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An Examination of the Yang-Baxter Equation

Alexandru Cibotarica
University of Southern Mississippi

The University of Southern Mississippi

EXAMINATION OF THE YANG-BAXTER EQUATION

by

Alexandru Cibotarica

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Approved:

Joseph Kolibal
Director

C.S. Chen

James V. Lambers

Susan A. Siltanen
Dean of the Graduate School

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ABSTRACT

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The Yang-Baxter equation has been extensively studied due to its application in numerous fields of mathematics and physics. This thesis sets out to analyze the equation from the viewpoint of the algebraic product of matrices, i.e., the composition of linear maps, with the intent of characterizing the solutions of the Yang-Baxter equation.

We begin by examining the simple case of 2×2 matrices where it is possible to fully characterize the solutions. We connect the Yang-Baxter equation to the Cecioni-Frobenius Theorem and focus on obtaining solutions to the Yang-Baxter equation for special matrices where solutions are more easily found. Finally, we derive a fixed point iteration algorithm to determine the Yang-Baxter complement of a given matrix, if it exists.

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Chapter 1
BACKGROUND

1.1 Introduction

One of the earliest introductions of the Yang-Baxter equation in statistical mechanics was in 1944, when Onsanger briefly mentioned the star-triangle relation in the introduction of his solution to the Icing model [16]. It was not until November and December of 1967, when C. N. Yang published two papers on a simple one-dimensional quantum many-body problem, and for the first time introduce the equation

$$A(u)B(u+v)A(v) = B(u)A(u+v)B(v), \tag{1.1}$$

where $A(u)$ and $B(v)$ are rational functions of u and v [9]. The same equation was also used by R. J. Baxter in 1972, when he was studying some classical statistical mechanics problems in two-dimensions, and discovered his solution of the eight-vertex model [3], [12]. The term Yang-Baxter equation was introduced in late 1970s by Faddeev to denote a principle of integrability in different fields of mathematics and physics [12]. In the past two decades, the Yang-Baxter equation has been studied extensively, and its implementations can be found in quantum mechanics, classical statistical mechanics, knot theory, braid theory, quantum groups, and other fields.

There are two forms of the YBE. The parameter-dependent form is given by

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u), \tag{1.2}$$

where $R(u)$ is a parameter-dependent invertible element of the tensor product $A \otimes A$, for some unital associative algebra A [2]. Parameters u and v are called spectral parameters. If we drop u and v in (1.2), we have the parameter-independent YBE

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{23}. \tag{1.3}$$

To introduce a more formal definition of the Yang-Baxter equation, we need to define the vector and matrix tensor products. For two vectors $u, v \in \mathbb{R}^3$, the tensor product of u and v, also known as the dyadic product [1], is given by

$$v \otimes w = \begin{pmatrix} v_1w_1 & v_1w_2 & v_1w_3 \\ v_2w_1 & v_2w_2 & v_2w_3 \\ v_3w_1 & v_3w_2 & v_3w_3 \end{pmatrix}. \tag{1.4}$$

Similarly, the tensor product of two matrices $A, B \in \mathbb{R}^3$, also known as the Kronecker product, is defined as

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B & A_{13}B \\ A_{21}B & A_{22}B & A_{23}B \\ A_{31}B & A_{32}B & A_{33}B \end{pmatrix}. \tag{1.5}$$

Thus, if A and B are $n \times n$ matrices, then $A \otimes B$ is a $nn \times nn$ block matrix. Using the definition of tensor product, let $\tau : V \otimes V \longrightarrow V \otimes V$ be the map such that $\tau(u \otimes v) = (v \otimes u)$, and let R_{12},R_{13}, and R_{23} be matrices defined as following

$$R_{12} = (R \otimes I) \tag{1.6}$$

$$R_{13} = (I \otimes \tau)(R \otimes I)(I \otimes \tau) \tag{1.7}$$

$$R_{23} = (I \otimes R). \tag{1.8}$$

The Yang-Baxter equation is given by

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \text{ [5]}. \tag{1.9}$$

As mentioned in [16], an example of a solution of (1.9) is

$$R = \begin{pmatrix} 1+u & 0 & 0 & 0 \\ 0 & u & 1 & 0 \\ 0 & 1 & u & 0 \\ 0 & 0 & 0 & 1+u \end{pmatrix}.$$

Equation (1.9) is a fundamental result with implementations in many different fields. It can be seen in Hopf algebras, statistical mechanics, and quantum groups, but one of it's most important applications is in the braid group theory. We will briefly introduce the Artin's braid groups and the representation of the Yang-Baxter equation in this field of mathematics.

An n-strand *Artin braid group* is generated by $\{\sigma_i | 1 \leq i \leq n-1\}$, with the following relationships

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},$$

for all $1 \leq i \leq n-1$, and

$$\sigma_i \sigma_j = \sigma_j \sigma_i,$$

when $|i-j| \geq 1$. If each n-strand braid represents a linear mapping from $V^{\otimes n}$ to $V^{\otimes n}$, with generator σ_i associated with the map $R : V \otimes V \longrightarrow V \otimes V$ such that:

$$\sigma_i \longmapsto I \otimes I \ldots \otimes R \otimes \ldots I \otimes I, \tag{1.10}$$

where I is the identity map and R is in the ith position [5], then the three strand case yields the Yang-Baxter equation

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R). \tag{1.11}$$

Since R is a linear map, it may be expressed as a matrix. An example [7] of a braid matrix is

$$R = \begin{pmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

1.2 Problem Framework

While the structure of the Yang-Baxter problem is typically found embedded in a comprehensive framework with problems of interest arising in numerous fields of mathematics, and mathematical physics, the matrix representation is also of interest. Rather than examining the more detailed algebraic structures resulting from the tensor products, along with the need to construct the R-matrices, the problem can be viewed entirely within the framework of the algebraic products of matrices, i.e., the composition of linear maps.

In this paper, we analyze the Yang-Baxter equation specialized to matrices $A : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, $B : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, having the following form

$$ABA = BAB. \tag{1.12}$$

We seek to characterize solutions of (1.12), including finding the necessary and if possible sufficient conditions under which distinct matrices A and B satisfy (1.12). In that regard, the approach is not too dissimilar to analyzing the structure of $AB = BA$, i.e., determining when two distinct matrices commute, and in that sense it seems appropriate to coin the usage that two distinct matrices form a braid, or more simply braid if they satisfy (1.12).

In examining double matrix product associated with the question of commutativity, or triple products associated with braiding, it is important that the matrices be square, i.e., that the row and column dimensions be equal. This is because $C_{mn} = A_{mn}B_{nm}A_{mn}$ while $D_{mn} = B_{nm}A_{mn}B_{nm}$, and C_{mn} and D_{nm} are not conforming matrices.

If A and B are invertible, an obvious necessary condition to have braiding matrices, i.e., to satisfy (1.12), is that the determinants of the resulting two matrices ABA and BAB be equal. Since the determinant of a product of matrices is equal to the product of the determinants of individual matrices, we have

$$\det(A)\det(B)\det(A) = \det(B)\det(A)\det(B).$$

This implies $\det(A) = \det(B)$, i.e., the determinants of the two matrices A and B must be equal. If we consider the fact that the determinant of a matrix is equal to the product of the eigenvalues of the matrix, then our necessary condition $\det(A) = \det(B)$ can be expressed as

$$\prod_i \lambda_i(A) = \prod_j \lambda_j(B),$$

where λ_i and λ_j are all the eigenvalues of A and B, including multiplicities.

1.3 Diagonal Matrices

The complexity of the Yang-Baxter equation is due the fact that the equation represents a combination of linear maps, i.e., ABA is equivalent to $F_A(F_B(F_A))$. If we attempt a direct computational approach to this problem based on multiplying out the coefficients of the matrices, the resulting system is computationally complex (as will be seen in Chapter 2), and finding solutions is computationally challenging. The complexity of the problem, however, is reduced greatly when we deal with special matrices. In the case of diagonal matrices, or for that matter simultaneously diagonalizable matrices, the Yang-Baxter equation becomes a scalar problem rather than a vector problem.

Theorem 1.3.1. *If A and B are two simultaneously diagonalizable matrices that satisfy the Yang-Baxter equation, then $A = B$.*

Proof. Suppose that matrices A and B are simultaneously diagonalizable, i.e., there exists a matrix P such that $\Lambda_A = P^{-1}AP$ and $\Lambda_B = P^{-1}BP$, where Λ_A and Λ_B are diagonal matrices. If we assume that $ABA = BAB$ then we have

$$P\Lambda_A P^{-1} P\Lambda_B P^{-1} P\Lambda_A P^{-1} = P\Lambda_B P^{-1} P\Lambda_A P^{-1} P\Lambda_B P^{-1}. \tag{1.13}$$

Since $P^{-1}P = I$, the above equation can be written as

$$P\Lambda_A \Lambda_B \Lambda_A P^{-1} = P\Lambda_B \Lambda_A \Lambda_B P^{-1}. \tag{1.14}$$

Multiplying both side by P^{-1} on the left and P on the right, the equation reduces to

$$\Lambda_A \Lambda_B \Lambda_A = \Lambda_B \Lambda_A \Lambda_B. \tag{1.15}$$

We can rewrite (1.15) as a system of n equations, where n is the size of A and B, and the i-th equation is given by

$$a_i^2 b_i = a_i b_i^2. \tag{1.16}$$

Here a_i and b_i are the i-th elements of the diagonals of Λ_A and Λ_B. Equation (1.16) implies $a_i = b_i$, i.e., $\Lambda_A = \Lambda_B$. This means that $A = B$. □

Other special matrices and their solutions of the Yang-Baxter equations are discussed in greater detail in Chapter 3. At this point, however we make an important observation concerning the structure of the Yang-Baxter equations.

1.4 The Cecioni Frobenius Theorem

Consider solving the Yang-Baxter equation, $ABA = BAB$, by letting

$$W = BA, \tag{1.17}$$

yielding the revised form of the problem

$$AW = WB. \tag{1.18}$$

Observe that by regrouping (1.17), an alternative factoring can be achieved by letting $V = AB$, yielding

$$VA = BV. \tag{1.19}$$

However, equations 1.18 and 1.19 are equivalent.

The following two theorems give us some information on the solutions of (1.18).

Theorem 1.4.1. *The equation $AX = XB$ has a non-zero solution if and only if A and B have a characteristic root in common [13].*

Theorem 1.4.2. *(Cecioni-Frobenius) The number of linearly independent solutions of the equation $AX = XB$ is $\sum_{i=1}^{n}\sum_{j=1}^{n} e_{ij}$, where e_{ij} is the degree of the greatest common divisor of the invariant factor a_i of $\lambda I - A$, and the invariant factor b_i of $\lambda I - B$ [10].*

Clearly, this applies to solutions of the Yang-Baxter equation, however the deeper problem of demonstrating the existence of a factorisable $W = BA$ remains. Note that if W is invertible, then

$$A = WBW^{-1} \text{ and } B = W^{-1}AW, \tag{1.20}$$

i.e., A is W similar to B and B is W similar to A. Obviously A and B will have a common eigenvalue which is consistent with Theorem 1.4.1.

Solving problems of the form alluded to in 1.4.1 and 1.4.2, i.e., $AW = WB$ will be denoted as the Cecioni-Frobenius equation (CFE). This form still poses difficulties in solving the YBE due to the requirements that a factorization of W must yield the matrices A and B when a solution W of the CFE is found, however for a class of problems involving the Yang-Baxter equation, this factorization can be achieved without much difficulty.

Suppose $A = (a_{ij})$ and $B = (b_{ij})$ are doubly-stochastic matrices, i.e., $\sum_{i=1}^{n} a_{ij} = \sum_{i=1}^{n} b_{ij} = 1$ and $\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} b_{ij} = 1$. In order to use (1.18) to construct doubly stochastic solutions of the Yang-Baxter equation, we need to find factorization such that $W = BA$. One such matrix is $W = U/n$, where U is the $n \times n$ unit matrix, i.e., the matrix in which all coefficients are ones.

Theorem 1.4.3. *Let A and B be $n \times n$ doubly stochastic matrices. If $B = W = U/n$, where U is the unit matrix, then A and B are solutions of 1.18.*

Proof. Since A and B are doubly-stochastic, the column vector $(1 \dots 1)^{\mathrm{T}}$ is a right eigenvector for A or B with eigenvalue 1, and the row vector $(1 \dots 1)$ is a left eigenvector for A or B with eigenvalue 1. Thus, the matrix U, in which every entry is one, is a left eigenvector matrix for A or B with eigenvalue 1, and is also a right eigenvector matrix for A or B with eigenvalue 1. This means that $W = U/n$, is also a right and left eigenvector matrix for A or B, i.e., $AW = BW = WA = WB = W$. Let $B = W$. Since $WA = W$, then for this choice of B, $BA = W$, and so $AW = WB$.

□

Observe that in Theorem 1.4.3, B is not only a symmetric doubly stochastic matrix, but also an idempotent matrix. Idempotent matrices are analyzed in more detail in Chapter 3.

Note that the set of all $n \times n$ doubly stochastic matrices is the convex hull of the set of all $n \times n$ permutation matrices, i.e., $A = \sum_{i}^{n} \alpha_i P_i$, where P_i is a permutation matrix.

Corollary 1.4.4. *If A and B are doubly stochastic matrices and at least one of them is equal to W, then A and B satisfy the Yang-Baxter equation.*

Proof. Assume $B = W$, then by Theorem (1.4.3), A and B are solutions of 1.18. Since $BA = W$ and $AW = WB$, then $ABA = BAB$. □

Another approach to analyzing (1.18) is by considering the eigenvector problem

$$(AW - \lambda WB)x_i = 0. \tag{1.21}$$

For the case when $\lambda = 1$, (1.21) becomes

$$(AW - WB)x_i = 0. \tag{1.22}$$

If $\{x_i\}_{i=1}^{n}$ are the linearly independent eigenvectors that span $\mathbb{R}^n$, then

$$(AW - WB)X = 0, \tag{1.23}$$

where X is the matrix formed from the column vectors x_i. Since $x_i \neq 0$, provided that W can be factored into BA, we have a solution of the Yang-Baxter equation.

1.5 Our Proposed Research

For the remainder of this thesis, we examine the Yang-Baxter equation in terms of algebraic product of matrices, rather than analyzing the detailed algebraic structures resulting from the tensor product as mentioned in sections (1.1) and (1.2). We analyze different cases in terms of algebraic properties to derive some general solutions.

In Chapter 2, we examine 2×2 triangular matrices and their structure, explicitly computing the general solutions for 2×2 matrices. The purpose of this exercise, as discussed, is to demonstrate the complexity of this direct approach, and to amply illustrate that this approach will not scale well for matrices of dimension larger than 2×2. Furthermore, there is some utility in this work, allowing some more interested results to be verified with some elementary 2×2 examples.

In Chapter 3 we build on some of the ideas introduced in this chapter related to special matrices, and use the properties of select special matrices to arrive at a general solution of the Yang-Baxter equation for these cases. Some special matrices which are considered include idempotent and nilpotent matrices. Finally, in Chapter 4, we derive a fixed point iteration algorithm that can be used to determine the Yang-Baxter complement of a given matrix, i.e., a matrix B such that for a given matrix A, A and B satisfy the Yang-Baxter equation, if there is a solution.

Chapter 2

THE 2 BY 2 CASE

2.1 Triangular Matrices

In this section, we look at the special case when A and B are triangular matrices, later generalizing the result for arbitrary 2×2 matrices. While it would seem pointless to do the triangular case given the more general case, our driving purpose has been to attempt to introduce simplifications which have the potential to be expanded beyond the case of 2×2 matrices.

Suppose that the matrices A and B are both lower triangular (the analysis for upper triangular matrices is similar). In our study of the 2×2 lower triangular matrices, there are several cases to consider.

Case 1: Both matrices are invertible

For simplicity, we first assume that the matrices are invertible, i.e., $a_1 \neq 0$, $a_2 \neq 0$, $b_1 \neq 0$, and $b_2 \neq 0$. Let

$$A = \begin{pmatrix} a_1 & 0 \\ c & a_2 \end{pmatrix} \text{ and } B = \begin{pmatrix} b_1 & 0 \\ d & b_2 \end{pmatrix}. \tag{2.1}$$

Note that c and d are not equal to zero, because in this case A and B will be diagonal (see sec. 1.3). Assuming that $ABA = BAB$, we get the following equation

$$\begin{pmatrix} a_1^2 b_1 & 0 \\ a_1^2 b_1 + a_1 cd + a_2 b_2 c & a_2^2 b_2 \end{pmatrix} = \begin{pmatrix} b_1^2 a_1 & 0 \\ a_1 b_1^2 + b_1 cd + a_2 b_2 d & a_2 b_2^2 \end{pmatrix}, \tag{2.2}$$

which results in a system of 3 equations with 6 unknowns

$$\left\{ \begin{aligned} a_1^2 b_1 &= a_1 b_1^2, && (2.3) \\ a_2^2 b_2 &= a_2 b_2^2, && (2.4) \\ a_1^2 b_1 + a_1 cd + a_2 b_2 c &= a_1 b_1^2 + b_1 cd + a_2 b_2 d. && (2.5) \end{aligned} \right.$$

We shall ignore the trivial case when $a_1 = b_1$, $a_2 = b_2$, and $c = d$, as this results in $A = B$.

Theorem 2.1.1. *If two $n \times n$ invertible triangular matrices A and B are solutions of the Yang-Baxter equation, then $a_{11} = b_{11}$ and $a_{nn} = b_{nn}$, where a_{ij} and b_{ij} are the corresponding entries of A and B.*

Proof. Suppose A and B are two $n \times n$ invertible lower triangular matrices (the proof is similar for upper triangular matrices). Since each matrix may be scaled by a single constant, let $A = a_{11}A'$ and $B = b_{11}B'$. The Yang-Baxter equation will be given by

$$a_{11}^2 b_{11} A'B'A' = a_{11} b_{11}^2 B'A'B'. \tag{2.6}$$

Both A' and B' have the first column starting with a one, followed by $n-1$ zeros. Since the product of two lower triangular matrices produces a lower triangular matrix, then $A'B'$ will be a lower triangular matrix with a one as the first entry of the main diagonal, due to the fact that $a'_1 \cdot b'_1 = 1$, where a'_1 is the first row of A', and b'_1 is the first column of B'. Similarly, $A'B'A'$ and $B'A'B'$ will have a one in the first entry of the main diagonal. This implies that $a_{11}^2 b_{11} = a_{11} b_{11}^2$, i.e., $a_{11} = b_{11}$.

On the other hand, if we scale A and B such that $A = a_{nn}A''$ and $B = b_{nn}B''$, then the Yang-Baxter equation will be

$$a_{nn}^2 b_{nn} A''B''A'' = a_{nn} b_{nn}^2 B''A''B''. \tag{2.7}$$

The lower triangular matrix $A''B''$ will have a one as the last entry of the main diagonal because $a''_n \cdot b''_n = 1$, where a''_n is the nth row of A'' and, b''_n is the nth column of B''. This means that $A''B''A''$ and $B''A''B''$ will also have ones in the last entries of the main diagonal, implying $a_{nn}^2 b_{nn} = a_{nn} b_{nn}^2$, i. e. $a_{nn} = b_{nn}$. □

For the 2×2 invertible triangular matrices, by Theorem 2.1.1, the entries on the main diagonal of A and B will be equal. We want to scale A and B so that $a_1 = b_1 = 1$. Since the two scaling constants are equal, we will ignore them in our further analysis of the invertible triangular matrices and will refer to a_1, b_1, a_2, b_2, c, and d as to the scaled version of the entries of the two matrices. Then the Yang-Baxter equation becomes

$$\begin{pmatrix} 1 & 0 \\ c + a_2 d + a_2 b_2 c & a_2^2 b_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ d + b_2 c + a_2 b_2 d & a_2 b_2^2 \end{pmatrix}, \tag{2.8}$$

giving us the following system of equations

$$\begin{cases} c + a_2 d + a_2 b_2 c = d + b_2 c + a_2 b_2 d, & (2.9) \\ a_2^2 b_2 = a_2 b_2^2. & (2.10) \end{cases}$$

From (2.10) we get that $a_2 = b_2$. Let $a_2 = b_2 = x$. We substitute it into (2.9) to obtain

$$c + xd + x^2 c = d + xc + x^2 d, \tag{2.11}$$

which is equivalent to

$$c(x^2 - x + 1) = d(x^2 - x + 1). \tag{2.12}$$

If $x^2-x+1\neq 0$, we obtain the trivial case $c=d$, which implies that $A=B$. On the other hand, if $x^2-x+1=0$, we have complex solutions $x=\frac{1}{2}\pm\frac{i\sqrt{3}}{2}$. In this case, our two matrices A and B are

$$A=\begin{pmatrix}1 & 0\\ c & \frac{1\pm i\sqrt{3}}{2}\end{pmatrix} \text{ and } B=\begin{pmatrix}1 & 0\\ d & \frac{1\pm i\sqrt{3}}{2}\end{pmatrix}, \tag{2.13}$$

and (1.12) holds for any values of c and d.

Now, assume $a_2=b_2=1$. The Yang-Baxter equation becomes

$$\begin{pmatrix}a_1^2b_1 & 0\\ a_1b_1c+a_1d+c & 1\end{pmatrix}=\begin{pmatrix}a_1b_1^2 & 0\\ a_1b_1d+b_1c+d & 1\end{pmatrix}, \tag{2.14}$$

From (2.14) we get the following system of equations

$$\begin{cases} a_1^2b_1 &= a_1b_1^2, & (2.15)\\ a_1b_1c+a_1d+c &= a_1b_1d+b_1c+d. & (2.16)\end{cases}$$

We notice that from (2.15) we get $a_1=b_1$, and the solution is the antitransverse of the solution (2.13). Thus, if we consider A and B to be matrices over the reals, we obtain the trivial case. On the other hand, if A and B are complex matrices, then we have $a_1=b_1=\frac{1\pm i\sqrt{3}}{2}$, and (1.12) holds regardless of the values of c and d. Therefore, the general solution for the first case when A and B are both invertible is

$$A=\begin{pmatrix}a & 0\\ c & \frac{1\pm i\sqrt{3}}{2}a\end{pmatrix} \text{ and } B=\begin{pmatrix}a & 0\\ d & \frac{1\pm i\sqrt{3}}{2}a\end{pmatrix}, \tag{2.17}$$

as is its antitranspose.

Case 2: Both matrices are non-invertible

We now analyze the case when both matrices are non-invertible. Since A and B are triangular matrices, they have to have a zero on the main diagonal in order to be non-invertible. Suppose A and B are non-invertible, lower triangular matrices, i.e. let $a_1=b_1=0$. Consequently, (1.12) becomes

$$\begin{pmatrix}0 & 0\\ a_2b_2c & a_2^2b_2\end{pmatrix}=\begin{pmatrix}0 & 0\\ a_2b_2d & a_2b_2^2\end{pmatrix}, \tag{2.18}$$

yielding the following system of equations

$$\begin{cases} a_2^2b_2 &= a_2b_2^2, & (2.19)\\ a_2b_2c &= a_2b_2d. & (2.20)\end{cases}$$

From this system, we see that $a_2=b_2$ and $c=d$, and thus, we have another trivial case.

If we let $a_2 = b_2 = 0$, then (1.12) becomes

$$\begin{pmatrix} a_1^2 b_1 & 0 \\ a_1 b_1 c & 0 \end{pmatrix} = \begin{pmatrix} a_1 b_1^2 & 0 \\ a_1 b_1 d & 0 \end{pmatrix}, \tag{2.21}$$

The solution of (2.21) is similar to the solution of (2.18), resulting in the same trivial case $A = B$.

Now, we consider the possibility when $a_1 = b_2 = 0$ (similar for $a_2 = b_1 = 0$). The Yang-Baxter equation simply becomes

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.22}$$

As we can see, in this case (1.12) holds regardless of the values of c, d, a_2, or b_1. Thus we have an infinite number of solutions. For example:

$$A = \begin{pmatrix} 0 & 0 \\ 1 & -154 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 0 \\ \pi & 0 \end{pmatrix},$$

with

$$ABA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } BAB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

or for example

$$A = \begin{pmatrix} -4 & 0 \\ 111 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 8 & 3 \end{pmatrix},$$

$$ABA = BAB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

From these examples we can conclude that for 2×2 triangular, non-invertible matrices, the Yang-Baxter equation holds whenever the zeros on the main diagonal for the two matrices are in opposite corners, regardless of the values of the rest of the elements of the matrices.

Case 3: Only one of the two matrices is non-invertible

The last case we shall analyze is when only one of the two matrices is non-invertible. For example, let

$$A = \begin{pmatrix} 0 & 0 \\ c & a_2 \end{pmatrix} \text{ and } B = \begin{pmatrix} b_1 & 0 \\ d & b_2 \end{pmatrix}. \tag{2.23}$$

In this case, (1.12) becomes

$$\begin{pmatrix} 0 & 0 \\ a_2 b_2 c & a_2^2 b_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b_1 b_2 c + a_2 b_2 d & a_2 b_2^2 \end{pmatrix}, \tag{2.24}$$

resulting in the following system of equations

$$\begin{cases} a_2^2 b_2 = a_2 b_1^2, & (2.25) \\ a_2 b_2 c = b_1 b_2 c + a_2 b_2 d. & (2.26) \end{cases}$$

From (2.25), it is obvious that $a_2 = b_2$. If we let $a_2 = b_2 = x$, then (2.26) simply becomes

$$cx^2 = b_1cx + dx^2, \tag{2.27}$$

which is equivalent to

$$(c-d)x^2 - b_1cx = 0, \tag{2.28}$$

or

$$x((c-d)x - b_1c) = 0. \tag{2.29}$$

The solutions of (2.29) are $x = 0$ and $x = \dfrac{b_1c}{c-d}$, where $c \neq d$. If $x = 0$, then we have

$$A = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} b_1 & 0 \\ d & 0 \end{pmatrix},$$

and the Yang-Baxter equation simply becomes

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.30}$$

with an infinite number of solutions for b_1, c, and d. On the other hand, if $x = \dfrac{b_1c}{c-d}$, then

$$A = \begin{pmatrix} 0 & 0 \\ c & \dfrac{b_1c}{c-d} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_1 & 0 \\ d & \dfrac{b_1c}{c-d} \end{pmatrix}. \tag{2.31}$$

For example, let

$$A = \begin{pmatrix} 0 & 0 \\ 2 & 6 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & 0 \\ 1 & 6 \end{pmatrix},$$

then $ABA = BAB$ yields

$$\begin{pmatrix} 0 & 0 \\ 72 & 216 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 72 & 216 \end{pmatrix}.$$

If we let $a_2 = 0$, then we have

$$A = \begin{pmatrix} a_1 & 0 \\ c & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} b_1 & 0 \\ d & b_2 \end{pmatrix}, \tag{2.32}$$

and the Yang-Baxter equation becomes

$$\begin{pmatrix} a_1^2b_1 & 0 \\ a_1b_1c & 0 \end{pmatrix} = \begin{pmatrix} a_1b_1^2 & 0 \\ a_1b_1d + b_1b_2c & 0 \end{pmatrix}, \tag{2.33}$$

yielding the following system

$$\begin{cases} a_1^2b_1 = a_1b_1^2, & (2.34) \\ a_1b_1c = b_1b_2c + a_1b_1d. & (2.35) \end{cases}$$

From (2.34) we have $a_1 = b_1$. Again, if we let $a_1 = b_1 = x$, then (2.35) can be written as

$$x^2c - b_2cx - dx^2 = 0, \tag{2.36}$$

which is equivalent to

$$x(x(c-d) - b_2c) = 0. \tag{2.37}$$

The solution of (2.37) is $x = 0$ and $x = \dfrac{b_2c}{c-d}$, provided that $c \neq d$. If we have $x = 0$, then (1.12) is simply (2.30). On the other hand, if $x = \dfrac{b_2c}{c-d}$, then A and B are

$$A = \begin{pmatrix} \dfrac{b_2c}{c-d} & 0 \\ c & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} \dfrac{b_2c}{c-d} & 0 \\ d & b_2 \end{pmatrix}, \tag{2.38}$$

and

$$ABA = BAB = \begin{pmatrix} \dfrac{b_2^3c^3}{(c-d)^3} & 0 \\ \dfrac{b_2^2c^3}{(c-d)^2} & 0 \end{pmatrix}, \tag{2.39}$$

for any real b_2, c, and d, where $c \neq d$.

Next, we look at the possibility when one of the two matrices is singular, with both the diagonal entries being zero. Note that if both matrices are singular with zeros on the main diagonal, then (1.12) is simply (2.30). Let

$$A = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} b_1 & 0 \\ d & b_2 \end{pmatrix}.$$

In this case the Yang-Baxter equation is

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b_1b_2c & 0 \end{pmatrix}. \tag{2.40}$$

We can see that in (2.40) at least one of the variables b_1, b_2, or c must be equal to zero. This means that either B is diagonal, or B is non-invertible as well. The analysis is similar for the case when

$$B = \begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix}.$$

Note that if matrices A and B are simultaneously triangularisable, i.e. there exists a matrix P such that $A = PT_AP^{-1}$ and $B = PT_BP^{-1}$, where T_A and T_B are triangular matrices, then the Yang-Baxter equation will be

$$PT_AP^{-1}PT_BP^{-1}PT_AP^{-1} = PT_BP^{-1}PT_AP^{-1}PT_BP^{-1}. \tag{2.41}$$

Since $PP^{-1} = P^{-1}P = I$, then (2.41) is reduced to

$$PT_AT_BT_AP^{-1} = PT_BT_AT_BP^{-1}, \tag{2.42}$$

which is equivalent to

$$T_AT_BT_A = T_BT_AT_B. \tag{2.43}$$

This means that simultaneously triangularisable matrices will have the same solutions of the Yang-Baxter equation as the triangular matrices.

2.2 Solution

In this section we give an explicit general solution of the Yang-Baxter equation for the 2×2 case. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

A direct approach of trying to solve the Yang-Baxter equation by matching the entries of the matrices ABA and BAB, yields a system of 8 equations and only 4 unknowns. This means that we would not have a unique solution, but rather a set of solutions that depends on 4 free parameters.

The system of equations that results from $ABA = BAB$ is

$$\begin{cases} a^2x + abz + acy + bcw = ax^2 + cxy + bxz + dyz, & (2.44) \\ acx + adz + c^2y + cdw = axz + cxw + bz^2 + dzw, & (2.45) \\ abx + b^2z + ady + bdw = axy + cy^2 + bxw + dyw, & (2.46) \\ bcx + bdz + dcy + d^2w = azy + cyw + bzw + dw^2. & (2.47) \end{cases}$$

A class of non-trivial solution of the above system can be characterized by by one of the two matrices having a column of zeros, while the other matrix has an opposite row of zeros, as seen in Section 2.1. For example, let

$$A = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \text{ and } B = \begin{pmatrix} e & f \\ 0 & 0 \end{pmatrix}. \tag{2.48}$$

It is easy to verify that (1.12) is zero, i.e.,

$$ABA = BAB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.49}$$

Other nontrivial solutions of the Yang-Baxter equation are more difficult to derive analytically. We used Maple 14 to find the rest of the nontrivial solutions. One of the sets of solutions given by Maple can be characterized by the following example

$$A = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \text{ and } B = \begin{pmatrix} e & \dfrac{b(d-e)}{d} \\ 0 & d \end{pmatrix}. \tag{2.50}$$

The Yang-Baxter equation for this example is given by

$$ABA = BAB = \begin{pmatrix} 0 & bd^2 \\ 0 & d^3 \end{pmatrix}. \tag{2.51}$$

We notice that A^{T} and B^{T} are also solutions of (1.12). This true in general, not just for this specific example. It is due to the fact that

$$(ABA)^{\mathrm{T}} = (BAB)^{\mathrm{T}} \implies (BA)^{\mathrm{T}}A^{\mathrm{T}} = (AB)^{\mathrm{T}}B^{\mathrm{T}} \implies A^{\mathrm{T}}B^{\mathrm{T}}A^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}B^{\mathrm{T}}. \tag{2.52}$$

The antitransposes of A and B are also solutions of the Yang-Baxter equation, i.e.,

$$A^{\mathrm{A}} = \begin{pmatrix} d & b \\ 0 & 0 \end{pmatrix} \text{ and } B^{\mathrm{A}} = \begin{pmatrix} d & \dfrac{b(d-e)}{d} \\ 0 & e \end{pmatrix} \tag{2.53}$$

satisfy (1.12), and by (2.52), so do $(A^{\mathrm{A}})^{\mathrm{T}}$ and $(B^{\mathrm{A}})^{\mathrm{T}}$.

Another set of solutions given by Maple is

$$A = \begin{pmatrix} 0 & -\dfrac{dh}{g} \\ 0 & d \end{pmatrix} \text{ and } B = \begin{pmatrix} e & \dfrac{he}{g} \\ g & h \end{pmatrix}. \tag{2.54}$$

In this case the Yang-Baxter equation is equal to 0. The rest of the solutions of the 2×2 case are given by the following matrices, provided that the denominators do not vanish:

(i)

$$A = \begin{pmatrix} \dfrac{1 \pm i\sqrt{3}}{2}h & 0 \\ c & h \end{pmatrix}, \quad B = \begin{pmatrix} \dfrac{1 \pm i\sqrt{3}}{2}h & 0 \\ g & h \end{pmatrix}; \tag{2.55}$$

(ii)

$$A = \begin{pmatrix} e+h & 0 \\ 0 & e+h \end{pmatrix}, \quad B = \begin{pmatrix} e & \dfrac{eh}{g} \\ g & h \end{pmatrix}; \tag{2.56}$$

(iii)

$$A = \begin{pmatrix} \dfrac{e+h}{2} & -\dfrac{(e+h)^2}{4g} \\ 0 & \dfrac{e+h}{2} \end{pmatrix}, \quad B = \begin{pmatrix} e & -\dfrac{(e-h)^2}{4g} \\ g & h \end{pmatrix}; \tag{2.57}$$

(iv)

$$A = \begin{pmatrix} e+h-d & -\dfrac{cd^2-cde-cdh+ceh+egh+g(e-d)^2}{cg+c^2+g^2} \\ c & d \end{pmatrix},$$

$$B = \begin{pmatrix} e & -\dfrac{cdh-cde-d^2g+ce^2-egh+deg+dgh}{cg+c^2+g^2} \\ g & h \end{pmatrix}; \tag{2.58}$$

(v)

$$A = \begin{pmatrix} a & \dfrac{ad}{c} \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} -\dfrac{dg}{c} & -\dfrac{dh}{c} \\ g & h \end{pmatrix}; \tag{2.59}$$

(vi)

$$A = \begin{pmatrix} a & \dfrac{(ga+hc-hg)hc}{g^3} \\ c & -\dfrac{cga-g^2a+c^2h-2gch}{g^2} \end{pmatrix},$$

$$B = \begin{pmatrix} \dfrac{ga+hc-hg}{g} & \dfrac{h(ga+hc-hg)}{g^2} \\ g & h \end{pmatrix}; \tag{2.60}$$

(vii)

$$A = \begin{pmatrix} a & -\dfrac{-ac^2h+a^2c^2-acgh+cgh^2+ag^2h}{c^2g+c^3+cg^2} \\ c & -\dfrac{ac-ag-ch}{c}, \end{pmatrix},$$

$$B = \begin{pmatrix} \dfrac{ag}{c} & -\dfrac{-ac^2h+a^2g^2-ag^2h+acgh+c^2h^2}{c^2g+c^3+cg^2} \\ g & h \end{pmatrix}; \tag{2.61}$$

(viii)

$$A = \begin{pmatrix} -\dfrac{ch}{g} & -\dfrac{dh}{g} \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} -\dfrac{dg^2+cdg-cgh}{c^2} & -\dfrac{dgh+cdh-ch^2}{c^2} \\ g & h \end{pmatrix}; \tag{2.62}$$

(ix)

$$A = \begin{pmatrix} a & b \\ \dfrac{-1 \pm i\sqrt{3}}{2}g & -a \end{pmatrix},$$

$$B=\begin{pmatrix} -\dfrac{1\pm i\sqrt{3}}{2}a & \dfrac{-2bg+a^2(3\pm i\sqrt{3})}{g(1\pm i\sqrt{3})} \\ g & \dfrac{1\pm i\sqrt{3}}{2}a \end{pmatrix}; \tag{2.63}$$

(x)

$$A=\begin{pmatrix} a & b \\ \dfrac{-1\pm i\sqrt{3}}{2}g & d \end{pmatrix},$$

$$B=\begin{pmatrix} -\dfrac{-1\pm i\sqrt{3}}{2}a+d & \dfrac{-2ad-2bg+a^2(1\pm i\sqrt{3})}{g(1\pm i\sqrt{3})} \\ g & \dfrac{1\pm i\sqrt{3}}{2}a \end{pmatrix}; \tag{2.64}$$

(xi)

$$A=\begin{pmatrix} a & b \\ \dfrac{-1\pm i\sqrt{3}}{2}g & d \end{pmatrix},$$

$$B=\begin{pmatrix} -(a-h)\dfrac{-1\pm i\sqrt{3}}{2} & \dfrac{2ah-2bg-2h^2+(-1\pm i\sqrt{3})(a^2-ah))}{g(1\pm i\sqrt{3})} \\ g & h \end{pmatrix}. \tag{2.65}$$

Note, that the totality of all the solutions include the transpose and antitranspose of all of the matrices listed in (i)–(xi).

We can see that these solutions for the general 2×2 case are consistent when the matrices A and B are symmetric. For example,

$$A=\begin{pmatrix} \dfrac{g^2+h^2}{h} & 0 \\ 0 & \dfrac{g^2+h^2}{h} \end{pmatrix} \text{ and } B=\begin{pmatrix} \dfrac{g^2}{h} & g \\ g & h \end{pmatrix} \tag{2.66}$$

are solutions of the Yang-Baxter equation and can be derived from (2.56) by letting $eh=g^2$, so that matrix B becomes symmetric. Another nontrivial example is

$$A=\begin{pmatrix} a & c \\ c & \dfrac{c^2}{a} \end{pmatrix} \text{ and } B=\begin{pmatrix} \dfrac{c^2h}{a^2} & -\dfrac{ch}{a} \\ -\dfrac{ch}{a} & h \end{pmatrix} \tag{2.67}$$

that is derived from (2.59) by letting $da=c^2$ and $-\dfrac{dh}{c}=g$.

Unfortunately we cannot derive symmetric solutions from examples like (2.58) analytically, because the two equations that we have to solve involve 5 unknowns, i.e.,

$$c = -\frac{cd^2 - cde - cdh + ceh + egh + g(e-d)^2}{cg + c^2 + g^2}, \tag{2.68}$$

and

$$g = -\frac{cdh - cde - d^2 g + ce^2 - egh + deg + dgh}{cg + c^2 + g^2}. \tag{2.69}$$

Solutions for (2.68) and (2.69) can be obtained using a symbolic algebra tool such as Maple, or Mathematica.

Also, we can see that the general solutions for 2×2 matrices is consistent with the solutions of the triangular matrices. For example, if we let $h = 1$ in (2.55), then it simply becomes (2.13).

2.3 Summary

Equating the coefficients associated with the matrix products of ABA and BAB clearly can be done for the 2×2 case because we are only having to work with four equation. Going to the 3×3 case, the complexity of the problem is more than doubled based on a count of the number of equations, and in terms of the folding the coefficients that occurs due to the matrix products, the solution approach quickly becomes untenable. Pushing the algebra through, while an intriguing exercise in complexity, and perhaps a measure of the robustness of modern symbolic algebraic tools, is certainly not of interest in developing an understanding of the underlying structure of matrix solutions of the Yang-Baxter equation. Our preferred approach using special matrices, in effect getting special solutions, and an approach using fixed point iterative methods seems potentially more productive, and thus becomes the focus of our efforts.

Chapter 3

SOLUTIONS FOR SPECIAL MATRICES

3.1 Introduction

In the previous chapter we provided the general solution of the Yang-Baxter equation for all 2×2 matrices. Even for the 2×2 case, a direct approach of trying to find solutions of the Yang-Baxter equation poses problems of complexity due to the fact that we have more unknowns than equations to solve, consequently the problem is under-determined, leading to solutions depending on free parameters. As we increase the size of the matrices, the difference between the number of equations we have to solve and the number of unknowns becomes even greater. While in principle the use of symbolic algebraic tools can provide these general solutions, the solutions of the Yang-Baxter equation for large $n \times n$ matrices becomes unwieldy even with these tools, and in general bogs down due to issues of computational complexity. Instead, we attempt to derive solutions to the Yang-Baxter equation using a different approach, examining special matrices which have properties that make a direct attack on solving the Yang-Baxter equations more amenable to solution.

Special matrices are those matrices that have certain symmetries or properties associated with their coefficients, and examples of some special matrices that we have already used to advantage in working on solutions of the Yang-Baxter equation include diagonal matrices and triangular matrices. Others that will prove interesting include nilpotent matrices, and idempotent matrices.

3.2 Circulant Matrices

Before we start the discussion on idempotent and nilpotent matrices, we will mention a special class of matrices, i.e., the circulant matrices. Circulant matrices are a particular kind of Toeplitz matrices with the same entries in each column, but each column is cyclically shifted one position downward compared to the previous column [11]. For example,

$$A = \begin{pmatrix} a & e & d & c & b \\ b & a & e & d & c \\ c & b & a & e & d \\ d & c & b & a & e \\ e & d & c & b & a \end{pmatrix}.$$

An important property of the circulant matrices is that they are diagonalized by the Fourier matrix. This property is used to prove the following theorem:

Theorem 3.2.1. *Given A and B two invertible circulant matrices, then A and B are solutions Yang-Baxter equation only when $A = B$.*

Proof. We will prove this theorem by contradiction. Assume that A and B are distinct and that they satisfy the Yang-Baxter equation. Since circulant matrices are diagonalized by the Fourier matrix, then $A = F^{-1}D_AF$ and $B = F^{-1}D_BF$, where F is the Fourier matrix. Thus, $AB = F^{-1}D_AFF^{-1}D_BF = F^{-1}D_AD_BF$. Since two diagonal matrices commute, $AB = F^{-1}D_BD_AF = F^{-1}D_BFF^{-1}D_AF = BA$. But since A and B are invertible, then $A = B$, resulting in a contradiction. □

3.3 Symmetry, Commutativity, and Invertibility

In our analysis of the Yang-Baxter equation, there are elementary cases that we can quickly dispense with. The two trivial solutions $A = B$ and $A = 0$ and/or $B = 0$ are not of interest to us, therefore when we say that a particular matrix does not have a Yang-Baxter complement, i.e., a matrix B such that for a given matrix A, A and B satisfy the YBE, we mean that it only has these two trivial solutions. A large class of matrices that yield a trivial solution are invertible matrices that commute. If A and B are invertible, and $AB = BA$, then A and B satisfy the Yang-Baxter equation because $ABA = BAB \implies ABA = ABB \implies BA = BB \implies A = B$. Another class of matrices that yield a similar trivial solution that is not necessarily obvious are symmetric orthogonal matrices and their inverses. An orthogonal symmetric matrix and its inverse satisfy the Yang-Baxter equation because by definition, $A = (A^{-1})^{\mathrm{T}}$, but since A is symmetric, $(A^{-1})^{\mathrm{T}} = A^{-1} \implies A = A^{-1}$.

Continuing the discussion on invertibility, when only one matrix is invertible, for example A, and A and B commute, then the simplified form of the Yang-Baxter equation

$$BA = BB, \tag{3.1}$$

still has the trivial solution $B = 0$. However, since there is no left cancellation, (3.1) may have nontrivial solutions. Such a class of nontrivial solutions of (3.1) can be obtained by requiring B to be an idempotent singular matrix. Note that the minimal polynomial of an idempotent matrix, i.e., the polynomial ψ of the smallest degree such that $\psi(B) = 0$ is $\psi = B^2 - B$. With the exception of the identity matrix, every idempotent matrix is singular. This trivial result is easily demonstrated, i.e.,

$$B^2 - B = 0 \implies B = I.$$

Therefore, for the remainder of this chapter, idempotent matrices will be assumed to be singular.

3.4 Idempotent Matrices

Idempotent matrices are of particular interest to us because they yield interesting nontrivial solutions of the Yang-Baxter equation.

Theorem 3.4.1. *Given a (singular) idempotent matrix B, $\exists A$, such that A and B commute, and A and B are solutions of the Yang-Baxter equation.*

Proof. We begin by constructing $A = aI + bB$. Since A is a polynomial function of B, then A and B must commute, i.e., $AB = BA$.

Next, we need to determine a and b such that (3.1) holds, i.e., we require that

$$(aI + bB)B(aI + bB) = B(aI + bB)B, \tag{3.2}$$

which yields

$$a^2B + 2abB^2 + b^2B^3 = aB^2 + bB^2. \tag{3.3}$$

Since $BB = B$, then (3.3) reduces to

$$(a+b)^2B = (a+b)B. \tag{3.4}$$

As a result, $(a+b) = (a+b)^2 \implies a+b = 1$, or $a+b = 0$. $\square$

Theorem 3.4.2. *Let matrix A be a polynomial function of degree n of an idempotent matrix B with $p(x) = a_0 + a_1x + \ldots + a_nx^n$, then A and B are solutions of the Yang-Baxter equation if $a_0 + a_1 + \ldots + a_n = 1$ or $a_0 + a_1 + \ldots + a_n = 0$.*

Proof. In Theorem 3.4.1, we have constructed A such that it is a linear polynomial function of B. If we can show that any polynomial function of B can be reduced to a polynomial of degree 1, then we can use the proof of 3.4.1 to prove that A and B are solutions of the Yang-Baxter equation, provided that A satisfies the coefficient requirement of Theorem 3.4.1.

Let $A = a_0I + a_1B + a_2B^2 + \ldots + a_nB^n$ be a polynomial function of B of degree n. Since B is idempotent, i.e., $BB = B$, $\forall n \geq 2$,

$$B^n = B^{n-2}BB = B^{n-2}B = B^{n-1}. \tag{3.5}$$

Then by induction, we have $B^n = B$, yielding

$$A = a_0I + a_1B + a_2B + \ldots + a_nB = a_0I + (a_1 + a_2 + \ldots + a_n)B = a_0I + qB, \tag{3.6}$$

where $q = a_1 + a_2 + \ldots + a_n$. If a_0 and q satisfy the algebraic expressions $a_0 + q = 1$ or $a_0 + q = 0$, then, by the proof of Theorem 3.4.1, A and B are solutions of the Yang-Baxter equation. □

To analyze the more general case when A is an arbitrary function of B, we need to introduce some definitions.

Definition 3.4.1. A function f is said to be defined on the spectrum of A if the values

$$f^{(j)}(\lambda_i), \quad j = 0, \ldots, n_i - 1, \quad i = 1, \ldots, s \tag{3.7}$$

exist, where λ_i are eigenvalues of A with the corresponding Jordan block index n_i, and s is the number of distinct eigenvalues. These are called the values of the function f on the spectrum of A [6].

Theorem 3.4.3. *Let f be defined on the spectrum of $S \in \mathbb{C}^{n\times n}$ and let ψ be the minimal polynomial of S. Then $f(S) = p(S)$, where p is the polynomial of degree less than*

$$\sum_{i=1}^{s} n_i = \deg \psi,$$

that satisfies the interpolation conditions

$$p^{(j)}(\lambda_i) = f^{(j)}(\lambda_i), \quad j = 0, \ldots, n_i - 1, \quad i = 1, \ldots, s. \tag{3.8}$$

There is a unique such p and it is know as the Hermite interpolating polynomial [6].

Using the Definition (3.4.1) and Theorem (3.4.3), we construct solutions of the Yang-Baxter equation A and B, where $A = f(B)$ is an arbitrary differentiable function of B.

Theorem 3.4.4. *Let matrix $A = f(B)$, where f satisfies the differentiability requirements of Theorem 3.4.3, and B is an idempotent matrix. If $f(1) = 1$, or $f(1) = 0$, then A and B are solutions of the Yang-Baxter equation.*

Proof. Since the idempotent matrix B is singular, the two eigenvalues of B are $\lambda_1 = 0$ and $\lambda_2 = 1$. Also, the degree of the minimal polynomial of B is 2. Thus, by Theorem 3.4.3, the polynomial p that interpolates $A = f(B)$ is a linear polynomial and is given by

$$p(B) = \frac{f(1) - f(0)}{1 - 0} B + f(0)I = (f(1) - f(0))B + f(0)I. \tag{3.9}$$

Thus, A is a linear polynomial of B, and A and B commute. By the proof or Theorem 3.4.1, A and B are solutions of the Yang-Baxter equation when $(f(1) - f(0)) + f(0) = 1$, or $(f(1) - f(0)) + f(0) = 0$, i.e., $f(1) = 1$, or $f(1) = 0$. □

For example, let

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

and let $A = f(B)$, where $f(x) = e^x$. Clearly, A and B are not solutions of the Yang-Baxter equation, because A does not satisfy the condition of Theorem 3.4.4, i.e., $f(1) \neq 1$ and $f(1) \neq 0$. On the other hand, if we choose f to be $\frac{e^x}{e}$, then

$$A = \begin{pmatrix} \frac{1}{e} & \frac{e-1}{e} \\ 0 & 1 \end{pmatrix},$$

and A and B satisfy the Yang-Baxter equation, i.e.,

$$ABA = BAB = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Similarly, if we let $f(x) = e^x - e$, then A is given by

$$A = \begin{pmatrix} 1-e & e-1 \\ 0 & 0 \end{pmatrix}.$$

Since in this case $f(1) = 0$, it is easy to see that A and B are indeed solutions of the Yang-Baxter equation, i.e.,

$$ABA = BAB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Note that if the function f satisfies the condition $f(1) = 1$, then the Yang-Baxter equation will always be equal to B. This is due to $ABA = BAB = BBA = BA = B(aI + bB) = aB + bB^2 = aB + bB = (a+b)B$, and since $f(1) = 1 \implies a + b = 1$, then $ABA = BAB = B$. On the other hand, if $f(1) = 0$, then $a + b = 0$ and $ABA = BAB = (a+b)B = 0$, i.e., the Yang-Baxter equation is always zero.

3.5 Nilpotent Matrices

Another group of interesting matrices are the nilpotent matrices. A nilpotent matrix is a square matrix A such that $A^k = 0$, for some positive integer k. The integer k is called the nilpotentcy index. Observe that if A and B are nonzero commuting matrices, and A and B are both nilpotent with index 2, then they satisfy the Yang-Baxter equation. This result is trivially derived from

$$ABA = BAB,$$

$$A^2B = AB^2 \implies 0 = 0.$$

The nilpotent matrices in triangular form are of particular interest to us. The canonical form of these matrices is the matrix that has ones on the superdiagonal, and the rest of the entries are zeros. The nilpotent index of a matrix in the canonical form can be found by adding one to the number of nonzero elements on the superdiagonal. As a conventional notation, general matrix diagonals can be specified by an index p measured relative to the main diagonal: the main diagonal has $p = 0$; the superdiagonal has $p = 1$; and the subdiagonal has $p = -1$. Nilpotent matrices of index k in upper triangular form that have $k-1$ nonzero diagonals counting from the n-th diagonal are special, because the k-fold product of such matrices is always zero, regardless of the values of the nonzero entries of the matrices. Thus, two nilpotent matrices of index 3 in such a triangular form will be solutions of the Yang-Baxter equation regardless of whether they commute or not. For example, let

$$A = \begin{pmatrix} 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 & 5 & \pi \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that A and B have to be both upper or lower triangular. This is only a special case of triangular nilpotent matrices of index 3. In general, A and B do not need to have the nonzero entries in the upper two diagonals in order to have a nilpotentcy index 3. In this case, A and B may not necessarily satisfy the Yang-Baxter equation. For example,

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 4 & 0 & 0 & 1 & 11 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The same statement is true for nilpotent matrices of index 2 or 3 in general form. For example, let

$$A = \begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix} \text{ and } B = \begin{pmatrix} 6 & 9 \\ -4 & -6 \end{pmatrix},$$

Then we have

$$ABA = \begin{pmatrix} 864 & -1296 \\ 574 & -864 \end{pmatrix}, \text{ but } BAB = \begin{pmatrix} 864 & 1296 \\ -574 & -864 \end{pmatrix}.$$

However, if either $AB = 0$ or $BA = 0$, then A and B satisfy the Yang-Baxter equation because both ABA and BAB contain them as factors.

3.6 Summary

While some of the results we examined are clearly rather elementary, others, i.e., the idempotent matrices, revealed some more interesting structure which led to some more interesting results. Simplifying the problem by exploiting the structural simplifications introduced through the use of special matrices has its merits, but it still does not address the more fundamental problem of how to find solutions to the Yang-Baxter equation for arbitrary matrices. For this, we need a more computational approach.

Chapter 4

FIXED POINT ITERATION

4.1 Introduction

Given a matrix A, we want to determine if there is a nontrivial matrix B, such that A and B are solutions of the Yang-Baxter equation. In general, a direct approach of trying to compute the matrix coefficients of the resulting equations when the matrices are multiplied out, becomes very laborious and much more complicated than in the case of 2×2 matrices. Instead, we want to develop a fixed point iteration algorithm that will guarantee that for a given matrix B, the initial matrix A will converge to the solution of the Yang-Baxter equation within a reasonable number of iterations, if such a solution exists. Recall from Section 1.4 that we can write the Yang-Baxter equation as

$$XA = BX, \tag{4.1}$$

where $X = AB$. Adding A to both sides of (4.1), we obtain $(X+I)A = BX + A$. Solving for A on the left side of the equation yields our fixed point iteration method[1]

$$A = (X+I)^{-1}(BX+A). \tag{4.2}$$

Equation (4.2) can be written as

$$A_n = (X_{n-1}+I)^{-1}(BX_{n-1}+A_{n-1}), \tag{4.3}$$

where $X_{n-1} = A_{n-1}B$, A_0 is the initial guess, and A_n is the computed solution.

The main difficulty of the algorithm in (4.3) is to ensure that $(X_{n-1}+I)$ remains invertible during the iteration. We could rewrite (4.3) as

$$A_n = (X_{n-1}+cI)^{-1}(BX_{n-1}+cA_{n-1}), \tag{4.4}$$

and for large enough values of c, the diagonal dominance of $(X_{n-1}+cI)$ will guarantee its invertibility. Another problem is how to choose the matrix A_0 so that we obtain a solution of the Yang-Baxter equation for a given B.

[1]The Matlab code is provided in the Appendix.

Numerical studies have shown that for a class of matrices B with eigenvalues less than or equal to 1, the choice of A_0 does not appreciably affect the rate of convergence of (4.3). At the same time, for matrices with eigenvalues larger than 1, the fixed point iteration method may, or may not converge. For example, if

$$B = \begin{pmatrix} 4 & 3 \\ 5 & 7 \end{pmatrix} \text{ and } A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

the fixed point iteration diverges. However, if instead

$$B = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}, \tag{4.5}$$

and we keep the same A_0, then (4.3) yields the approximate solution

$$A_n = \begin{pmatrix} 4.199998653 & 1.599999237 \\ 1.599999327 & 1.799999663 \end{pmatrix} \tag{4.6}$$

in 5 iterations. We can see that in both cases, the maximum eigenvalue of B is greater than 1, but in the first case the iteration diverges, while in the second it converges. Note that (4.5) and (4.6) are a particular example of (2.60) with $a = 4.2$, $c = 1.6$, $g = 2$, and $h = 1$, which shows that the solutions obtained from the fixed point iteration are consistent with the general solution presented in Section 2.2. Still, an important question is for which matrices does the fixed point iteration guarantee to find an approximate solution of the Yang-Baxter equation. We discuss such a class of matrices in the next section.

4.2 Fixed Point Iteration For Stochastic Matrices

The class of stochastic matrices is important because the fixed point iteration method given in (4.3) is guaranteed to find a solution. If we let A and B be stochastic matrices, then $\|A\|_\infty = \|B\|_\infty = 1$. Recall that the product of two stochastic matrices yields a stochastic matrix. Thus, $\|BX + A\|_\infty = 2$. Also, notice that $\left\|(X+I)^{-1}\right\|_\infty \geq 1/2$. Therefore,

$$\left\|(X+I)^{-1}(BX+A)\right\|_\infty \leq \left\|(X+I)^{-1}\right\|_\infty \|BX+A\|_\infty = 2p, \tag{4.7}$$

where $p \geq \dfrac{1}{2}$. If the infinity norm of the mapping (4.3) is equal to 1 at each step of the iteration, since the unit ball in $\mathbb{R}^n$ is closed, then by the Brouwer Fixed Point Theorem, our method will converge to a fixed point, which is a solution of the YBE. We will see that in some of the examples considered that this is the case.

Even though we have proved that for both A_0 and B stochastic, (4.3) is guaranteed to converge to a solution of the YBE, numerical results show that it is sufficient that only B is stochastic, and for any choice of A, the fixed point iteration will converge to a stochastic solution. However, the choice of A is important, as it may cause the fixed point iteration to increase or slow down the speed of convergence. For example, if we have the fixed matrix

$$B = \begin{pmatrix} 0.33 & 0.21 & 0.11 & 0.35 \\ 0.41 & 0.16 & 0.23 & 0.2 \\ 0.17 & 0.03 & 0.5 & 0.3 \\ 0.22 & 0.33 & 0.11 & 0.34 \end{pmatrix},$$

then for the initial choice

$$A_0 = \begin{pmatrix} 1 & 8 & 3 & 1 \\ 4 & 23 & 5 & 1 \\ 1 & 3 & 45 & 2 \\ 1 & 2 & 35 & 6 \end{pmatrix}, \tag{4.8}$$

the fixed point iteration will yield an approximate solution of the Yang-Baxter equation

$$A_n = \begin{pmatrix} 0.345897590 & 0.250356630 & 0.103689445 & 0.300056334 \\ 0.281596163 & 0.203086596 & 0.207993615 & 0.307323625 \\ 0.116252167 & 0.078676946 & 0.484729286 & 0.320341602 \\ 0.326827745 & 0.230916299 & 0.140813336 & 0.301442620 \end{pmatrix},$$

within an error of 10^{-4} in 234534 iterations.

A similar approximate solution can be obtained more quickly if

$$A_0 = \begin{pmatrix} 0.1 & 2 & 3 & 1 \\ 0 & 0.2 & 1 & 4 \\ 0 & 0 & 0.55 & 2 \\ 0 & 0 & 0 & 0.13 \end{pmatrix}. \tag{4.9}$$

For this A_0, the fixed point iteration takes 71817 iterations to find

$$A_n = \begin{pmatrix} 0.346171330 & 0.250208321 & 0.103747523 & 0.299872826 \\ 0.279873763 & 0.203811846 & 0.207669297 & 0.308645093 \\ 0.115571763 & 0.078941282 & 0.484605552 & 0.320881403 \\ 0.328177006 & 0.230393902 & 0.141058350 & 0.300370742 \end{pmatrix}.$$

Finally, for a doubly stochastic choice of the matrix A_0

$$A_0 = \begin{pmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{pmatrix}, \tag{4.10}$$

the algorithm (4.3) takes only 16 iterations to provide the solution

$$A_n = \begin{pmatrix} 0.276987314 & 0.197407476 & 0.219249228 & 0.306364982 \\ 0.276987314 & 0.197407476 & 0.219249228 & 0.306364982 \\ 0.276987314 & 0.197407476 & 0.219249228 & 0.306364982 \\ 0.276987314 & 0.197407476 & 0.219249228 & 0.306364982 \end{pmatrix}.$$

From the examples above, we can see that the choice of A_0 not only determines the speed of convergence, but also the matrix that the algorithm converges to. Also, we notice that for the choice of A_0 as in (4.8), the infinity norm of A_n after the first iteration is 25.54, and it increases before it finally goes to 1 after more than 230000 iterations. If we choose A_0 as in (4.9), the infinity norm of A_n starts at 3.30 and it increases again before it drops to 1 after 70000 iterations. Finally, for the choice of A_0 as in (4.10), $\|A_n\|_\infty = 1$ at each step of the iteration. This seems to indicate that the rate of convergence of the fixed point algorithm (4.3) is strongly dependent on the magnitude of the norm of the matrix A_n at each step of the iteration. Consequently, when $\|A_n\|_\infty = 1$ for each n, the fixed point seems to converge rapidly.

In the case of a doubly stochastic matrix B, we choose another doubly stochastic matrix as our initial guess for A_0, and see that fixed point iteration converges fast. For example, for

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0.2 & 0 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0 & 0.2 & 0.2 & 0.2 & 0.2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

the fixed point iteration converges to an approximate solution

$$A = \begin{pmatrix} -0.000000097 & 1 & 0.000000024 & 0.000000024 & 0.000000024 & 0.000000024 \\ 0.999999792 & 0 & 0.000000052 & 0.000000052 & 0.000000052 & 0.000000052 \\ 0.000000076 & 0 & 0.249999981 & 0.249999981 & 0.249999981 & 0.249999981 \\ 0.000000076 & 0 & 0.249999981 & 0.249999981 & 0.249999981 & 0.249999981 \\ 0.000000076 & 0 & 0.249999981 & 0.249999981 & 0.249999981 & 0.249999981 \\ 0.000000076 & 0 & 0.249999981 & 0.249999981 & 0.249999981 & 0.249999981 \end{pmatrix},$$

in 22 iterations with an error of 10^{-6}. Note that if either A_0 or B is equal to U/n, where U is the unit matrix, then by Theorem 1.4.3, A_0 and B are solutions of the Yang-Baxter equation already. However, if $B = U/n$, and A_0 is just a row stochastic matrix rather than doubly stochastic, then the fixed point iteration rapidly converges to a non-trivial solution. For example, if

$$B = \begin{pmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{pmatrix} \text{ and } A_0 = \begin{pmatrix} 0.25 & 0.3 & 0.29 & 0.16 \\ 0.15 & 0.45 & 0.2 & 0.2 \\ 0.5 & 0.1 & 0.2 & 0.2 \\ 0.1 & 0.05 & 0.7 & 0.15 \end{pmatrix},$$

then the fixed point algorithm yields the following approximate solution

$$A_n = \begin{pmatrix} 0.250000000 & 0.324999905 & 0.192500372 & 0.232499723 \\ 0.150000000 & 0.474999905 & 0.102500372 & 0.272499723 \\ 0.500000000 & 0.124999905 & 0.102500372 & 0.272499723 \\ 0.100000000 & 0.074999905 & 0.602500372 & 0.222499723 \end{pmatrix},$$

in 18 iterations.

4.3 Summary

It is important to note that the fixed point iteration scheme we presented is, as is usually the case with such problems, only one of many such schemes which can be devised. The purpose of this exercise, then, was not to be definitive, instead it was meant to be illustrative of the potential utility of this approach. Finally, as with the case of attempting to find solutions of the Yang-Baxter equation using special matrices, the use of special matrices in the fixed point iteration scheme is also a significant contributor to the computability of the solutions we obtained.

Chapter 5

CONCLUSIONS

5.1 Assessing Various Solution Approaches

As stated in the introduction of this paper, the Yang-Baxter equation has a wide range of application in physics and mathematics, including but not limited to statistical mechanics, quantum mechanics, knot theory, braid theory, and quantum groups. The general form of the equation is

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u) \tag{5.1}$$

Rather than examining the detailed algebraic structures resulting from tensor products, along with the need to construct R-matrices, we analyzed the Yang-Baxter equation within the framework of algebraic product of matrices.

In particular, the product of matrices corresponds to the composition of linear maps. Even considering only two matrices A and B, the question concerning the k-fold products of these rapidly becomes very difficult. For the two-fold product, we clearly are considering the concept of commutativity when we ask if $AB = BA$. Another question is when does $AA = BB$? For the three-fold products, we have the larger question of when $AAA = AAB$, or $AAB = ABB$, or $ABB = BBB$, and so on. The Yang-Baxter equation is clearly only one of these, i.e., $ABA = BAB$.

A different approach to the Yang-Baxter problem can be obtained by letting $W = BA$ and analyze the equation $AW = WB$. Despite the fact that Theorems 1.4.1 and 1.4.2 give us some useful information about the solution of $AW = WB$, in particularly 1.4.1 states the necessary condition that A and B have a common eigenvalue, the deeper problem of demonstrating the existence of a factorisable $W = BA$ remains. For some doubly stochastic matrices, this factorization is trivially achieved, as demonstrated in Section 1.4.

The analysis of the Yang-Baxter equation from the view point of combination of linear maps creates some difficulties. As we have seen in Chapter 2, a direct computational approach to the problem is laborious, and even for the simple 2×2 case, the general solution is expressed in term of free parameters due to the fact that we have more unknowns than equations to solve. Even though we obtained the general solutions for 2×2 matrices, the derivation of general $n \times n$ solutions using the properties of some special matrices is more important.

As we have seen in Chapter 3, some of these special matrices like diagonal matrices, nilpotent matrices, and circulant matrices, yield rather trivial solutions, and thus are not of great importance to us. On the other hand, idempotent matrices are of more interest since they yield nontrivial solutions for $n \times n$ matrices. These are also consistent with the general solution of the Yang-Baxter equation for the 2×2 case. Because of the special properties of idempotent matrices, we were able to derive a class of nontrivial solutions of the Yang-Baxter equation for a given idempotent matrix. This result motivates the direction of future research.

5.2 Research Directions

After analyzing the Yang-Baxter equation from different perspectives, as well as looking at the properties of some special matrices, an important question arises: given a matrix A, when does the Yang-Baxter equation have a solution? More importantly, when is this solutions unique? As we have seen in this paper, some matrices, like the circulant ones, do not have solutions, while others, like the idempotent matrices, can have interesting solutions. In Chapter 4, we derived a fixed point iteration algorithm that works particularly well with stochastic matrices. As we saw, given any stochastic matrix B, we can easily find its Yang-Baxter complement A. However, the method is not very practical for non-stochastic matrices.

There are clearly several avenues for productive further research into the matrix solutions of the Yang-Baxter equation. All of these involve attempting to find solutions by imposing some appropriate restrictions which allow the demonstration or computation of these solutions. The use of iterative algorithms for approximating solutions to the Yang-Baxter equation would appear to provide a means for obtaining some class of non-trivial solutions. A faster fixed point algorithm that will be able to find solutions of the YBE in the general case, if such exist, is clearly desirable. All of this, however, needs to be done within a framework in which the computational approaches are developed concurrently with a deeper understanding of analytically derived solutions.

The ultimate objective is a full characterization of all solutions, as well as a means for computing these efficiently. We have made some very limited progress, however the bulk of the work remains unfinished.

Appendix A

MATLAB CODE

The code in Section A.1 is used to compute the fixed point iteration sequence described in Sec 4.1.

A.1 Fixed Point Iteration Code

```
function ceciteration(A,B)
C=eye(size(A));
maxiteration=20000;
epsilon=0.000001;
X=A*B;
k=0;
while (norm(A*B*A-B*A*B,Inf)>epsilon)
    if (k>maxiteration)
        display('Matrix A does not seem to converge to the solution
            within the specified number of iterations')
        break;
    end
    if (det(X+C)<eps)
        A=inv(X+pi*C)*(B*X+pi*A);
    else
    A=inv(X+C)*(B*X+A);
    end
    if (any(isnan(A)))
        display('There is no solutions of the YBE for this matrix');
        break;
    end
    X=A*B;
    k=k+1;
end
if (k<=maxiteration & ~any(isnan(A)))
    display('The matrix ABA is:');
    display(A*B*A);
    display('The matrix BAB is:');
    display(B*A*B);
    display('The number of iteration is:');
    display(k)
    display('The approximate solution is:');
    display(A);
end
```

BIBLIOGRAPHY

[1] Vector and Tensor Mathematics. Copyright 2001, PolymerProcessing.com.

[2] Yang-Baxter Equation. `http://en.wikipedia.org/wiki/Yang-Baxter_equation`.

[3] Shiing-Shen Chern. Yang-Mills Equations and Yang-Baxter Equations. *Chinese Journal of Physics*, 30(7):952, December 1992.

[4] Gustav W. Delius, Mark D. Gould, and Yao-Zhong Zhang. On the construction of trigonometric solutions of the Yang-Baxter equation. April 1994.

[5] H. A. Dye. Unitary solutions to the Yang-Baxter Equation in Dimension Four. pages 4–9, June 2003.

[6] Nicholas J. Higham. *Functions of Matrices: Theory and Computation*. SIAM, 2002.

[7] Christian Kassel and Vladimir Turaev. *Braid Groups*. Graduate Texts in Mathematics. Springer Science + Business Media, LLC, 2008. pp. 94.

[8] Toshitake Kohno. Braids and 3-dimensional geometry interacting with physics. *IMPU News*, (4), December 2008.

[9] Bing-An Li and Yuefan Deng. Chen Ning Yang. translated from "Biographies of Contemporary Chinese Scientists", Volume 3, 1992.

[10] Cyrus Colton McDufee. *The Theory of Matrices*. Courier Dover Publications, 2004.

[11] Carl Dean Meyer. *Matrix Analysis and Applied Linear Algebra*. SIAM, 2000.

[12] Jacques H. H. Perk and Helen Au-Yang. Yang-Baxter Equations. June 2006. arXiv:math-ph/0606053v1.

[13] Sylvester. *C.R. Acad. Sci., Paris*, 99:67–71 and 115–116, 1884.

[14] F. Y. Wu. The Yang-Baxter Equation in Knot Theory. *International Journal of Modern Physics B*, 7(20,21), 1993.

[15] C. N. Yang. Some exact results for the many-body problem in one dimension with repulsive delta-function interaction. *Physical Review letters*, 19, November 1967.

[16] Chen Ning Yang and Mo-Lin Ge. *Braid group, knot theory, and statistical mechanics II*. World Scientific Publishing Co. Pte. Ltd., 1994.