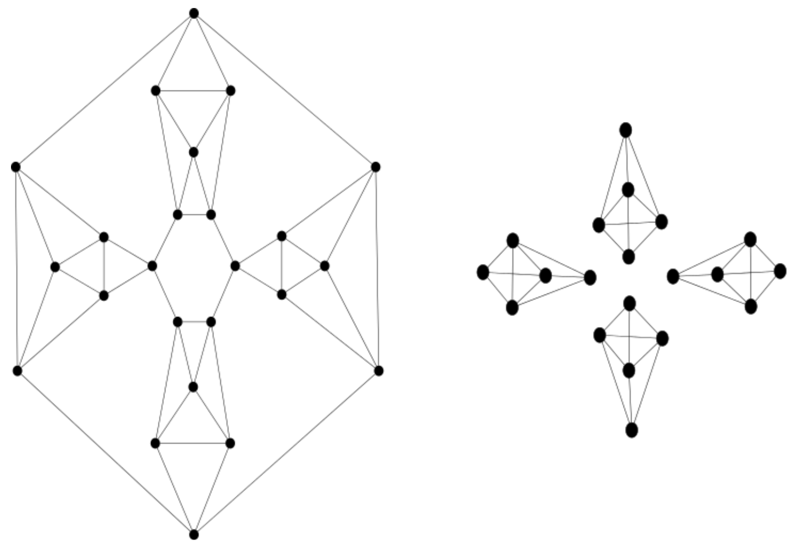


Figure 2.2: 4-regular graph G with its non-regular clique graph $cl_3(G)$

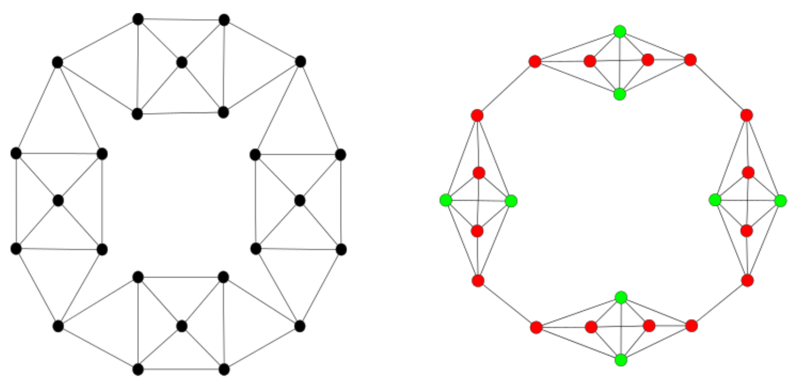


Figure 2.3: Graph G satisfying all 4 conditions with its non-regular clique graph $cl_3(G)$

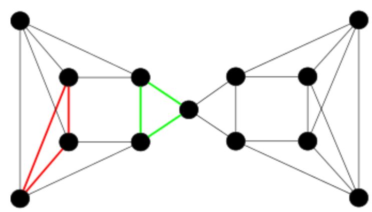
While not all edges of the graph G in Figure 2.2 are in a 3-clique, the graph G in Figure 2.3 satisfies all of the following 4 conditions, and its clique graph $cl_3(G)$ is not regular.

- G is non-complete.
- G is regular.
- Every edge in G is contained in a t-clique.
- $t = 3$ is close to δ.

Therefore, the clique graphs $cl_3(G)$ of graphs G which satisfy (and of graphs G which do not satisfy) the 4 stated conditions do not necessarily have to be regular.

In Figure 2.3, the degree difference, which is defined as the difference between the maximum degree, denoted by $\Delta(G)$, and the minimum degree, denoted by $\delta(G)$, of the clique graph $cl_3(G)$ is 1, since the green vertices have degree $\Delta(cl_3(G)) = 5$ and the red vertices have degree $\delta(cl_3(G)) = 4$.

Therefore, the degree difference of the clique graph can be 0 (if the clique graph is regular) and 1 (See Figure 2.3). The following question arises: Could the degree difference of the clique graph $cl_3(G)$ be arbitrarily large?



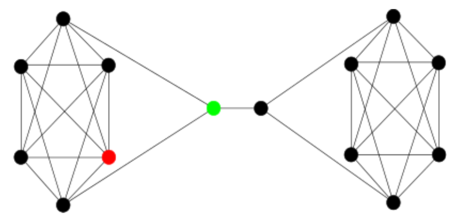


Figure 2.4: Graph G and $cl_3(G)$

To begin, 2 examples are presented for which the degree difference is 2. The graph in Figure 2.4 satisfies all 4 conditions (regular with degree δ, non-complete, t close to δ and every edge is in a t-clique) with $\delta = 4$. The red (green) triangle in the original graph becomes the red (green) vertex in the clique graph. Since the degree of the red vertex is 5, $\Delta = 5$, and the degree of the green vertex is 3, $\delta = 3$, the degree difference is $\Delta(cl_3(G)) - \delta(cl_3(G)) = 2$.

Similarly, the degree difference of the clique graph of the graph in Figure 2.5 is also 2, with $\Delta(cl_3(G)) = 5$ and $\delta(cl_3(G)) = 3$.

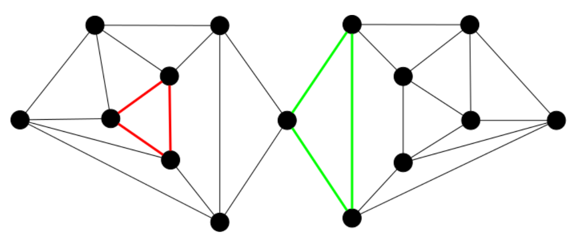


Figure 2.5: Graph G

Therefore, we now know that the degree difference of the clique graph can be 0, 1, or 2. In order to answer the original question ("Could the degree difference of the clique graph be arbitrarily large?"), we need to build graphs where 1 triangle (3-clique) shares a vertex with many triangles. Table 2.1 summarizes the maximal values that $\Delta(cl_t(G))$ can assume, for given $\delta(G)$, when G is a complete graph.

$\delta(G)$	$n(G)$	Original graph G	$n(cl_3(G))$	$\Delta(cl_3(G))$	$\frac{\Delta}{n}$
1	2	K_2	0		
2	3	K_3	1		
3	4	K_4	4	3	0.75
4	5	K_5	10	9	0.90
5	6	K_6	20	18	0.90
6	7	K_7	35	30	0.86
7	8	K_8	56	45	0.80
8	9	K_9	84	63	0.75
9	10	K_{10}	120	84	0.70
10	11	K_{11}	165	108	0.65

Table 2.1: How to maximize $\Delta(cl_3(G))$

In order to find the values for the 4th column in Table 2.1, the number of triangles in G, $\binom{n(G)}{3}$ is calculated. Similarly, in order to find the values for the fifth column, $\Delta(cl_3(G))$, $(\binom{n(G)}{3}-1)-\binom{n(G)-3}{3}$ is calculated. This table indicates, for instance, for $\delta=5$, for any graph G, $\Delta(cl_3(G))$ is likely less than 18. It may be beneficial to consider graphs G where $\Delta(cl_3(G))$ is as close to these values as possible. Recall that the goal is to maximize the degree difference.

For instance, if 1 of the 2 graphs in Figure 2.6 is a subgraph of a larger graph G with $\delta=5$, we have $\Delta(cl_3(G))\geq 10$ because the red triangle has at least 1 vertex in common with 10 other triangles; therefore, the corresponding vertex in the clique graph will have degree 10. This value for the maximum degree of the clique graph is relatively close to the upper bound for $\delta=5$ which is 18.

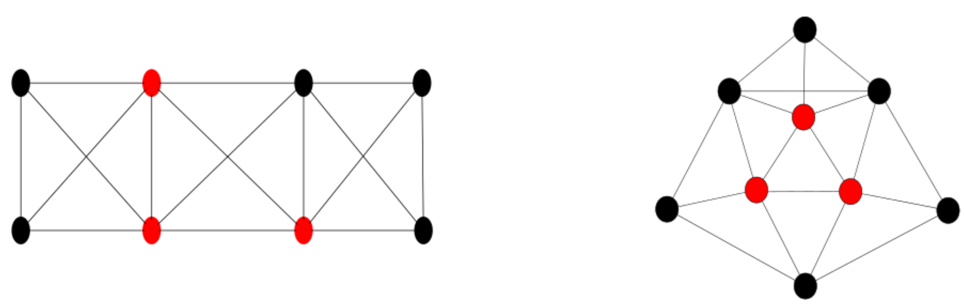


Figure 2.6: Subgraph G' and H'

Furthermore, if the graph G' in Figure 2.7 is a subgraph of a larger graph G with $\delta=5$, we have $\Delta(cl_3(G))\geq 12$, since the red triangle shares a vertex with 12 other triangles (9 from K_5 and 3 from K_4) in the subgraph G'; therefore, the corresponding red vertex in the clique graph will have degree ≥ 12.

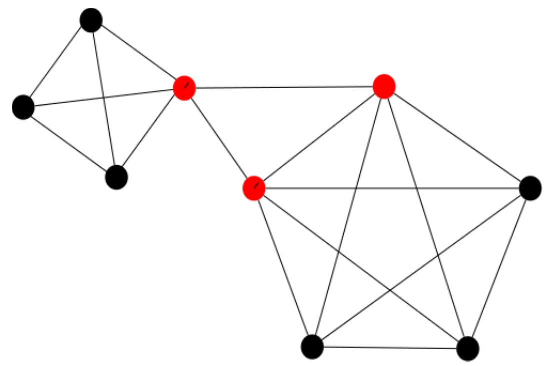


Figure 2.7: Subgraph G'

Although more and more subgraphs G' where $\Delta(cl_3(G'))$ is close to 18 can be found, it is sometimes extremely difficult to build a graph G that contains G'. This fact is primarily due to the fact that G has to satisfy the 4 conditions. The conditions that G has to be regular and that every edge in G has to be in a 3-clique are particularly difficult to satisfy for most such graphs. Additionally, even if a graph G (that contains G' and satisfies all 4 conditions) is found, the minimum degree of the clique graph, $\delta(cl_3(G))$, should be made as small as possible.

The graph in Figure 2.8 does satisfy all 4 conditions, and $\Delta(cl_3(G))$ is as large as possible (1) while keeping $\delta(cl_3(G))$ as low as possible (2). Namely, for (1), the red triangle shares a vertex with as many triangles as possible (13) and thereby bounding the maximum degree below, $\Delta(cl_3(G)) \geq 13$, and, for (2), the green triangle shares a vertex with as few triangles as possible (6) and thereby bounding the minimum degree above, $\delta(cl_3(G)) \leq 6$. Therefore, the graph illustrates an example where the degree difference is at least 7, $\Delta - \delta \geq 7$, for $\delta = 5$.

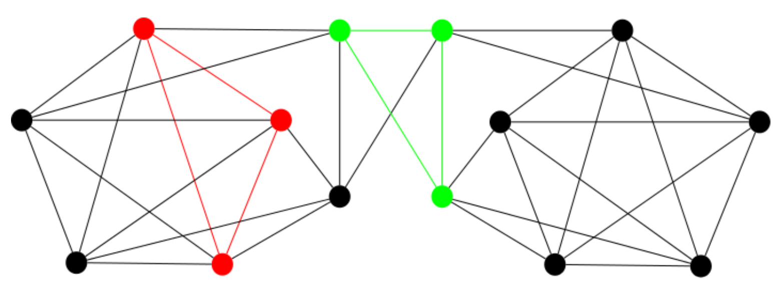


Figure 2.8: Graph G whose clique graph's degree difference is 7 for $\delta = 5$

In sum, when building graphs with the purpose of maximizing the degree difference, we want graphs where 1 triangle shares a vertex with as many other triangles as possible (in order to maximize $\Delta(cl_3(G))$), and 1 triangle that shares a vertex with as few triangles as possible (in order to minimize $\delta(cl_3(G))$).

At the beginning of the discussion, the following question was asked: Could the degree difference of the clique graph $cl_3(G)$ be arbitrarily large? An example for the degree difference to be 7 for $\delta = 5$ was found (Figure 2.8). This example gives evidence to support the conjecture that the degree difference of $cl_3(G)$ can be arbitrarily large, and this is shown in the next sections for $\delta = 3$ and $\delta > 3$.

2.3 Degree difference of $cl_3(G)$ using 3-regular graphs

While the strategy of maximizing the degree difference by building graphs where $\Delta(cl_3(G))$ is as large as possible and $\delta(cl_3(G))$ is as small as possible certainly makes sense, these graphs that are built in the previous section do not have a common structure. Therefore, it is very difficult to prove that the clique graph's degree difference can be arbitrarily large by using these examples. In contrast, the following discussion illustrates 3 steps that can be followed in order to easily build a graph whose clique graph can serve as an example to demonstrate that the degree difference can be made arbitrarily large. At the end of the discussion, formulas for calculating $\Delta(cl_3(G))$, $\delta(cl_3(G))$ and the degree difference will be derived and analyzed.

In the following, the 3 steps to build graphs whose clique graph's degree difference is arbitrarily large are presented:

1. Let $\delta = 3$. Let $k \geq 6$, k be even and $\frac{3}{2}k$ be an integer.

2. Build 2, disconnected 3-regular graphs G_k and H_k on $k = n_G = n_H$ vertices.

    - We will construct G_k to be triangle-free. In order to build the triangle-free graph G_k, begin with the cycle C_k and number the vertices of $V(G_k)$. Then, pair each odd numbered vertex with an even numbered vertex. Since no 2 even numbered vertices are adjacent (and no 2 odd numbered vertices are adjacent), G_k is triangle-free.

    - We will construct H_k to have exactly 2 disjoint triangles. In order to build graph H_k, begin with the cycle C_k and number the vertices of $V(H_k)$. Then, connect vertices 1 and 3 and vertices $\frac{n}{2}+1$ and $\frac{n}{2}+3$. Add an edge between v_2 and $v_{2+\frac{n}{2}}$. If there are vertices v_k (for $k \leq \frac{n}{2}$) with degree $\delta - 1$ remaining, then add edges between v_4, ..., v_k and $v_{k+\frac{n}{2}}$. The resulting graph H contains 2 triangles which do not share a common vertex.

3. Add all edges of the form (u,v) where $u \in V(G_k)$ and $v \in V(H_k)$. The resulting graph R_k is going to have $n_R = 2k$ vertices and degree $\delta_R = k+3$.

The graph R_k in Figure 2.9 is built by using these 3 steps. Every graph R on n_R vertices that is built by using these 3 steps has the following 3 key structural features.

1. The subgraph G_k of R_k has no triangles.

2. The subgraph H_k of R_k has exactly 2 triangles.

3. Graph R_k satisfies 3 of the 4 conditions - δ-regular, non-complete, and every edge is in a 3-clique.

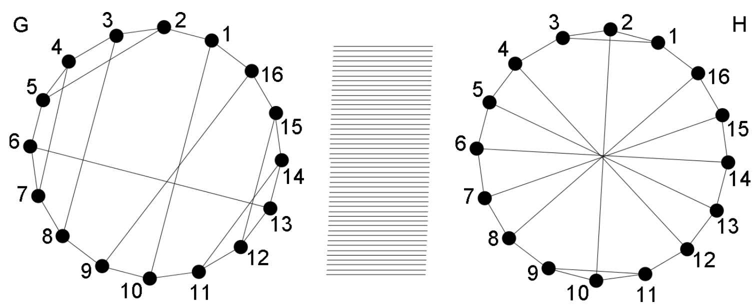


Figure 2.9: Graph R ($\delta_R = 19$, $n_R = 32$) consisting of G and H

An example of R on 12 vertices (Figure 2.10) will be analyzed in order to construct the general formulas for calculating $\Delta(cl_3(R))$, $\delta(cl_3(R))$, and the degree difference. These formulas will be used to prove that the degree difference of the clique graph can indeed be arbitrarily large.

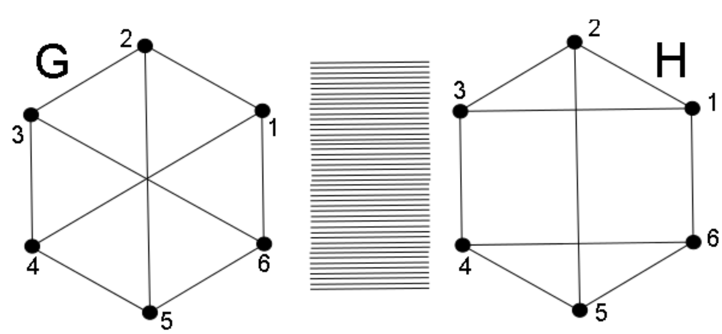


Figure 2.10: Graph R ($\delta_R = 9$, $n_R = 12$) consisting of G and H

Since we are interested in the degree difference of $cl_3(R)$, we will choose different triangles in R (that will turn into vertices in $cl_3(R)$) and count the triangles with whom they share at least 1 vertex, which we will call neighboring triangles. The number of neighboring triangles defines their corresponding vertex's degree in $cl_3(R)$. Then, in order to provide a bound on the degree difference in $cl_3(R)$, we will choose 2 particular triangles: the one with many neighboring triangles (for a lower bound on $\Delta(cl_3(R))$) and the one with fewer neighboring triangles (for an upper bound on $\delta(cl_3(R))$).

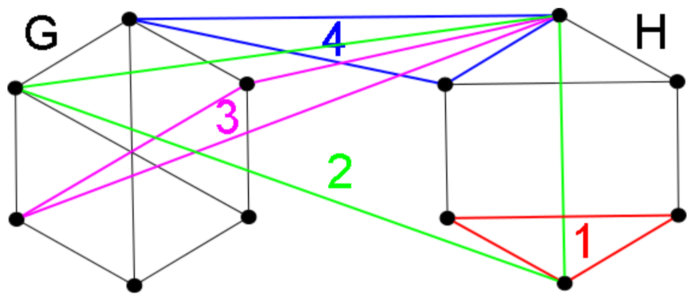


Figure 2.11: 4 different types of triangles in R

For the graph R in Figure 2.11, we can consider 4 different types of triangles whose neighboring triangles we are going to count in order to find their corresponding vertex's degree in $cl_3(R)$. Let the base of a triangle be defined as the edge, such that both vertices are in G or in H except for the 2 triangles whose vertices are all in H. Let the apex of a triangle be defined as the vertex corresponding to the base of a triangle.

- Type 1: The vertices of the triangle are in H.
- Type 2: The base of the triangle is in H and its apex in G. A type 2 triangle shares a vertex with 2 triangles in H.
- Type 3: The base of the triangle is in G and its apex in H.
- Type 4: The base of the triangle is in H and its apex in G. A type 4 triangle shares a vertex with 1 triangle in H.

In the following, the neighboring triangles of a type 1 triangle (Figure 2.12) are counted.

First of all, if we let the base of the neighboring triangle be in H, then we have 3 possibilities as to how the base of the neighboring triangle is related to a type 1 triangle.

1. The base of the neighboring triangle is an edge of the chosen triangle (Figure 2.13, 1.). In this case, we have 3 edges of the chosen triangle that can serve as a base of the neighboring triangle. At the same time, we have 6 vertices in G that can serve as an apex of the neighboring triangle. Therefore, we have $6 \cdot 3 = 18$ neighboring triangles. In R_k (for $\delta = 3$), there would be $3k$ neighboring triangles.

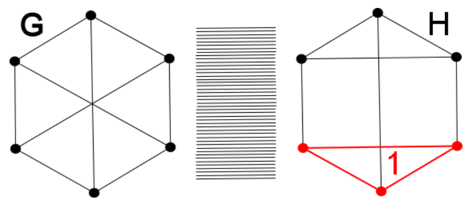


Figure 2.12: Type 1: The vertices of the triangle are in H

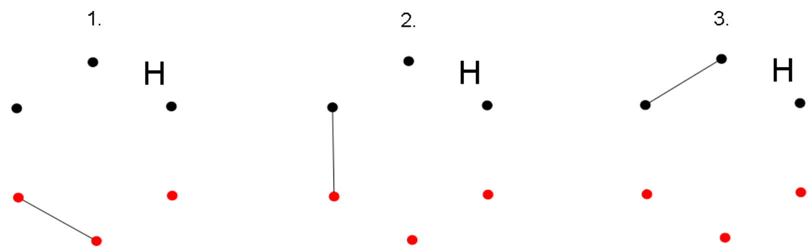


Figure 2.13: Type 1: Base of the neighboring triangle in H

2. The base of the neighboring triangle shares 1 vertex with the chosen triangle (Figure 2.13, 2.). In this case, since $\delta = 3$ and 2 edges of the chosen triangle are used to build the triangle, we have 1 edge per vertex of the chosen triangle that can serve as the base of a neighboring triangle. As seen before, we have 6 vertices in G that can serve as an apex of the neighboring triangle. Therefore, we have $1 \cdot 3 \cdot 6 = 18$ neighboring triangles. In R_k (for $\delta = 3$), we have $(\delta - 2) \cdot 3 \cdot k = 3k$ neighboring triangles.

3. The base of the triangle does not share an edge or a vertex with the chosen triangle (Figure 2.13, 3.). Then, no neighboring triangles can be built.

Secondly, if we let the base of the neighboring triangle be in G, then we have 1 possibility how the base of the neighboring triangle is related to a type 1 triangle.

1. The base of the neighboring triangle does not share an edge or a vertex with the chosen triangle (Figure 2.14, 1.) (because the neighboring triangle's base is in G and the chosen triangle is in H). Therefore, for every base in G (whose number is equivalent to the number of edges in G), the apex of the neighboring triangle has to be 1 of the 3 vertices of the chosen triangle. Since there are $\frac{3 \cdot 6}{2} = 9$ edges in G, we have $9 \cdot 3 = 27$ neighboring triangles. In R_k (for $\delta = 3$), we have $\frac{\delta k}{2} \cdot 3 = \frac{9}{2}k$ neighboring triangles.

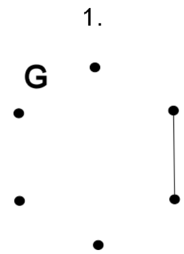


Figure 2.14: Type 1: Base of the neighboring triangle in G

Altogether, each of the 2 triangles in H has $18+18+27=63$ triangles with whom it shares at least 1 common vertex (neighboring triangles). This implies that the vertex (in $cl_3(R)$) corresponding to a type 1 triangle has degree $=63$, and that $\delta(cl_3(R_k)) \leq 63$.

Subsequently, the neighboring triangles of a type 2 triangle (Figure 2.15) are counted.

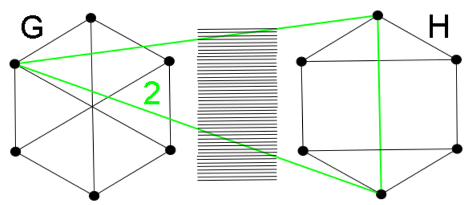


Figure 2.15: Type 2: The base of the triangle is in H and its apex in G

First of all, if we let the base of the neighboring triangle be in G, then we have 2 possibilities as to how the base of the neighboring triangle is related to the type 2 triangle.

1. The base of the neighboring triangle shares 1 vertex with the type 2 triangle (Figure 2.16, 1.). Therefore, the apex of the neighboring triangle can be any vertex in H. In this case, we have $3 \cdot 6 = 18$ neighboring triangles. In R_k (for $\delta = 3$), we have $\delta k = 3k$ neighboring triangles.

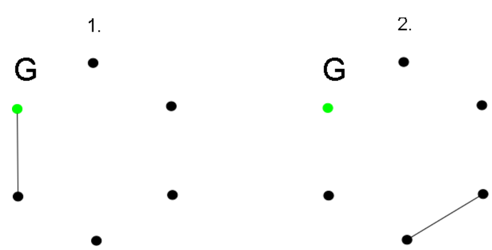


Figure 2.16: Type 2: Base of neighboring triangle in G

2. The base of the neighboring triangle does not share an edge or a vertex with the type 2 triangle (Figure 2.16, 2.). Since we have a total of 9 edges in G and 3 edges are already used in 1., we have 6 edges that can serve as the base of a neighboring triangle. For all these 6 bases, the apex has to be 1 of the 2 vertices of the type 2 triangle in H in order for the resulting triangle to be a neighboring triangle. In this case, we have $6 \cdot 2 = 12$ neighboring triangles. In R_k (for $\delta = 3$), we have $(\frac{\delta k}{2} - \delta) \cdot 2 = 3k - 6$ neighboring triangles.

Secondly, if we let the base of the neighboring triangle be H, then we have 3 possibilities as to how the base of the neighboring triangle is related to the type 2 triangle.

1. The base of the neighboring triangle is the edge between the 2 type 2 triangle's vertices that are in H (Figure 2.17, 1.). The vertices of G can serve as the apex of the neighboring triangle. Since taking 1 of these vertices as the apex would result in the type 2 triangle, and therefore must be neglected, we can build exactly 5 neighboring triangles. In R_k (for $\delta = 3$), we have $k - 1$ neighboring triangles.

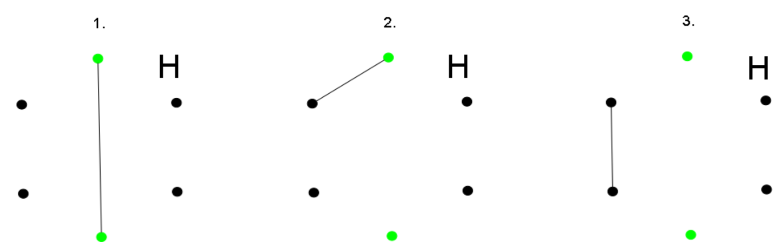


Figure 2.17: Type 2: Base of the neighboring triangle in H

2. The base of the neighboring triangle shares a vertex with either 1 of the type 2 triangle's vertices (Figure 2.17, 2.). Since 1 edge is already taken for connecting the 2 vertices, we have 2 possible bases for each of the 2 vertices. Each of these 4 bases is paired with any of the 6 vertices from G that serve as apex of the neighboring triangles. Therefore, we have $2 \cdot 2 \cdot 6 = 24$ neighboring triangles. In R_k (for $\delta = 3$), $(\delta - 1) \cdot k \cdot 2 = 4k$ neighboring triangles.

3. The base of the neighboring triangle does not share an edge or a vertex with the type 2 triangle (Figure 2.17, 3.). Since we have 9 edges in H and $1 + 4 = 5$ edges are already used in 1. and 2., we have 4 edges left that can serve as bases of a neighboring triangle. In order to be a neighboring triangle, the corresponding top of the triangle must be the type 2 triangle's vertex in G. Therefore, we have $4 \cdot 1 = 4$ neighboring triangles. In R_k (for $\delta = 3$), we have $(\frac{\delta k}{2} - (2\delta - 1)) = \frac{3}{2}k - 5$ neighboring triangles.

Altogether, for the type 2 triangle, we have $18+12+5+24+4=63$ neighboring triangles which contain an edge from G_k to H_k. Since the type 2 triangle also shares a vertex with the 2 triangles in H (Figure 2.15), the total number of neighboring triangles of the type 2 triangle is $63+2=65$. A similar analysis gives the values for the type 3 and 4 triangles (which is 64). This implies that the vertex in $cl_3(R_k)$ corresponding to the type 2 triangle has degree $=65$, and that $\Delta(cl_3(R_k)) \geq 65$. Therefore, the degree difference of $cl_3(R_k)$ is $65-63=2$ since $\delta(cl_3(R_k)) \leq 63$ (from type 1 triangle) and $\Delta(cl_3(R_k)) \geq 65$ (from type 2 triangle).

While this finding is not very impressive (since a larger degree difference is found in Figure 2.8) the process of building the graph R and of counting the neighboring triangles for different initial triangles serves as a great starting point for having a structured approach for coming up with examples of graphs for which the clique graph's degree difference is arbitrarily large. On the following pages, we will come up with bounds on $\delta(cl_3(R_k))$, $\Delta(cl_3(R_k))$ and the degree difference, $\Delta-\delta$, that will allow us to easily and quickly build graphs for any desired clique graph's degree difference.

As long as $k=n_G=n_H \geq 6$ and k is even, independent on the k we choose, the process of deriving the number of neighboring triangles for the type 1, 2, 3 and 4 triangles in R_k is exactly the same. Based on this observation, formulas to count the number of neighboring triangles which contain an edge from G_k to H_k will be derived.

First, consider a type 2 triangle (Figure 2.15). Let $\delta=\delta_G=\delta_H=3$ and $k=n_G=n_H=\frac{n_R}{2}$. Let X be the number of neighboring triangles (which contain an edge from G_k to H_k) of a type 2 triangle. Then, we have

$$
\begin{aligned}
X &= \delta k + \left(\binom{\delta k}{2} - \delta\right)\cdot 2 + (k-1) + (\delta-1)\cdot k \cdot 2 + \left(\binom{\delta k}{2} - (2\delta-1)\right), \\
&= \left(\frac{9}{2}\right)\delta k - 4\delta - k, \\
&= \left(\frac{25}{2}\right)k - 12. \qquad (2.1)
\end{aligned}
$$

In the previous discussion (Figure 2.15), we manually counted 63 neighboring triangles for the type 2 triangle. In this case, we have $k=6$, so the equation (2.1) $\frac{25}{2}k-12=\frac{25}{2}\cdot 6-12=63$ confirms the number of neighboring triangles which contain an edge from G_k to H_k.

Secondly, consider a type 1 triangle (Figure 2.12). Let $\delta=\delta_G=\delta_H=3$ and $k=n_G=n_H=\frac{n_R}{2}$. Let Y be the number of neighboring triangles (which contain an edge from G_k to H_k) of a type 1 triangle. Then, for $\delta=3$, we have

$$\begin{aligned} Y &= 3k + (\delta - 2)\cdot 3k + \left(\frac{\delta k}{2}\right)\cdot 3, \\ &= \left(\frac{9}{2}\right)\cdot k\delta - 3k, \\ &= \left(\frac{9}{2}\right)\cdot k \cdot 3 - 3k, \\ &= \left(\frac{21}{2}\right) k. \end{aligned} \tag{2.2}$$

In the previous discussion (Figure 2.12), we manually counted 63 neighboring triangles for the type 1 triangle. In this case, we have $k = 6$, so the equation (2.2) $\frac{21}{2}k = \frac{21}{2}\cdot 6 = 63$ confirms the number of neighboring triangles which contain an edge from G_k to H_k.

After having derived the formulas for X and Y, further analysis will be done. First of all, the following question will be answered: What are the conditions on k for $X \geq Y$? Using equations 2.1 and 2.2 for X and Y respectively, we can solve to find

$$\begin{aligned} \frac{25}{2}k - 12 &\geq \frac{21}{2}k, \\ 25k - 24 &\geq 21k, \\ 4k &\geq 24, \\ k &\geq 6. \end{aligned}$$

Therefore, if $k = 6$, then $X = Y$. For R_k in Figure 2.10 (where $k = 6$), we found that $X = Y = 63$ which confirms the analysis. If $k > 6$, X is always going to be greater than Y. This implies that, if $k > 6$, the number of neighboring triangles (which contain an edge from G_k to H_k) of the type 2, 3, and 4 triangles (Figure 2.15) will always be greater than the number of neighboring triangles (which contain an edge from G_k to H_k) of the type 1 triangle (Figure 2.12).

Finally, let $\Delta(cl_3(R_k)) = X + x$ and $\delta(cl_3(R_k)) = Y + y$ where x and y are the number of the chosen type triangle's neighboring triangles in H. For instance, recall that $X = Y = 63$ in Figure 2.11. Since a type 2 triangle has 2 neighboring triangles in H, $x = 2$ and $\Delta(cl_3(R_k)) = 63 + 2 = 65$. In contrast, since a type 1 triangle has 0 neighboring triangles in H, $y = 0$ and $\delta(cl_3(R_k)) = 63 + 0 = 63$.

Furthermore, independent on the choice of k, the maximum degree in the clique graph will always be $\Delta(cl_3(R)) \geq X + 2$ while the minimum degree will always be $\delta(cl_3(R)) \leq Y + 0$ for type 2 and type 1 triangles, respectively. This generalization is only possible because the process of building R_k is strictly defined by the 3 steps process which results in R_k that contains the triangle-free subgraph G_k and the subgraph H_k that contains 2 vertex-disjoint triangles (Figure 2.9, 2.10).

Therefore, the degree difference formula for $cl_3(R_k)$ (using equation 2.1 for X and equation 2.2 for Y) is

$$\begin{aligned}\Delta - \delta &= (X+2) - (Y+0),\\ &= (\frac{25}{2}k - 12 + 2) - (\frac{21}{2}k),\\ &= 2k - 10.\end{aligned} \tag{2.3}$$

We previously found that the degree difference is 2 for Figure 2.10 (where $k = 6$) which is confirmed by equation 2.3.

The degree difference formula 2.3 implies that as k increases, the degree difference in $cl_3(R_k)$ also increases. Ultimately, the previous discussion implies that the degree difference of $cl_3(R_k)$ can be arbitrarily large for R_k on $2k$ vertices, with $\delta_R = k+3$ and where every edge of R_k is in a 3-clique.

2.4 Degree difference of $cl_3(G)$ using δ-regular graphs, for $\delta > 3$

In the following, we will construct R_k in such a way that for any $\delta > 3$, the degree difference will be arbitrarily large.

1. Let $\delta > 3$. Let $k \geq 6\delta - 8$, k be even and $\frac{k\delta}{2}$ be an integer.

2. Build 2, disconnected $\delta = \delta_G = \delta_H$-regular graphs G_k and H_k on $k = n_G = n_H$ vertices.

    - We will construct G_k to be triangle-free. In order to build the triangle-free graph G_k, begin with the cycle C_k and number the vertices of $V(G_k)$. Then, in addition to the edges of the cycle, pair each odd numbered vertex v_k with the next $\delta - 2$ even numbered vertices. Since no 2 even numbered vertices are adjacent (and no 2 odd numbered vertices are adjacent), G is triangle-free. In sum, G_k is a δ-regular, bipartite graph (Figure 2.18).

    - We will construct H_k to have triangles. In order to build graph H_k, begin with the cycle C_k and number the vertices of $V(H_k)$. Add the missing edges to build 2 copies of K_δ in H_k by using the vertices v_1, ... , v_δ and $v_{\frac{k}{2}+1}$, ... , $v_{\frac{k}{2}+\delta}$. For all vertices v_s ($s < \delta$) (in the first copy of K_δ) that have degree $\delta - 1$, add the edge between v_s and $v_{s+\frac{k}{2}}$. Lastly, connect odd numbered vertices to even numbered

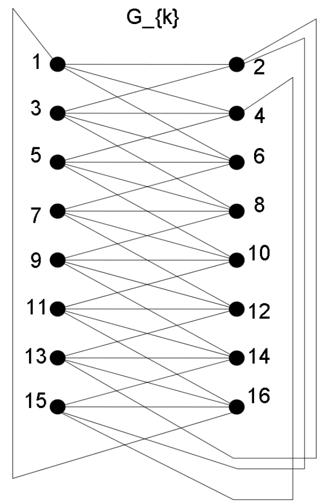


Figure 2.18: How to build the triangle-free graph G_k for $k = 16$ and $\delta = 4$

vertices until all vertices in H_k have degree δ. The resulting graph H_k contains 2 complete subgraphs on the top and bottom and no triangles in the center.

3. Add all edges of the form (u, v) where $u \in V(G_k)$ and $v \in V(H_k)$. The resulting graph R_k is going to have $n_R = 2k$ vertices and degree $\delta_R = k + \delta$.

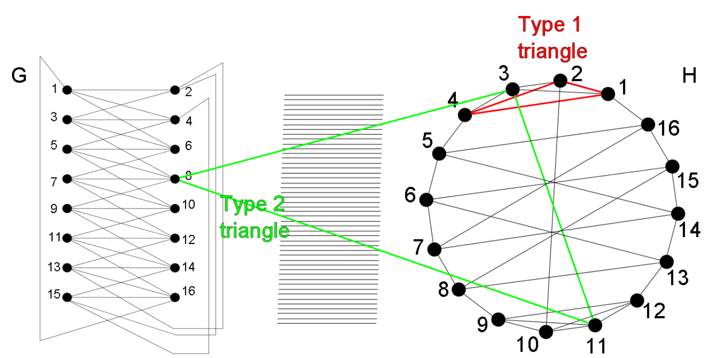


Figure 2.19: Graph R_k with $\delta = 4$ and $k = 16$

For example, the graph R_k with $\delta = 4$ and $k = 16$ in Figure 2.19 is built by following these steps.

A similar analysis as in section 2.3 gives the number of neighboring triangles (which contain an edge from G_k to H_k) of a type 1 triangle

$$\binom{9}{2} k\delta - 3k,$$

and of a type 2 triangle

$$\left(\frac{9}{2}\right)k\delta - 4\delta - k,$$

for $\delta > 3$. Therefore, for $\delta = 4$ and $k = 16$, the red triangle (type 1 triangle) in Figure 2.19 has $\frac{9}{2} \cdot 16 \cdot 4 - 3 \cdot 16 = 240$ neighboring triangles (which contain an edge from G_k to H_k), and the green triangle (type 2 triangle) has $\frac{9}{2} \cdot 16 \cdot 4 - 4 \cdot 4 - 16 = 256$ neighboring triangles. If the number of neighboring triangles coming from H_k is neglected, the degree difference in $cl_3(R_k)$ is $256 - 240 = 16$.

However, in contrast to the section 2.3, the number of the neighboring triangles coming from H_k does play a significant role. The green, type 2 triangle has

$$2 \cdot \left(\binom{\delta}{3} - \binom{\delta - 1}{3}\right)$$

neighboring triangles coming from H_k, whereas the red, type 1 triangle has

$$\binom{\delta}{3} - \binom{\delta - 3}{3} - 1$$

neighboring triangles coming from H_k.

In sum, for $k = 16$ and $\delta = 4$, the green, type 2 triangle in Figure 2.19 has a total of $\frac{9}{2}k\delta - 4\delta - k + 2 \cdot (\binom{\delta}{3} - \binom{\delta-1}{3}) = 256 + 6 = 262$ neighboring triangles in R_k, whereas the red, type 1 triangle has a total of $\frac{9}{2}k\delta - 3k + \binom{\delta}{3} - \binom{\delta-3}{3} - 1 = 240 + 3 = 243$ neighboring triangles in R_k with degree difference $262 - 243 = 19$. Therefore, the degree difference in the clique graph $cl_3(R_k)$ is increased by $6 - 3 = 3$ when considering the neighboring triangles coming from H_k.

Table 2.2 compares the number of neighboring triangles in H_k of the green, type 2 triangle and the red, type 1 triangle. If $\delta \geq 8$, the number of neighboring triangles of the red, type 1 triangle in H_k is larger than the number of neighboring triangles of the green, type 2 triangle.

Let $k = 6\delta - 8$. Let $C = \frac{9}{2}k\delta - 4\delta - k + 2 \cdot (\binom{\delta}{3} - \binom{\delta-1}{3})$ be the number of neighboring triangles of the green, type 2 triangle in R_k. Let $D = \frac{9}{2}k\delta - 3k + \binom{\delta}{3} - \binom{\delta-3}{3} - 1$ be the number of neighboring triangles of the red, type 1 triangle in R_k. Then, Table 2.3 illustrates the degree difference in R_k.

The Table 2.3 implies that if $k = 6\delta - 8$, for $\delta < 23$, the green, type 2 triangle has more neighboring triangles in R_k than the red, type 1 triangle. For $\delta = 23$, both triangles have the same number of neighboring triangles and therefore the degree difference is 0 in $cl_3(R_k)$. For $\delta > 23$, the red, type 1 triangle has more neighboring triangles in R_k than the green, type 2 triangle. Therefore, for instance, if $\delta = 100$, the degree difference in $cl_3(R_k)$ will be $3,773$.

	Green, type 2 triangle	Red, type 1 triangle	Difference	Ratio
δ	$A = 2 \cdot (\binom{\delta}{3} - \binom{\delta-1}{3})$	$B = \binom{\delta}{3} - \binom{\delta-3}{3} - 1$	$A - B$	$\frac{A}{B}$
4	6	3	3	2.000
5	12	9	3	1.333
6	20	18	2	1.111
7	30	30	0	1.000
8	42	45	-3	0.933
9	56	63	-7	0.889
14	156	198	-42	0.788
19	306	408	-102	0.750
23	462	630	-168	0.733
28	702	975	-273	0.720
78	5852	8550	-2698	0.684
100	9702	14259	-4557	0.680
150	22052	32634	-10582	0.676
200	39402	58509	-19107	0.673

Table 2.2: The number of neighboring triangles of the triangle in H_k

	k	Type 2 triangle	Type 1 triangle	Degree difference
δ	$k = 6 \cdot \delta - 8$	C	D	$\lvert C - D \rvert$
4	16	262	243	19
5	22	465	438	27
6	28	724	690	34
7	34	1039	999	40
8	40	1410	1365	45
9	46	1837	1788	49
14	76	4812	4758	54
19	106	9187	9153	34
23	130	13695	13695	0
28	160	20590	20655	65
78	460	166540	168630	2090
100	592	275110	278883	3773
150	892	622660	632058	9398
200	1192	1110210	1127733	17523

Table 2.3: The number of neighboring triangles of the triangle in R_k and the degree difference in the clique graph

In contrast to the previous section, if $\delta \gg 23$ and $k = 6\delta - 8$, the vertex corresponding to the red, type 1 triangle in R_k will have much larger degree in $cl_3(R_k)$ than the vertex corresponding to the green, type 2 triangle in R_k. This implies that the degree difference in the clique graph $cl_3(R_k)$ can be arbitrarily large for any $\delta \gg 23$.

Chapter 3

INDEPENDENT SETS OF THE CLIQUE GRAPH

Recall that an independent set is a set of vertices in a graph G of which no pair is adjacent. The independence number $\alpha(G)$ of a graph G is the order of the largest independent set of G. This continues the same theme as the previous chapter, namely what properties of G are carried into $cl_t(G)$. It is a folklore result that the independence number of a regular graph is less than or equal to $\frac{n}{2}$, where n is the number of vertices in the graph. The goal of this section is to investigate whether this property (no independent set on more than half the vertices of the graph) carries over to $cl_t(G)$ if G is regular.

3.1 Independence number of the clique graph - Preliminary discussion

In order to understand what are the necessary conditions for $\alpha(cl_t(G)) > \frac{|cl_t(G)|}{2}$ to hold or not to hold, multiple cases that all have different restrictions on G and its clique graph will be analyzed.

First of all, before considering the question of $\alpha(cl_t(G)) > \frac{|cl_t(G)|}{2}$, consider the following examples. If $G = C_7 = cl_2(C_7)$, then $\alpha(cl_2(C_7)) = 3 < \frac{n_{cl_t(G)}}{2} = \frac{7}{2} = 3.5$. And, if $G = C_6 = cl_2(C_6)$, then $\alpha(cl_2(C_6)) = 3 = \frac{6}{2} = \frac{n_{cl_t(G)}}{2}$. In these examples which satisfy the 4 conditions, the bound $\alpha(cl_t(G)) \leq \frac{|cl_t(G)|}{2}$ indeed holds for G.

Secondly, the consequences on $\alpha(cl_t(G))$ of not enforcing all of the 4 conditions on G will be analyzed.

1. If the original graph G does not necessarily have to be regular, $\alpha(cl_t(G)) > \frac{|cl_t(G)|}{2}$ is possible. Let G be P_6, and the clique graph be $cl_2(G) = P_5$ (Figure 3.1). Since P_5 has $n = 5$ vertices and $\alpha(P_5) = 3$, we have $\alpha(cl_2(G)) = 3 > \frac{n_{cl_t(G)}}{2} = 2.5$.

Figure 3.1: Non-regular graph $G = P_6$ and its clique graph $cl_2(G) = P_5$

2. If not all of the edges of G necessarily have to be in a t-clique, then Figure 3.2 and Figure 3.3 illustrate examples where $\alpha(cl_t(G)) > \frac{|cl_t(G)|}{2}$ holds.

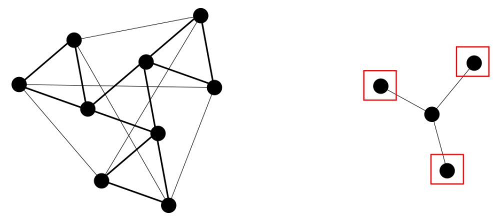


Figure 3.2: Graphs G (whose edges are not all in a t-clique) and $cl_3(G)$

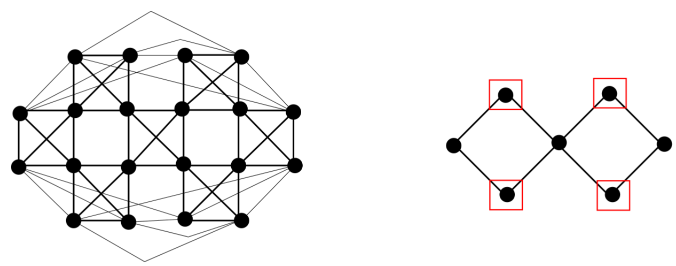


Figure 3.3: Graphs G (whose edges are not all in a t-clique) and $cl_4(G)$

In Figure 3.2, the original graph G does satisfy 3 of the 4 conditions (regular, non-complete, $t = 3$ close to $\delta = 4$) and $\alpha(cl_3(G)) = 3 > \frac{|cl_t(G)|}{2} = \frac{4}{2} = 2$. Similarly, in Figure 3.3, the original graph G also satisfies 3 of the 4 conditions (regular, non-complete, $t = 4$ close to $\delta = 6$) and $\alpha(cl_4(G)) = 4 > \frac{|cl_t(G)|}{2} = \frac{7}{2} = 3.5$.

3. If not all of the edges of G necessarily have to be in a t-clique and the clique graph $cl_t(G)$ does not have to be connected, then Figure 3.4 gives an example of a clique graph $cl_3(G)$ of a regular graph G where $\alpha(cl_3(G)) = 2 > \frac{n_{cl_t(G)}}{2} = \frac{2}{2} = 1$ holds.

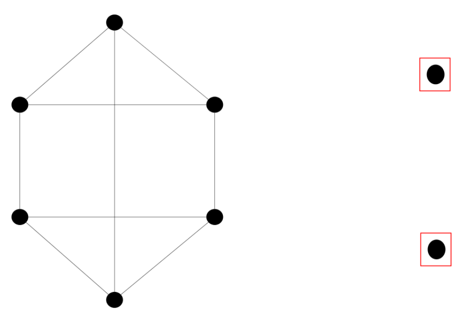


Figure 3.4: Non-connected clique graph $cl_3(G)$ of a regular graph G whose edges are not all in a 3-clique

Therefore, the inequality $\alpha(cl_t(G)) > \frac{|cl_t(G)|}{2}$ is possible if any 1 of the 4 conditions on G and $cl_t(G)$ is replaced with 1 of the following:

1. G is not regular ($G = P_6$ and $cl_2(G) = P_5$).

2. Not all of the edges of G are in a t-clique (Figures 3.2, 3.3).

3. Not all of the edges of G are in a t-clique, and $cl_t(G)$ is not connected (Figure 3.4).

After having established these special cases, the remaining part of the discussion will be spent on analyzing if $\alpha(cl_t(G)) > \frac{|cl_t(G)|}{2}$ is possible for graphs G that satisfy all of our 4 conditions. In order to possibly find a clique graph, such that $\alpha(cl_t(G)) > \frac{|cl_t(G)|}{2}$, the desired (and undesired) structural features of the clique graph will be analyzed.

1. If the clique graph is a complete graph, $cl_t(G) = K_n$, then $\alpha(cl_t(G)) = \alpha(K_n) = 1$. While α remains constant, $\frac{n_{cl_t(G)}}{2} = \frac{|cl_t(G)|}{2} = \frac{K_n}{2}$ increases with increasing $n_{cl_t(G)}$. Therefore, the clique graph should not be complete.

2. As discussed previously, if the clique graph is a cycle, $cl_t(G) = C_n$, then $\alpha(cl_t(G)) = \alpha(K_n) = \frac{n}{2}$ and $\frac{|cl_t(G)|}{2} = \frac{C_n}{2} = \frac{n}{2}$. Since $\alpha(cl_t(G)) = \frac{|cl_t(G)|}{2}$, this case is an improvement compared to the K_n case. Therefore, we would like the clique graph to have more structural features of the cycle than the complete graph.

3. If the clique graph is a star on $n_{cl_t(G)}$ vertices (Figure 3.5), then $\alpha(cl_t(G)) = n_{cl_t(G)} - 1$. If $n_{cl_t(G)} > 2$, $\alpha(cl_t(G)) = n_{cl_t(G)} - 1 > \frac{n_{cl_t(G)}}{2}$. Additionally, the inequality can be arbitrarily large if $n_{cl_t(G)}$ is large. Therefore, if possible, we might like the clique graph to have structural features of a star.

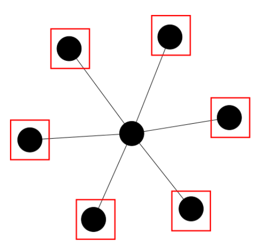


Figure 3.5: Desired clique graph $cl_t(G)$ with $\alpha(cl_t(G)) = 6$

Remark: If the clique graph cannot be a "true" star (due to conditions' limitations on G), then we would like the clique graph to have a lot of leaves, as illustrated in Figure 3.6, in order to drive the independence number arbitrarily large.

In sum: Figure 3.7 illustrates the spectrum of possible cases that were just discussed. Whereas on the left side, we have, for $cl_t = K_n$, the case where the independence number is

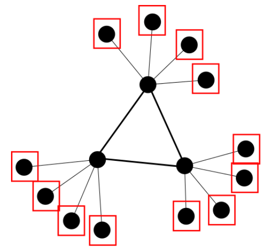


Figure 3.6: Desired clique graph $cl_t(G)$ with $\alpha(cl_t(G)) = 12$

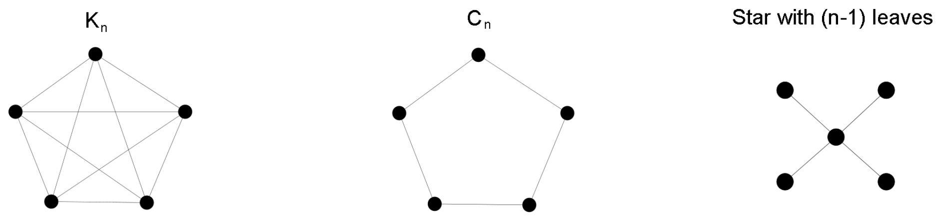


Figure 3.7: What structural features should the clique graph have for $\alpha(cl_t(G)) > \frac{|cl_t(G)|}{2}$ to be possibly true?

arbitrarily smaller than half of the graph's vertices, that is $\alpha = 1 \ll \frac{|cl_t(G)|}{2}$, the right side illustrates the case where the reverse holds, that is $\alpha \gg \frac{|cl_t(G)|}{2}$, if $cl_t(G)$ is a star. The center of the figure shows the case where $\alpha = \frac{|cl_t(G)|}{2}$ if $cl_t(G) = C_n$.

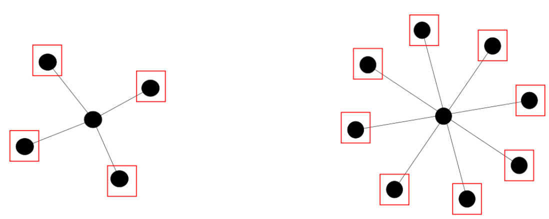


Figure 3.8: Implications of increasing degree difference on the independence number

As seen in Figure 3.7, both graphs K_n and C_n (which are graphs that do not satisfy the condition for the clique graph that we are trying to prove) are regular. In contrast, the stars in Figure 3.8 have degree difference 3 and 7, and their respective difference between independence number and half of the vertices, $\alpha - \frac{n}{2}$, is 1.5 and 3.5. For stars, the larger the degree difference, the larger the difference between the inequality between α and $\frac{n}{2}$. In the following, the folklore result mentioned at the beginning of the chapter will be proven in order to better understand the relationship between regularity of the graph and the independence number of the graph.

Theorem 3.1.1. *For any regular graph G,* $\alpha(G) \leq \frac{|G|}{2}$.

Proof. Let I be the largest independent set in G. The number of outgoing edges from I are $E(I, G-I) = |I| \cdot \delta$ (Figure 3.9). The number of outgoing edges from $(G-I)$ are at most $E(G-I, I) \leq |(G-I)| \cdot \delta$ because vertices in $(G-I)$ could be adjacent. Therefore, we have

$$|I| \cdot \delta = E(I, G-I) \leq |(G-I)| \cdot \delta,$$

that simplifies to

$$|I| \leq \frac{|G|}{2},$$

which is what we want to show.

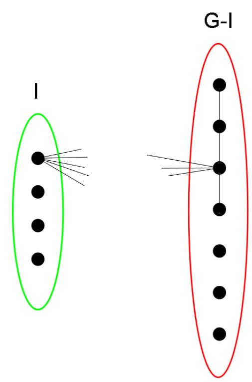


Figure 3.9: Doublecounting argument to show that $|I| = \alpha(G) \leq \frac{|G|}{2}$ in a regular graph

Therefore, we now know that the clique graph in question cannot be regular if $\alpha(cl_t(G)) > \frac{|cl_t(G)|}{2}$. This finding is consistent with the previously stated observation that K_n and C_n are both regular and are not possible clique graphs for $\alpha(cl_t(G)) > \frac{|cl_t(G)|}{2}$ to be true.

However, does this finding imply that large enough degree difference may cause $\alpha(G) > \frac{n}{2}$ to be true?

3.2 Independence number and degree difference

This is not the case and can be shown by considering the construction from the last chapter. As shown in Chapter 2, the clique graph in Figure 3.10 has degree difference 1. Therefore, this graph is slightly irregular. Additionally, the clique graph has the independence number 8, $\alpha(cl_3(G)) = 8$, and order 24, $|cl_3(G)| = 24$. Therefore, $\alpha = 8 \not> \frac{n}{2} = \frac{24}{2} = 12$. Therefore, not every irregular clique graph satisfies $\alpha(cl_t(G)) > \frac{|cl_t(G)|}{2}$. Next, the family of graphs R_k illustrating arbitrarily large degree difference from Chapter 2 will be analyzed.

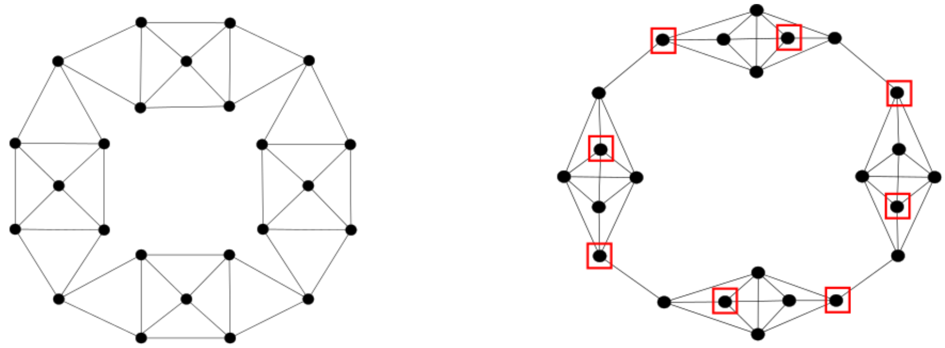


Figure 3.10: Independence number of slightly irregular clique graph $cl_3(G)$

In the following, we will derive formulas for calculating the number of vertices in $cl_3(R)$, $|cl_3(R)|$, and $\alpha(cl_3(R))$. Then, we will be able to compare $\alpha(cl_3(R))$ and $\frac{|cl_3(R)|}{2}$.

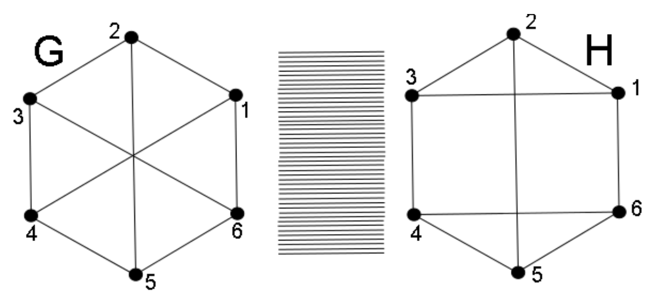


Figure 3.11: Graph R_k with regular subgraphs G_k and H_k

First of all, for $\delta = \delta_G = \delta_H = 3$, since $k = n_G = n_H$, there are

$$\frac{3}{2}k \cdot k + \frac{3}{2}k \cdot k + 2 = 3 \cdot k^2 + 2$$

total triangles in R_k (Figure 3.11), because there are $\frac{n_G \delta_G}{2}$ choices for a base in G and n_H choices for an apex in H, with a similar expression for H. Therefore, the clique graph $cl_3(R)$ has $(3 \cdot k^2 + 2)$ vertices. For example, in Figure 3.11, the clique graph $cl_3(R)$ has $n_{cl_3(R)} = 3 \cdot (6)^2 + 2 = 110$ vertices.

Secondly, $\alpha(cl_t(R))$ is the same as the maximum number of vertex-disjoint t-cliques in R. In this particular case, if $t = 3$, the maximum number of vertex-disjoint triangles in R are counted. In order to maximize the number of vertex-disjoint triangles in R, the best combination of bases and apexes of triangles in G and H needs to be found. Since the edges in G and H that do not belong to C_n (Figure 3.12) are covered by the vertices and edges of the respective C_n, these edges in G and H can be neglected.

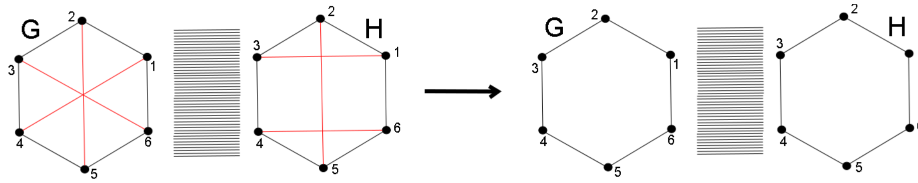


Figure 3.12: Edges within G or H can be neglected when finding the maximum number of vertex-disjoint 3-cliques in R

Since a total of 3 vertices from both G and H are needed to build a triangle, we will need 2 vertices and their corresponding edge for the base of the triangle in either G or H and the third vertex as the apex of the triangle from the other subgraph. For $k = n_G = n_H$, if we use $\left\lceil \frac{\frac{2}{3}k}{2} \right\rceil \cdot 2$ of G's vertices and their corresponding edges as bases of triangles and the remaining vertices of G as apexes of triangles, it is ensured that the subgraph H has enough vertex-disjoint bases and vertices to build a total of $\left\lfloor \frac{2}{3}k \right\rfloor$ vertex-disjoint triangles in R_k. This implies that $\alpha(cl_3(R_k)) = \left\lfloor \frac{2}{3}k \right\rfloor$.

Remark: Since R has $2k$ vertices, it can have at most $\left\lfloor \frac{2}{3}k \right\rfloor$ disjoint triangles (each triangle accounts for 3 vertices).

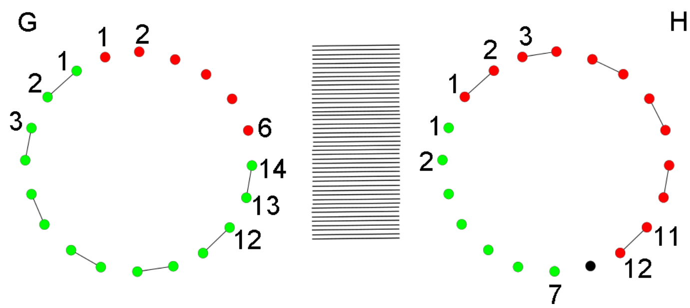


Figure 3.13: Finding the maximum number of vertex-disjoint triangles between G and H when $G_k = H_k = C_{20}$

For instance, for $k = 20$ (Figure 3.13), we would use $\left\lceil \frac{\frac{2}{3}20}{2} \right\rceil \cdot 2 = 7 \cdot 2 = 14$ (green) vertices in G and their corresponding edges as bases of 7 triangles. The remaining $20 - 14 = 6$ (red) vertices in G can serve as apexes of triangles. Conversely, in H, we will use 7 (green) vertices (corresponding to the 7 bases in G) as the apexes of triangles and $6 \cdot 2 = 12$ (red) vertices as bases of triangles. In total, we will therefore have a total of $7 + 6 = 13$ vertex-disjoint triangles for $k = 20$. This finding is confirmed by $\alpha(cl_3(R)) = \left\lfloor \frac{2}{3}k \right\rfloor = \left\lfloor \frac{2}{3}20 \right\rfloor = \lfloor 13.33 \rfloor = 13$.

	A: Order of $cl_3(R)$	B: Independence number of $cl_3(R)$	Ratio	Difference
k	$\lvert cl_3(R)\rvert = 3k^2+2$	$\alpha(cl_3(R)) = \lfloor \frac{2}{3}k \rfloor$	$\frac{\frac{A}{2}}{B}$	$\frac{A}{2} - B$
6	110	4	13.75	51
8	194	5	19.40	92
10	302	6	25.17	145
12	434	8	27.13	209
28	2354	18	65.39	1159
30	2702	20	67.55	1331
32	3074	21	73.19	1516
34	3470	22	78.86	1713

Table 3.1: Comparison of $\alpha(cl_3(R))$ and $\frac{|cl_3(R)|}{2}$

In sum: In Chapter 2 (degree difference formula, 2.3), it was found that for increasing k, the degree difference becomes arbitrarily large in $cl_3(R_k)$. For increasing k, Table 3.1 illustrates the increase of $|cl_3(R)|$ and of $\alpha(cl_3(R))$ and compares them in the last 2 columns. Previously, the following question was asked: Does a large enough degree difference may cause $\alpha(cl_t(G)) > \frac{|cl_t(G)|}{2}$ to possibly be true? The result of Table 3.1 indicate that this is not the case. Even if the degree difference of the clique graph, $cl_3(R)$, becomes arbitrarily large for increasing k, the independence number of $cl_3(R)$ is not necessarily greater than the clique graph's order divided by 2. In contrast, both the ratio of $\frac{\frac{A}{2}}{B}$ and the difference $\frac{A}{2} - B$ actually increase with increasing k.

3.3 Independence number of line graphs and of $cl_t(G)$ for $t > 2$

In this section, the independence number of line graphs and $cl_t(G)$ for $t > 2$ will be analyzed. To begin, we will show that the independence number of line graphs is smaller or equal to half the number of its vertices. Subsequently, we begin extending this proof to $cl_t(G)$ for $t > 2$.

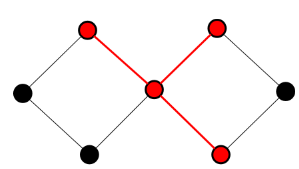


Figure 3.14: Graph with $\alpha(G) > \frac{n}{2}$ that contains the claw $K_{1,3}$

The graph in Figure 3.14) is non-regular and satisfies $\alpha(cl_t(G)) > \frac{|cl_t(G)|}{2}$. However, the line graph $cl_2(G)$ cannot contain the claw $K_{1,3}$ as an induced subgraph (Section 1.3.4 and

Figure 1.19). Therefore, the line graph cannot be a star, S_n, or the graph in Figure 3.14 since these graphs contain the claw $K_{1,3}$.

Theorem 3.3.1. *Let $L(G)$ be a line graph of a regular graph G. Then, $\alpha(L(G)) \leq \frac{|L(G)|}{2}$.*

Proof. In Theorem 2.1.1, we showed that the line graph $L(G)$ of a regular graph G is regular. In Theorem 3.1.1, we showed that for any regular graph G, $\alpha(G) \leq \frac{1}{2}|V(G)|$. Therefore, any line graph $L(G)$ of a regular graph G has $\alpha(L(G)) \leq \frac{|L(G)|}{2}$.

The following question arises: Can this result be extended to $cl_t(G)$ for $t > 2$? Furthermore, what properties of a line graph might translate to $cl_t(G)$, and can we base a proof off of these properties?

First of all, although we do not know that $cl_t(G)$ is regular, the explanation of why the line graph $cl_2(G)$ does not contain the claw $K_{1,3}$ presented in Section 1.3.4 and Figure 1.20 can be extended to the general case t. Therefore, we do know that $cl_t(G)$ does not contain $K_{1,1+t}$.

Additionally, we can show that cl_t contains no leaves.

Lemma 3.3.2. *Let G be a regular graph. Then, $cl_t(G)$ cannot have any leaves.*

Proof. Let K be a copy of K_t in G corresponding to a leaf in $cl_t(G)$, and let K' be the copy of K_t that intersects K (The red and green cliques in Figure 3.15). Let $v_1,...,v_k$ be the k vertices of K that are not in the intersection of K and K'. Let $q_1,...,q_t$ be the t vertices of K' that are not in the intersection of K and K'. Let $u_1,...,u_s$ be the s vertices that are in the intersection of K and K'. The vertices $v_1,...,v_k$ are adjacent only to vertices in K since if they were adjacent to a vertex outside the clique, the corresponding edge could be extended to another clique, which in turn would make the vertex in $cl_t(G)$ corresponding to K no longer a leaf. The vertices $u_1,...,u_s$ are adjacent to the vertices $v_1,...,v_k$ in K and to the vertices $q_1,...,q_t$ in K'. Therefore, the degree of the vertices $u_1,...,u_s$ is larger than the degree of the vertices $v_1,...,v_k$, making the graph G a non-regular graph. Therefore, since G is regular and every edge is in a t-clique, then $cl_t(G)$ cannot have a leaf.

Relaxing the condition that our graph must be a line graph of a regular graph, we can just assume that the clique graph contains no copy of $K_{1,3}$ as an induced subgraph and contains no leaves.

Theorem 3.3.3. *Let G contain no copy of $K_{1,3}$ as an induced subgraph and contain no leaves. Then, $\alpha(G) \leq \frac{|G|}{2}$.*

Proof. Assume that $\alpha(G) > \frac{n}{2}$ (with $n \geq 2$). Then, there must be some vertex in $G - I$ adjacent to 2 or more vertices in I (Figure 3.16). If this was not the case, then there would

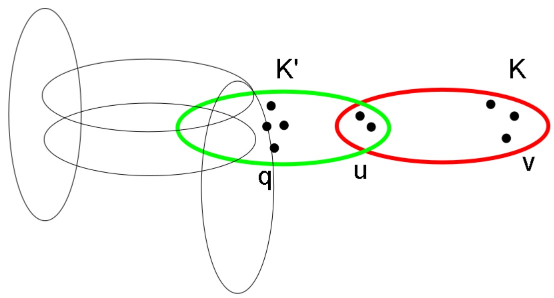


Figure 3.15: Graph G with 6 t-cliques

be at least as many vertices in $G-I$ as in I. Remove "alternating" paths P_i from G that alternate between I and $G-I$ (Figure 3.17), such that each path P_k is of maximum length in $G-(P_1 \cup P_2 \cup \cdots \cup P_{k-1})$. Then, find the first path P_l with more vertices in I than in $G-I$. This has to happen since $|I| > |G-I|$. Consider the end vertex $u \in I$. Since the end vertex u (green vertex in Figure 3.17) cannot be a leaf, it has to be adjacent to another vertex $s \in (G-I)$ in another path. The vertex u must be adjacent to a vertex s in a previous path, or the path containing u could be extended. The vertex u cannot be adjacent to an end vertex of a previous path (red vertices in Figure 3.17), or that path could have been extended. Therefore, the vertex u has to be adjacent to a non-end vertex $s \in (G-I)$ (blue vertices in Figure 3.17) in a previous path. If the edge (u,s) is added to the graph, then G contains a copy of the claw $K_{1,3}$, which is a contradiction.

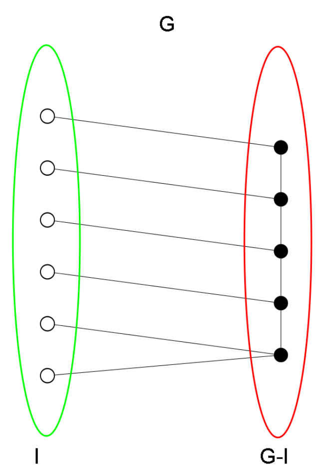


Figure 3.16: Graph G with $|I| > |G-I|$

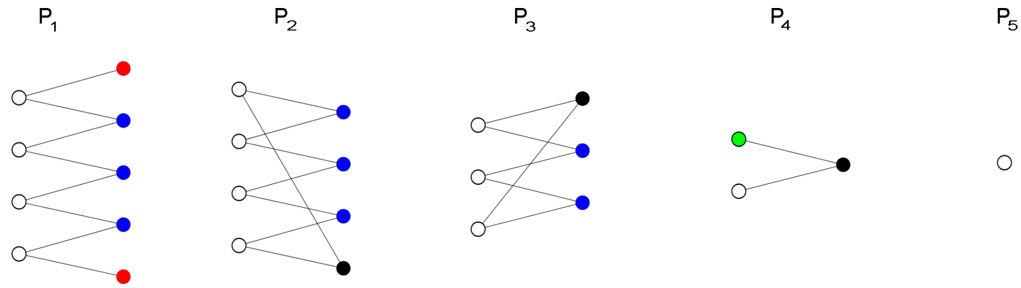


Figure 3.17: The possible alternating paths that can be removed from G

Therefore, we have shown that if G is regular and its clique graph $cl_t(G)$ does not contain a copy of $K_{1,3}$ as an induced subgraph, then $\alpha(cl_t(G)) \leq \frac{|cl_t(G)|}{2}$.

Chapter 4

REMAINING QUESTIONS ABOUT THE CLIQUE GRAPHS

The following chapter discusses 2 topics that are related to the main theme of this thesis and some of the open questions.

4.1 On the completeness of the clique graph $cl_t(G)$ and the original graph G

In this section, the following question will be addressed: If the clique graph $cl_t(G)$ is complete, does the original graph G have to be complete? First of all, let $cl_t(G) = K_3$. For what G and corresponding t is $cl_t(G) = K_3$? The following examples and the discussion below answer this question.

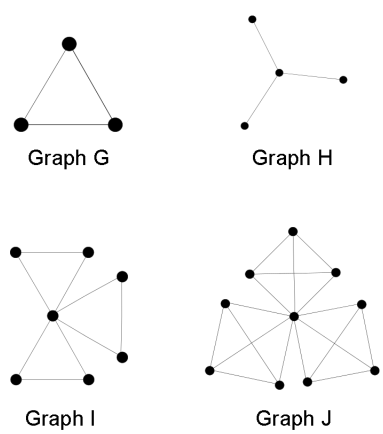


Figure 4.1: Examples for $cl_2(G) = cl_2(H) = cl_3(I) = cl_4(J) = K_3$

As illustrated in Figure 4.1, we have $cl_2(G) = cl_2(H) = cl_3(I) = cl_4(J) = K_3$. Therefore, we now know that if the clique graph $cl_t(G)$ is complete, the original graph does not necessarily have to be complete, as the graphs H, I, and J indicate.

After having found that if the clique graph $cl_t(G)$ is complete, the graph G does not necessarily have to be complete, we want to know for what conditions G will be complete.

1. Could it be true that for certain choices of t or n, if the clique graph $cl_t(G)$ is complete, then the graph G will also be complete? The following argument shows that this statement is also not true.

Given any K_n as the complete clique graph, you can always find a non-complete graph G, such that $cl_t(G) = K_n$ for any t. In order to find the graph G that satisfies the previous statement, you have to draw n t-cliques that all have 1 vertex in common. Therefore, you will have 1 vertex that dominates all the t-cliques.

2. The non-complete graphs H, I, and J are not regular. So, could it be true that if $cl_t(G)$ is complete and G is regular, then G will be complete? In order to disprove this statement, a counterexample that satisfies the following conditions needs to be found:

   - G is regular.
   - G is non-complete.
   - $cl_t(G)$ is complete.

   Figure 4.2 serves as a counterexample.

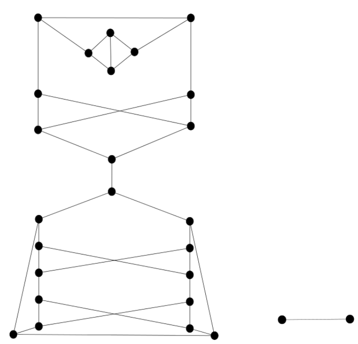


Figure 4.2: Graph K and $cl_3(K) = K_2$

First of all, the clique $cl_3(K) = K_2$ is a complete graph. Furthermore, graph K is both regular and not complete. Therefore, it is not true that if $cl_t(G)$ is complete and graph G is regular, then graph G will be complete.

As a next step, other conditions that could possibly imply that if $cl_t(G)$ is complete, then G will also be complete would have been analyzed, including requiring that every edge be contained in a copy of K_t.

4.2 On the number of induced cycles and paths in the clique graph

As an introduction to the following analysis, a result by Wolf will be presented and discussed. Let P_k (respectively, C_k) denote an induced path (respectively, cycle) on k vertices in $cl_t(G)$. In [11], Wolk showed that if G is a connected graph, then either G contains an induced copy of P_4 (the path on 4 vertices), C_4, or a dominating vertex. As the following examples suggest (Figures 4.3, 4.4), for some connected graphs G, just 1 of the criteria – P_4, C_4, or a dominating vertex – is met whereas other connected graphs exhibit all of the above criteria.

If $n < 4$ (Figure 4.3), every connected graph has a dominating vertex and does not contain an induced copy of P_4 and C_4.

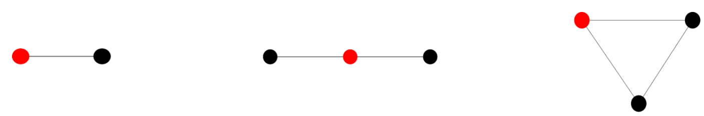


Figure 4.3: Graphs with dominating vertex ($n < 4$)

If $n \geq 4$ (Figure 4.4), the connected graph could also contain an induced copy of P_4 and C_4. The complete graph on 4 vertices, K_4, even contains all 4 criteria.

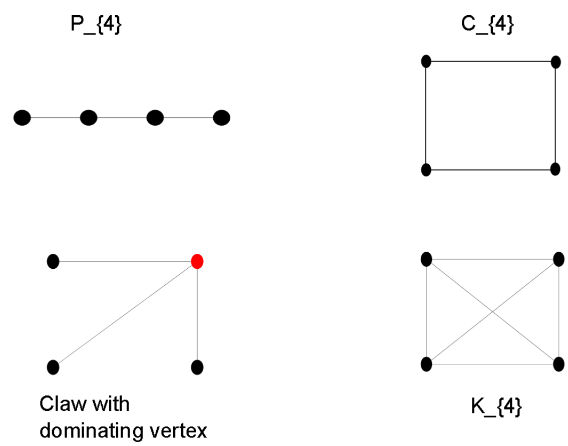


Figure 4.4: Graphs with dominating vertex, P_4 and C_4 ($n \geq 4$)

Lyle [7] proved that if G is a connected, δ-regular graph with $t > \frac{3}{5}(\delta+1)$, then there is some $k \geq 4$, such that $cl_t(G)$ contains an induced copy of C_k. For instance, the 8-regular graph G and its clique graph $cl_6(G) = C_7$ (Figure 4.5) satisfy the theorem's statement since $t = 6 > \frac{3}{5}(8+1) = \frac{27}{5} = 5.4$.

1 of the reasons why we are interested in the existence of induced cycles in G is the following bound on t: If G is a δ-regular graph and $cl_t(G)$ contains an induced copy of C_k, then $t \leq \frac{2}{k}n$. In the example (Figure 4.5), we have $t \leq \frac{2}{k}n = \frac{2}{7} \cdot 21 = 6$. Therefore, since $t = 6$, the bound is sharp.

In the following, we will discuss what conditions have to be met in order for the clique graph to be a cycle. First of all, a couple of definitions will be discussed and presented.

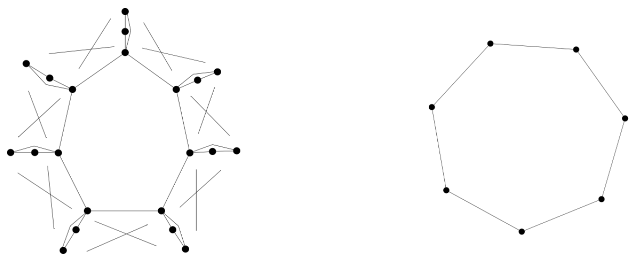


Figure 4.5: Graph G ($\delta = 8$) and its clique graph $cl_6G) = C_7$

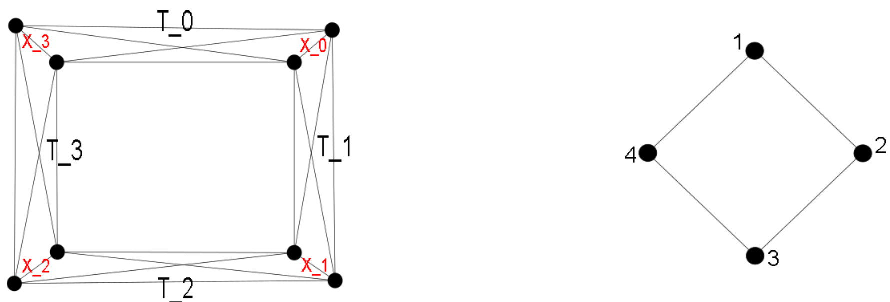


Figure 4.6: Graph G ($\delta = 5$) and its clique graph $cl_4G) = C_4$

Let T_0, T_1,..., T_{k-1} be cliques corresponding to the vertices of the path P_k (respectively, cycle C_k). Let $X_i = T_i \cap T_{i+1(modk)}$ for $i = 0$,..., $k-2$ (respectively, $k-1$). The following example (Figure 4.6) illustrates these definitions. For instance, T_0 corresponds to vertex 1 of the cycle C_4 in $cl_4(G)$. Furthermore, $X_0 = T_0 \cap T_{1(mod4)}$, which is equivalent to the 2 top right vertices of graph G with their corresponding edge.

Additionally, for each vertex $v \in V(G)$, $I(v) = (i : N(v) \cap X_i \neq \emptyset)$. For instance (Figure 4.7), if we analyze vertex $v_1 \in V(G)$, we get $I(v_1) = (0, 2, 3)$. To see this, we have $N(v_1) = v_2, v_3, v_4, v_7, v_8$, and $X_0 = v_3, v_4$, $X_1 = v_5, v_6$, $X_2 = v_7, v_8$, and $X_3 = v_1, v_2$.

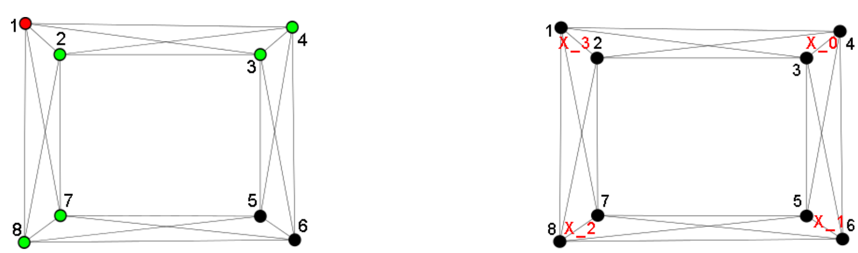


Figure 4.7: Conditions for the clique graph to be a cycle

Since $N(v_1) \cap X_0 = v_3, v_4 \neq \emptyset$, $N(v_1) \cap X_1 = \emptyset$, $N(v_1) \cap X_2 = v_7, v_8 \neq \emptyset$ and $N(v_1) \cap X_3 = v_2 \neq \emptyset$, we have $I(v_1) = (0, 2, 3)$, So, $|I(v)| = 3$.

After analyzing the conditions for existence of induced cycles and paths in the clique graph, we will be discussing how many induced cycles and paths are in the clique graph.

First of all, in order to better understand the question, a couple of examples are presented. Let the clique graph $cl_t(G)$ be a graph with 2 disjoint 4-cycles (Figure 4.8). Then, for different t's, the respective graph G is presented.

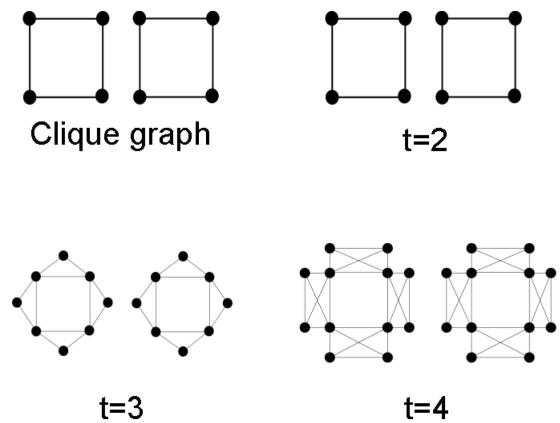


Figure 4.8: Graphs G for different ts of a given clique graph $cl_t(G)$

Secondly, let the clique graph $cl_t(G)$ be a graph with 3 4-cycles (Figure 4.9). Then, for different t's, the respective graph G is presented.

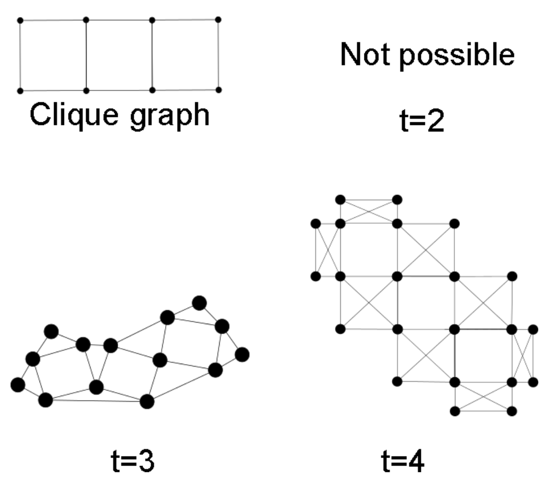


Figure 4.9: Graphs G for different t's of a given clique graph $cl_t(G)$

Thirdly, let the clique graph $cl_t(G)$ be a graph with a 5-cycle and a slightly changed 4-cycle (Figure 4.10). Then, for different t's, the respective graph G is presented.

The analysis of the previous examples serves as a first step in understanding the question "How many induced cycles and paths are in the clique graph?" Therefore, these examples

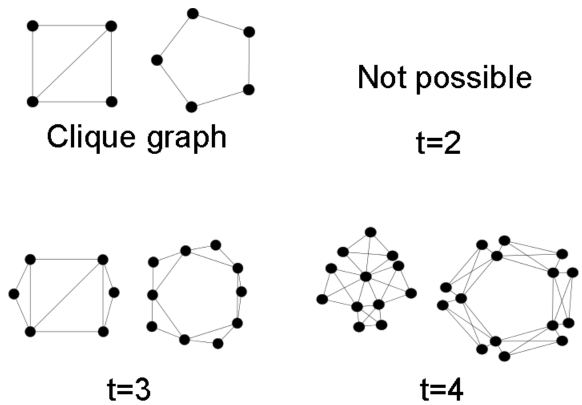


Figure 4.10: Graphs G for different t's of a given clique graph $cl_t(G)$

should be analyzed closely. First of all, the approach of coming up with examples was in reverse order than what we have done so far. In this section, as a first step, the clique graph $cl_t(G)$ was chosen, and subsequently the graph G for different values of t was found. In the initial approach - building the clique graph from a given original graph -, the clique graph $cl_t(G)$ could always be found for $t = 2$ (as long as the original graph is connected) and for any subsequent t as long as the graph G contains at least 1 t-clique. In contrast, for a couple of the previous examples (Figures 4.9, 4.10), it was not possible to find the graph G (of a given clique graph $cl_t(G)$) for $t = 2$ since the clique graph has a claw as an induced subgraph. Therefore, this clique graph $cl_2(G)$ (the line graph) does not exist; therefore, G cannot be found for $t = 2$ for these examples.

Secondly, whereas the degree of the given clique graph $cl_t(G)$ is obviously unchanging, with increasing value of t, the degree of the graph G is also increasing. Furthermore, if the clique graph $cl_t(G)$ is not connected, then the graph G is also not connected. The reverse also holds. Recall that when building a clique graph $cl_t(G)$, cliques of order t in the original graph G become vertices in the clique graph which are adjacent if the t-cliques in G have at least 1 vertex in common. Therefore, for a vertex or set of vertices of the clique graph to not be disjoint, the corresponding clique or collection of cliques in the original graph cannot have a vertex in common with the other cliques of the graph, assuming that every edge of G is in a t-clique. Therefore, in order for the clique graph not to be connected, the original graph has to be disconnected.

Thirdly, recall that in order to find an induced subgraph of G, choose a subset of the vertices of G and then choose all edges in G whose endpoints (corresponding to vertices) are both in the chosen set. For instance, if the red vertices in the graph G in Figure 4.11 are chosen to be in the subset of the vertices, then the induced subgraph of these vertices is

Figure 4.11: Number of induced cycles and paths of the graph G

shown on the right in Figure 4.11. Therefore, the number of induced cycles and paths is 1 since we have 1 C_5 and 1 P_5 in the induced subgraph.

In order to find the number of induced cycles or paths in the clique graph, the previous examples and clique graphs of graphs that satisfy the 4 conditions need to be further analyzed. As seen in the other chapters, the implications of the regularity and the "every edge has to be in a t-clique" conditions on G have to be closely analyzed. Furthermore, what are the differences of the number of induced cycles or paths if $t = 2$ compared to $t > 3$?

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