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2-Domination and Annihilation Numbers

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2-DOMINATION AND ANNIHILATION NUMBERS

by

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ABSTRACT

Using information provided by Ryan Pepper and Ermelinda DeLaVina in their paper *On the 2-Domination number and Annihilation Number*, I developed a new bound on the 2-domination number of trees. An original bound, $\gamma_2(G) \leq \frac{n+n_1}{2}$, had been shown by many other authors. Our goal was to generate a tighter bound in some cases and work towards generating a more general bound on the 2-domination number for all graphs. Throughout the span of this project I generated and proved the bound $\gamma_2(T) \leq \frac{1}{3}(n+2n_1+n_2)$. To prove this bound I first proved that the 2-domination number of a tree was less than or equal to the sum of two sub-trees formed by the deletion of an edge: $\gamma_2(T) \leq \gamma_2(T_1)+\gamma_2(T_2)$. From there, I proved our bound by showing that a minimum counter-example did not exist. A large portion of the results involves cases where a graph T is considered the minimum counter-example for the sake of contradiction. From there, I showed that if T was a counter-example, then a sub-tree T_1 was also a counter-example, meaning that T would no longer be the minimum counter-example. The last portion of the results is a section comparing the two bounds on the 2-domination number with respect to the number of edges and the degrees of those edges.

Key Words: Annihilation number of a graph, 2-Domination number of a graph, trees

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

Figure

Chapter 1

Introduction

Ermelinda DeLaVina and Ryan Pepper wrote an article [Delavina and Pepper, 2008] in 2008 entitled "2-Domination and Annihilation Number" which describes results from a computer program named Graffiti.pc. The paper presents a conjecture which suggests that the 2-domination number of a graph G, denoted by $\gamma_2(G)$, is less than or equal to the annihilation number of a graph, denoted by $a(G)$, plus one [Delavina and Pepper, 2008]. Precisely,

$$\gamma_2(G) \leq a(G) + 1.$$

A significant amount of progress has been made by such authors as Wyatt Desormeaux, Michael Henning, Douglas Rall, and Anders Yeo when G is a tree [Desormeaux et al., 2014]. This research project attempts to expand upon their work and the work of many others by analyzing a bound on the 2-domination number suggested by this conjecture and proving a special case. In doing so, this project helped to expand the field of graph theory which has varied practical applications in the areas of biosystematics, social networking analysis, and topology.

According to DeLaVina, Graffiti.pc is a "conjecture-making computer program" that has been used to help support, form, and discredit predictions made primarily in the fields of graph theory and chemistry [DeLaVina, 2002]. It is used to search through many graphs and make conjectures about the relationships between graph parameters. Graffiti.pc is the second generation of this computer program. The original version, Graffiti, was created by Siemion Fajtlowicz and later enhanced by DeLaVina to make a more user friendly version [DeLaVina, 2005].

This project is in the field of mathematics called graph theory. A graph, in the context of this project, is a set of vertices and a set of edges. A graph is often denoted by $G = (V, E)$, where E is the set of edges, and V is the set of vertices. The edge set is a subset of the ordered pairs of $V \times V$. In this thesis, only simple, loopless graphs are considered; that is, no vertex is in an edge with itself and there are no repeated edges. Two vertices are considered adjacent if there is an edge between them. For a subset S of vertices in the graph to be considered a k-dominating set, there must be k or more adjacent (or neighboring) vertices in S for each vertex not in the subset S. The value of k is set as $k = 2$ for this project. In order for S to satisfy the definition of a 2-dominating set, each vertex that is not in set S must have two neighbors that are in the set S. This is more easily understood with a few examples. Take the graph in Figure 1.1:

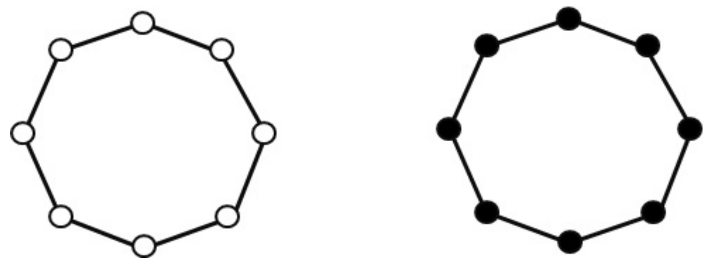

Figure 1.1: Even Cycle, C_8

It has 8 vertices and 8 edges. The degree of a vertex v is defined as the number of edges incident to a vertex. The total number of edges in a graph is denoted by the letter m, and the total number of vertices for a graph is denoted by n. The total number of vertices of degree d is denoted by n_d, where d is the degree of that vertex. When the graph under consideration is not clear, I used $n(G)$and $n_d(G)$ to denote the number of vertices of G and the number of vertices from G of degree d. For example, in Figure 1.1, $n_2 = n$ because all of the vertices have exactly 2 edges incident with them. There are many sets of vertices that can be chosen to form a 2-dominating set. For example, in Figure 1.1, suppose every vertex is chosen. Then, by defintion, S is a 2-dominating set, because every vertex is in the set S. Every vertex not in S has two neighbors by default because there are no vertices which are in this category.

Even though choosing every vertex is the simplest way of achieving a 2-dominating set, it serves very little purpose and is not an interesting choice. Something more difficult to determine is the minimum number of vertices in S such that S is still a 2-dominating set. This is called the 2-domination number, and it is denoted by $\gamma_2(G)$. An analogous parameter for k-dominating sets is called the k-domination number and is denoted by $\gamma_k(G)$. An attempt to minimize the amount of vertices in a 2-dominating set for the graph in Figure 1.1 would result in the set of vertices indicated by Figure 1.2. Shaded vertices indicate vertices in the 2-dominating set. All of the vertices are either in the set or have two neighbors which are in the set, so this set satisfies the definition of a 2-dominating set. Because four vertices are in the set S, $\gamma_2(G) = 4$. It is simple to figure out the 2-domination number of this graph, but making a generalization is a little more difficult.

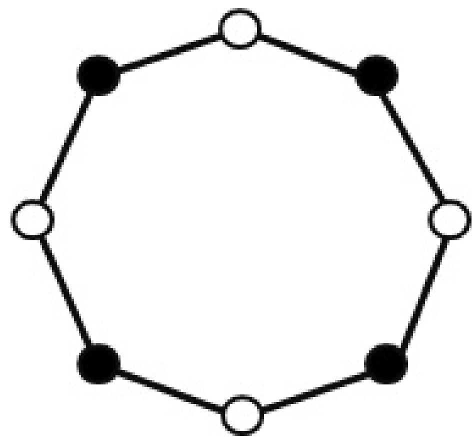

Figure 1.2: A 2-Dominating Set of Minimum Order for the Graph in Figure 1.1

A cycle is defined to be a graph that "consists of a sequence of vertices starting and ending at the same vertex, with each two consecutive vertices in the sequence adjacent to each other in the graph" [Imrich et al., 2008]. In particular, an even cycle has an even number of vertices. Cycles do not have any chords, which are edges that divide a cycle of four or more edges into two smaller cycles. For such a specific set of graphs, an easy result about $\gamma_2(G)$ can be made. Take, for example, the even cycle graphs in Figure 1.1. In these two graphs, along with the graph from 1.2, we can choose S such that $\gamma_2(G) \leq \frac{n_2}{2}$. In fact, $\gamma_2(G) = \frac{n}{2}$ when G is an even cycle [DeLaVina et al., 2011].

Another term that needs to be defined is the annihilation number of a graph. In order to obtain the annihilation number, arrange all degrees of the vertices in non-decreasing order. The annihilation number of a graph is the maximum number of degrees that can be added together in ascending order to approach but not go over the total number of edges of that graph. It is represented by $a(G)$. Figure 1.3 provides an example of calculating the annihilation number. The numbers inside each vertex in Figure 1.3 represent the degrees. This graph has several edges not in the main 7-vertex cycle and chord. Notice that every vertex with degree 1 must be chosen for the 2-dominating set. This is because these vertices, referred to as leaves, cannot have two neighbors in the set S. Thus, leaves must be included in any 2-dominating set.

Fact 1. Any 2-dominating set S of a graph must contain all leaves.

Fact 2. The annihilation number can also be described as: $k : \sum_{i \leq k} d_i \leq \sum_{i>k} d_i$ where $d_1, \ldots, d_n$ are the vertex degrees in non descending order. [Delavina and Pepper, 2008]

For the graph in Figure 1.3, the degrees arranged in order are: 1, 1, 1, 2, 2, 2, 2, 3, 4, and 4. The number of edges is $m = 11$. After adding the first seven terms, the value is exactly

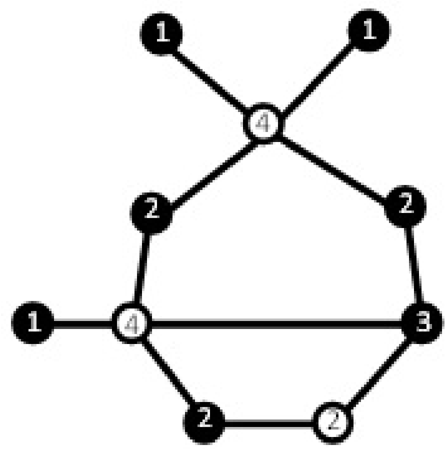

Figure 1.3: Reference for Annihilation Number Discussion with Degrees Marked

11, so $a(G) = 7$. The 2-domination number for the graph in Figure 1.3 is $\gamma_2(G) = 7$. As can be seen here, there is already possible evidence for a correlation between the 2-domination number and the annihilation number. For example, in even cycles, the annihilation number will always be equal to the 2-domination number. This is because every vertex in a cycle has degree two. In fact, the annihilation number is $\frac{n}{2}$ because every edge supplies a value of 2, which means that it will take half of the degrees to add up to the total number of edges. The 2-domination number for an even cycle is also $\frac{n}{2}$.

This correlation is interesting to note, but the next section focuses on the following conjecture [Delavina and Pepper, 2008].

$$\gamma_2(G) \leq a(G) + 1$$

This project builds on the progress made by other mathematicians and work to explore more conjectures related the 2-domination and annihilation numbers. Those include, but are not limited to:

$$\gamma_2(G) \leq \frac{1}{2}(m + n_1)$$

$$\gamma_2(G) \leq \frac{1}{3}(m + 2n_1 + n_2).$$

And the more general:

$$\gamma_2(G) \leq \frac{1}{k}(m + (k-1)n_1 + (k-2)n_2 + ... + (k-(k-1))n_{k-1}).$$

Chapter 2

Literature Review

As discussed in the introduction, Graffiti.pc was used to generate the potential correlation between the annihilation number and the 2-domination number. Graffiti.pc has been used for numerous research projects. Barbara Chervenka, a student of DeLaVina, described her experiences with the program in a more anecdotal account entitled "Graffiti.pc Red Burton Style: A Student's Perspective." She describes a specific project that she was working on and how the program helped develop and support her claims. Chervenka says Red Burton Style describes a methodology developed by Fajtlowicz for using the Graffiti program. She also tells how the program was utilized as a learning tool [Chervenka, 2002].

When describing how Graffiti.pc functions, DeLaVina explains the importance of "Dalmatian Heuristics." There is a large collection of past conjectures which the program stores and keeps track of. When a new conjecture is introduced, it is compared to the database of previous conjectures to see if it is compatible and genuine. After this stage of processing is finished, Graffiti.pc tests this conjecture against a large database of graphs of all forms to see if it is still worthwhile. Every conjecture generated by Graffiti is expressed as an inequality. [DeLaVina, 2002] All of the subsequent formulas that were derived for the sake of this project are also represented as inequalities.

The annihilation number was first introduced in [Pepper, 2004] by Ryan Pepper. This concept of the annihilation number was introduced in tandem with the Havel-Hakimi process, and Pepper's dissertation worked to explore those topics [Pepper, 2004]. The domination number of graphs has also sparked many mathematicians' interest, with its origins in the 1977 paper "Towards a Theory of Domination of Graphs" [Cockayne and Hedetniemi, 1977]. This article explains the concept of domination and works to relate it to graph coloring along with other mathematical concepts. Graph coloring is a branch of graph theory in which mathematicians attempt to find out the minimum number of colors necessary to color a map or graph so that no adjacent countries or vertices have the same color. There are many articles published on graph coloring. The chromatic number is denoted by $\chi(G)$ and is defined as the minimum number of colors needed to color all of the vertices in a graph and have no vertices with the same colors adjacent to one another [Catlin, 1978].

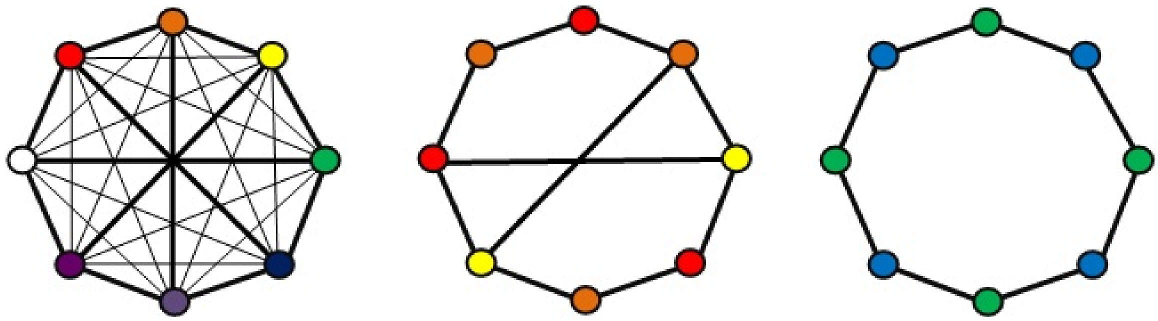

Figure 2.1: Examples of the Chromatic Number of a Graph

For example, consider the first graph in Figure 2.1. A graph in which all vertices are adjacent to another must have a different color on every vertex. Therefore, the chromatic number of the first graph in Figure 2.1 is $\chi(G) = 8$. As you start to remove edges from a graph, the amount of colors needed for the graph may decrease. The chromatic number for the second graph in Figure 2.1 is $\chi(G) = 3$, and the chromatic number for third graph in Figure 2.1 is $\chi(G) = 2$. Catlin presents a simple inequality for the chromatic number $\chi(G) \leq \Delta(G) + 1$ as a starting block for his theories [Catlin, 1978], where $\Delta(G)$ represents the maximum degree of any vertex in a graph G.

Wyatt Desormeaux, Michael Henning, Douglas Rall, and Anders Yeo work to explain the conjecture from [Delavina and Pepper, 2008] when it is restricted to trees in an article [Desormeaux et al., 2014] titled *Relating the Annihilation Number and the 2-Domination Number of a Tree*. This paper proves the fact that the inequality: $\gamma_2(G) \leq a(G) + 1$ works for degrees of 3 or greater and proves that the inequality works for degrees of 1 and 2 if the graph in question is a tree [Desormeaux et al., 2014]. A tree is a undirected graph in which any two vertices are connected by exactly one simple path [Golumbic, 1980]. Alternatively, a tree can be defined as a connected graph without a cycle. An important theorem that was shown in [Desormeaux et al., 2014] *Relating the Annihilation Number and the 2-Domination Number of a Tree* is the following, which has been shown by many authors:

Theorem 1 (Fink et. al [Fink and Jacobson, 1985]). For any tree T,

$$\gamma_2(T) \leq \frac{1}{2}(n + n_1).$$

The term n_1 refers to the number of vertices that are of degree 1. These vertices of degree one are also called leaves. These particular vertices are set apart from the other vertices

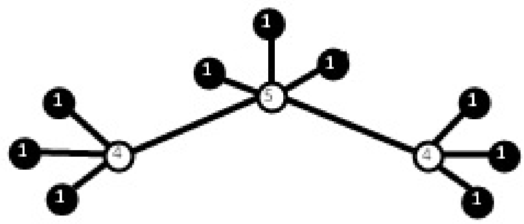

Figure 2.2: Example of a 2-Dominating Set

because they are required to be in any 2-dominating set S. This is because, as discussed earlier, leaves are not capable of having two neighbors in any set S. This formula is a starting point for the work done in this project. Figure 2.2 shows an example of a tree such that the shaded vertices form a 2-dominating set. The numbers inside each vertex represent the degree of each vertex. Notice that every vertex with a degree of one, a leaf, is shaded. A tree can have vertices with any degree as long as no cycles are formed. The value for n_1 of this graph would be 9 because there are 9 vertices of degree one.

The formula listed above came from work done by DeLaVina, Pepper, and Vaughan in their manuscript titled *On the 2-Domination Number of a Graph* [DeLaVina et al., 2010]. They presented the following theorem for a graph G with $n \geq 3$ vertices.

Theorem 2 (DeLaVina et. al [DeLaVina et al., 2011]). Let G be a graph and S_2 be the set of vertices with degree at most 2, then

$$\gamma_2(T) \leq \frac{1}{2}(n + \alpha(G[S_2])).$$

The variable $\alpha(G)$ in the above theorem represents the independence number, which is the order of the largest independent set [DeLaVina et al., 2010]. This formula only considers the vertices of degree 2 or less. This formula reappears in a subsequent paper titled "Graffiti.pc on the 2-domination number of a graph" [DeLaVina et al., 2010] written by DeLaVina, Larson, Pepper, and Waller. This paper is comprehensive and includes many inequalities that relate the 2-domination number to other prominent values. At the end of their discussion, there is a list of notable conjectures. The two listed are the most relevant to this project.

Conjecture 1 (Delavina et. al [DeLaVina et al., 2010]). Let G be a connected n-vertex graph. Then,

$$\gamma_2(G) > n - m(G) + 1.$$

where $m(G)$ represents is the upper or statistical median of the degree sequence of G.

Conjecture 2 (DeLaVina et. al [DeLaVina et al., 2010]). Let G be a connected graph. Then,

$$\gamma_2(G) \leq a(G) + 1$$

This project uses the information gained from the above resources as a starting block for research and as a source of mathematical perspective when generating new proofs and explanations.

Chapter 3

Methods

In order to generate and begin analyzing any formulas pertaining to the original inequality suggested by DeLavina and Pepper, there are many preliminary calculations and proofs that must be done.

In Section 4.1, I provide an alternative proof for the bound $\gamma_2(T) \leq \frac{n+n_1}{2}$ which was shown in [Desormeaux et al., 2014]. This proof relies on layering trees in such a way that I can find a 2-domination set. I first choose a vertex from the tree as a "root". The distance between two vertices, denoted by $d(u,v)$ for vertices u and v, is the length of the shortest path in G between u and v, where $d(u,u) = 0$. I use the distance from the root to layer the vertices of a tree and choose 2-dominating sets based on this layering.

In Section 4.2, I conjecture a new family of bounds on the 2-domination number using motivation from the conjecture regarding the annihilation number. I start by arranging the degree values necessary to generate the annihilation number, with k being the largest term needed. Then I multiply the number of terms for each degree by the degree value they represent and set that greater than or equal to m or the total number of edges. After some algebraic manipulations, I am left with the desired formula with respect to k and the total number of edges of each degree (or n_d). In order to consider a particular instance of this formula for analysis, k is set equal to 3.

Before proving the main result, I first prove a lemma about the 2-domination number of vertex-disjoint subgraphs of a graph. Section 4.3 provides a proof for the inequality $\gamma_2(T) \leq \gamma_2(T_1) + \gamma_2(T_2)$. This shows that when a particular tree is split at one edge, the original 2-domination number for T is less than or equal to the sum of the 2-domination numbers for each of the individual parts left after the edge was taken out. This section is a direct proof, and this lemma is utilized in the following section where I show the main result: $\gamma_2(T) \leq \frac{1}{3}(n + 2n_1 + n_2)$.

The proof of this bound relies on considering a minimum counter-example T and proof by contradiction. I remove a segment of graph T to yield T_1 and T_2. The rest of the proof relies on proving that T_1 is also a counter-example. If T_1 is also a counter-example, then T is not the *minimum* counter-example. The goal of this type of proof is to show that there is no minimum counter-example by using this technique over different cases with smaller and more specific graphs of T. The cases are considered to put more and more restrictions on the specific graph being considered. For the first example, the graph of a tree with a maximum path ending with four leaves was considered. This is the first example necessary because a terminal degree of anything larger would not add any terms to the formula for the boundary because it only considers terms of degree 2 or less. The amount of vertices removed to generate T_1 is based on the ease with which the proof of contradiction can be generated. If a certain choice in removal does not generate a contradiction, then a different set of vertices were chosen. With different sets of vertices being removed and steadily more specific graphs being considered, more specific cases were considered in order to cover every potential source of a minimum counter-example. Another similar proofing technique is used in this section, and it also relies on a proof by contradiction in which T is a the minimum counter-example. Because T is the minimum counter-example, T_1 is not a counter-example, as stated before. The approach to this proof is different though because it works to added edges back to T_1 in order to regenerate T. When T is regenerated, it was no longer be a counter-example, which is a contradiction. Because of the contradiction, T would not be the minimum counter-example. This very similar proof technique was utilized because some of the more specific cases involving a tree with a maximum path ending with only one leaf did not cooperate with the initial proof method.

After this formula has been proven, some time is devoted to comparing the family of conjectured bounds for the 2-domination number. In particular, I consider the cases in which $k = 2$ and $k = 3$. The two bounds are compared by setting them equal to one another and solving for n. When n is greater than the suggested value, the bound for $k = 3$ provides a more accurate boundary. When n is less than the suggested value, the bound for $k = 2$ provides a more accurate boundary. Several examples are provided which show when each formula provides a better estimate.

Chapter 4

Results

4.1 Alternate Proof of $\gamma_2(T) \leq \frac{n+n_1}{2}$

To begin with, I provide an alternate proof of the following theorem.

Theorem 3. Let T be a tree. Then,

$$\gamma_2(T) \leq \frac{n+n_1}{2}$$

where n_1 is the total number of vertices with degree one and n is the total number of vertices.

Proof. For the sake of this proof, I considered a graph T which has one vertex selected as the root. In order to begin generating a 2-domination set for T, the root is put in set S_1 along with every leaf in the graph. Layer L_1 is defined as all of the vertices adjacent to the root. Each consecutive layer, L_l, is defined as every vertex adjacent to a vertex in L_{l-1} not including those in L_{l-2} where l is the index of each layer. L_0 is considered the root. The number of layers varies depending on the size and structure of the tree. Alternatively, L_l is described as the vertices at distance l from the root.

After selecting the root and every leaf to be in set S_1, I chose all of the vertices in each layer with an even index to also be in set S_1. Each vertex that is not included in set S_1, meaning the index of the layer containing the vertex is odd, has a vertex in set S_1 adjacent to it in the layers L_{l-1} and L_{l+1}. Because set S_1 contains every leaf, the root, and every vertex in an even numbered layer, it is a 2-domination set by definition.

For a visual of the layers of graph T along with the chosen layers to be included in set S_1, reference the first two graphs presented in Figure 4.1.

To find a different 2-domination set S_2, a similar method is used. The root and the layers are the same, but instead of choosing the even layers to be in the 2-domination set, I chose the odd layers to be in the 2-domination set. That being said, the root may or may not be in set S_2. As before, all of the leaves need to be included in the 2-domination set, so S_2 is a 2-domination set by definition. The graph on the right in Figure 4.1 represents the layers chosen to complete the 2-domination set for S_2. The rooted vertex for the graph on the right in Figure 4.1 is shaded gray to signify that the rooted vertex may be a leaf and thus would be included in set S_2.

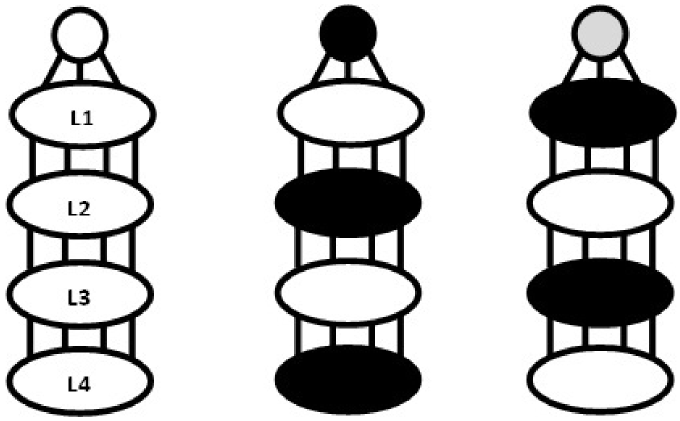

Figure 4.1: Alternating set S

Because the 2-domination number is the minimum number of vertices needed for set S to be a 2-domination set, $\gamma_2(T) \leq S_1$. Likewise, the number of vertices in S_2 must be less than or equal to S. So we have:

$$\gamma_2(T) \leq |S_1|$$
$$\gamma_2(T) \leq |S_2|$$

Because both of the values for S_1 and S_2 are separately greater than or equal to the 2-domination number of graph T, their sum should be at least twice the value of $\gamma_2(T)$. So,

$$2\gamma_2(T) \leq |S_1| + |S_2|$$
$$\gamma_2(T) \leq \frac{1}{2}\left(|S_1| + |S_2|\right)$$

Collectively, S_1 and S_2 contain every vertex in the graph at least once. Every leaf was included in both set S_1 and S_2. Every vertex with a degree greater than 1 was either included in S_1 or S_2, but they were not included in both due to the alteration between even and odd layers being chosen to be in the 2-domination set. Therefore, if n is the total sum of vertices in T and n_1 is the number of leaves, then

$$|S_1| + |S_2| = n + n_1,$$

By substituting into the relationship between $\gamma_2(T)$ and $|S_1|$ and $|S_2|$, we get

$$\gamma_2(T) \leq \frac{n + n_1}{2}$$

□

4.2 Generating a New Formula: $\frac{1}{3}(n+2n_1+n_2)$

By definition, the annihilation number is the minimum number of non-decreasing degrees that are necessary to reach but not go over the total number of edges. That being said, $a(G) \leq m$. The connection between the notation n_k and $a(G)$ is illustrated by Figure 4.2.

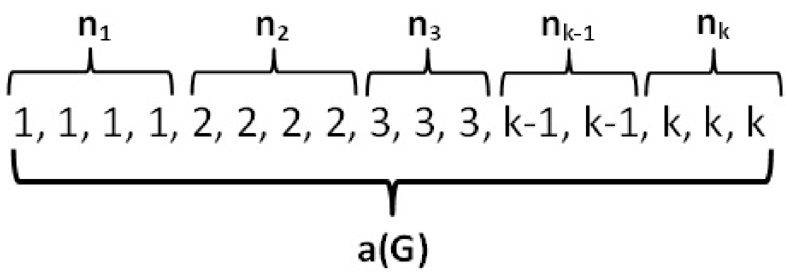

Figure 4.2: Annihilation Number

In Figure 4.2, k is the degree of the vertex with the largest degree considered for the sum of degrees that generate annihilation number. There may be other vertices with degree k that are not necessary to consider when attempting to reach the total number of edges. Let n_k is the number of terms with degree k, and let m denote the number of edges in G. Now suppose we choose a minimum t and a c such that $n_1+2n_2+\ldots+t(n_t-c)>m$. Then, from the definition of annihilation number and from the inequality $\gamma_2(G) \leq a(G)+1$, $a(G)+1 = n_1+n_2+\ldots+(n_t-c)$. From there we can say,

$$\begin{aligned}
t(a(G)+1) &= tn_1+tn_2+\ldots+t(n_t-c) \\
&= [n_1+2n_2+\ldots+t(n_t-c)]+(t-1)n_1+(t-2)n_2+\ldots+(n_t-c) \\
&> m+(t-1)n_1+(t-2)n_2+\ldots+n_{t-1}
\end{aligned}$$

Then, $a(G)+1 > \dfrac{1}{t}(m+(t-1)n_1+(t-2)n_2+\ldots+n_{t-1})$

In order to find a related boundary, I considered how values of t change the bound. The following function defines such a bound with the variable k.

$$p(k,G) = (m+1)+(k-1)n_1+(k-2)n_2+\ldots+n_{k-1}$$

If $k=t$ and G is a graph such that $c=0$, then,

$$\frac{1}{t}p(t,G) \leq a(G)+1.$$

From there, I generated the following conjecture using the function just defined, which I subsequently proved.

Theorem 4. For any connected tree T with n vertices, n_1 leaves and n_2 vertices with degree 2,

$$\gamma_2(T) \leq \frac{1}{3}p(3,T) = \frac{1}{3}(n+2n_1+n_2).$$

4.3 Proving $\gamma_2(T) \leq \gamma_2(T_1) + \gamma_2(T_2)$

In this proof I am working to show that the inequality $\gamma_2(T) \leq \gamma_2(T_1) + \gamma_2(T_2)$ is true for a graph T which is a tree. Graph T_1 and T_2 are two sub-trees of the original tree T that are left when one edge of the graph is deleted. If one edge of the graph is deleted, there must be exactly two separate graphs because there are no cycles by definition of a tree. Both of the new graphs are trees because deleting an edge does not form a cycle, and therefore both of the new subgraphs also do not have cycles in them.

Conjecture 3. Let T be a tree, and let T_1 and T_2 be the two sub-trees formed by deleting an edge. Then $\gamma_2(T) \leq \gamma_2(T_1) + \gamma_2(T_2)$.

Proof. Let T be a tree, and let v_1 and v_2 be adjacent vertices. Deleting the edge between these two vertices would result in two disconnected sub-trees, T_1 and T_2. Let S_1 and S_2 be minimum 2-domination sets for the corresponding sub-trees. Now let $S = S_1 \cup S_2$ and choose a vertex $v \in T$. Either $v \in S$ or $v \notin S$. If $v \notin S$, I assumed without loss of generality that $v \in T_1$. Then, the edge is reinserted between v_1 and v_2. Because S_1 is a 2-domination set of T_1, v must have two neighbors in S_1. This could also be said for any vertex in T_2 that is not in S. Therefore S is a 2-domination set for T. So the 2-domination number for T is less than or equal to the number of vertices in S. Therefore,

$$\gamma_2(T) \leq |S| = |S_1| + |S_2| = \gamma_2(T_1) + \gamma_2(T_2)$$

Note that one or more unnecessary vertices may be taken out of the 2-domination set of T to create a 2-domination set with the minimum amount of vertices. So, the 2-domination set for T could be less than S.

□

4.4 Minimizing the bounds on the counter-example for $\gamma_2(T) \leq \frac{1}{3}(n+2n_1+n_2)$

This section is working to prove that the inequality $\gamma_2(T) \leq \frac{1}{3}(n+2n_1+n_2)$ is true by proving that a counter-example does not exist. This was done by taking more and more

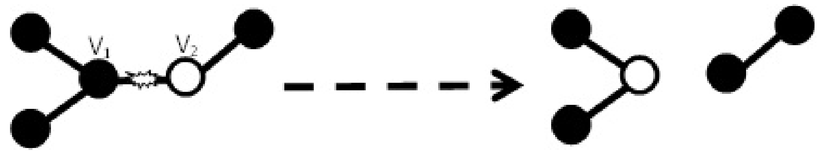

Figure 4.3: Splitting a Tree Example

specific cases and proving that there must be a smaller counter-example than the potential example provided if a counter-example exists. This method is called "Proof by Minimal Counter-example." Each case is distinguished by the degree of vertex x, which is the penultimate vertex in a maximum path P of the graph T.

To start, let T be a tree such that $\gamma_2(T) > \frac{n+2n_1+n_2}{3}$. This would be saying that the suggested formula does not provide an upper bound as previously suggested. I looked at some potential scenarios that would prove the above counter statement true. For the first example, we have a graph T with four leaves on the terminal end of the path with the maximum length in the graph. See Figure 4.4.

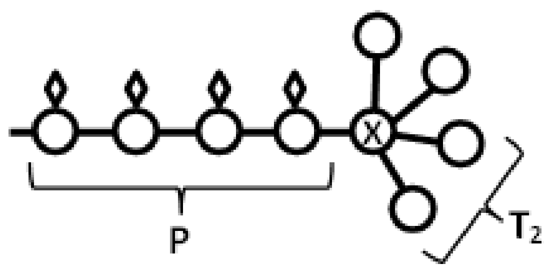

Figure 4.4: Graph T: Maximum Path with 4 Leaves

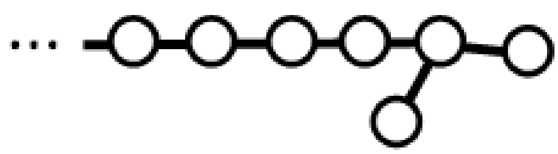

Figure 4.5: Graph: T_1: Maximum Path with 4 Leaves Minus 2 Leaves

Claim 1. The degree of x is less than or equal to 5.

Theorem 5. Let T be a tree. Then, $\gamma_2(T) \leq \frac{n+2n_1+n_2}{3}$.

Proof. For sake of contradiction, let T be a tree which is the minimum counter-example of the inequality $\gamma_2(T) \leq \frac{n+2n_1+n_2}{3}$. In this tree, let P be a path of maximum length, and let x be the penultimate vertex on P. I showed that T does not exist through a series of claims.

Vertices in P may have adjacent vertices not in the maximum path so long as they do not generate a path exceeding the length of P. These potential extensions are denoted as small diamonds in the following images.

First, I took away two of the four or more leaves on x, as displayed above in Figure 4.5 to form the graph which I called T_1, and show that if the original tree was a counter-example, the new tree with two less leaves must also be a counter-example. T_2 is the removed portion of the graph. Some values for the original graph T are provided:

1. $n(T) =$ total number of vertices
2. $n_1(T) =$ total number of leaves
3. $n_2(T) =$total number of vertices with degree 2

If two leaves are removed from graph T, the values n, n_1, and n_2 of T_1, are written with respect to values for the graph of T.

1. $n(T_1) = n(T) - 2$ because two vertices were taken away from the graph of T to form T_1.
2. $n_1(T_1) = n_1(T) - 2$ because two leaves were taken away from the graph of T to form T_1.
3. $n_2(T_1) = n_2(T)$ because no vertices of degree two were generated or removed.

The 2-domination number of the graph dropped by two when the two vertices were removed because they were both leaves and all leaves must be included in set S. Because of the rule proven in the previous section ($\gamma_2(T) \leq \gamma_2(T_1) + \gamma_2(T_2)$), I said that the original 2-domination number of T is less than or equal to the sum of the new parts.

$$\gamma_2(T) \leq \gamma_2(T_1) + 2$$

So

$$\gamma_2(T_1) \geq \gamma_2(T) - 2$$

And using the counter-example above, I said:

$$
\begin{aligned}
\gamma_2(T_1) \geq \gamma_2(T_) - 2 > \frac{1}{3}(n(T) + 2n_1(T) + n_2(T)) - 2, so \\
> \frac{1}{3}(n(T) + 2n_1(T) + n_2(T) - 6) \\
= \frac{1}{3}((n(T) - 2) + 2(n_1(T) - 2) + n_2(T)) \\
> \frac{1}{3}(n(T_1) + 2n_1(T_1) + n_2(T_1))
\end{aligned}
$$

The above statement says that the graph of T_1 also provides a counter-example. If T_1 is a counter-example, then T is not be the minimum counter-example, and thus the original claim is a contradiction.

□

Claim 2. Now the degree of x is less than 4.

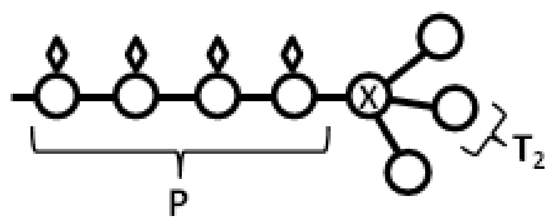

Figure 4.6: Graph T: Maximum Path P, Vertex x with degree 3

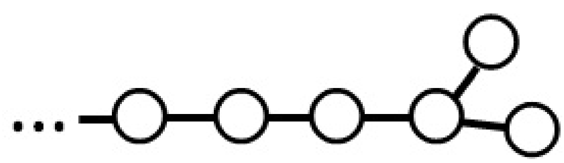

Figure 4.7: Graph T_1: Maximum Path with 3 Leaves Minus One Leaf

Proof. Suppose x has three leaves. See Figure 4.6. For the sake of contradiction, let us say that graph T in Figure 4.6 provides a minimum counter-example for our inequality. In order to show that it is not a minimum counter-example, I took off one of the three leaves to generate T_1, as displayed in Figure 4.6, and show that if T is a counter-example, then T_1 must also be a counter-example. If one leaf is removed from the graph,

1. $n(T_1) = n(T) - 1$ because one vertex was removed.

2. $n_1(T_1) = n_1(T) - 1$ because one leaf was removed.

3. $n_2(T_1) = n_2(T)$ because no vertices with degree 2 were altered.

$$\gamma_2(T) \leq \gamma_2(T_1) + \gamma_2(T_2) \Rightarrow \gamma_2(T) \leq \gamma_2(T_1) + 1$$
$$\gamma_2(T) \leq \gamma_2(T_1) + 1 \Rightarrow \gamma_2(T_1) > \gamma_2(T) - 1$$

Then,

$$\begin{aligned}
\gamma_2(T_1) \geq \gamma_2(T) - 1 &> \frac{1}{3}(n(T) + 2n_1(T) + n_2(T)) - 1 \\
&= \frac{1}{3}(n(T) + 2n_1(T) + n_2(T) - 3) \\
&= \frac{1}{3}((n(T) - 1) + 2(n_1(T) - 1) + n_2(T))
\end{aligned}$$

I then substitute these values of T for the previously mentioned values of T'.

$$\gamma_2(T_1) > \frac{n(T_1) + 2n_1(T_1) + n_2(T_1)}{3}$$

The above statement states that the graph of T' also provides a counter-example. If T' is a counter-examples, then T is not the minimum counter-example, and thus I have shown that the claim is true.

□

Claim 3. The degree of x is less than 3.

Proof. For this claim, I consider several different cases. For these examples, I did not remove leaves. For this proof I removed a larger segment including both of the leaves and the vertex they are attached to and considered the vertex preceding x on P, which I called u. Our cases are based on the degree of u.

1. **Case 1:** The degree of u is one.

   Because only one tree satisfies this set of circumstances, I just substituted the values into the formula and see if it is a counter-example. The 2-domination number of the graph in Figure 4.8 is 3, as displayed in Figure 4.9.

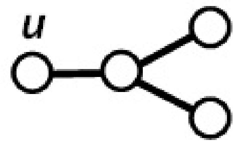

Figure 4.8: Maximum Path with 2 Leaves

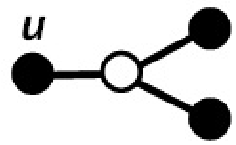

Figure 4.9: Maximum Path with 2 Leaves in 2-Domination Set

So plugging into the formula $\gamma_2(T) \leq \frac{1}{3}(n+2n_1+n_2)$ gives $\frac{1}{3}((4)+2(3)+0) = 3\frac{1}{3}$. This number is greater than the 2-domination number of 3, so this graph in Figure 4.8 is not a counter-example.

2. **Case 2:** The degree of u is 2. Such a graph is displayed in Figure 4.10.

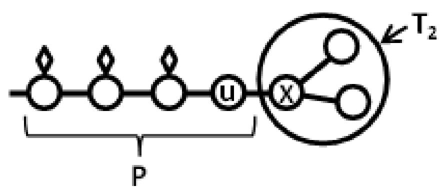

Figure 4.10: Graph T: Maximum Path with 2 Leaves where the degree of u is 2

Let T_2 be x and its 2 neighbors that are leaves. T_1 is the remaining sub-tree and is displayed in Figure 4.11.

Then,

(a) $n(T_1) = n(T) - 3$ because 3 vertices were removed.

(b) $n_1(T_1) = n_1(T) - 1$ because two leaves were removed and one leaf was created.

(c) $n_2(T_1) = n_2(T) - 1$ because one vertex with degree 2 was removed (u no longer has degree 2.)

Also,

$$\gamma_2(T) \leq \gamma_2(T_1) + \gamma_2(T_2) = \gamma_2(T_1) + 2$$

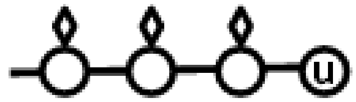

Figure 4.11: Graph T_1: Maximum Path with 2 Leaves where the degree of u is 2

Therefore,

$$\begin{aligned}\gamma_2(T_1) &\geq \gamma_2(T) - 2 \\ &> \frac{1}{3}(n(T) + 2n_1(T) + n_2(T)) - 2 \\ &= \frac{1}{3}((n(T) - 3) + 2(n_1(T) - 1) + (n_2(T) - 1)) \\ &= \frac{1}{3}(n(T_1) + 2n_1(T_1) + n_2(T_1))\end{aligned}$$

Thus, if T is a counter-example, T_1 is also a counter-example, thus making T not the minimum counter-example.

3. **Case 3:** The third case to consider is when the degree of u equals 3. An example of such a graph is displayed in Figure 4.12.

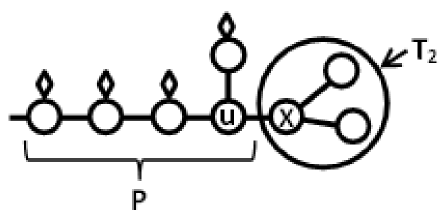

Figure 4.12: Graph T: Maximum Path with 2 Leaves where $d(u) = 3$

When the selected segment to the right of u in Figure 4.12 is removed, the image in Figure 4.13 is generated.

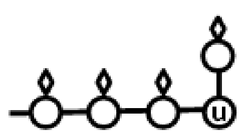

Figure 4.13: Graph T_1: Maximum Path with 2 Leaves where $d(u) = 3$ with Segment Removed

(a) $n(T_1) = n(T) - 3$ because 3 vertices are removed.

(b) $n_1(T_1) = n_1(T) - 2$ because two leaves are removed from T.

(c) $n_2(T_1) = n_2(T) + 1$ because one vertex of degree 2 was generated (the vertex u).

Then,

$$\begin{aligned} \gamma_2(T) &\leq \gamma_2(T_1) + \gamma_2(T_2) \\ &= \gamma_2(T_1) + 2 \\ \text{Then, } \gamma_2(T_1) &\geq \gamma_2(T) - 2 \\ &> \frac{1}{3}(n(T) + 2n_1(T) + n_2(T)) - 2 \\ &= \frac{1}{3}((n(T) - 3) + 2(n_1(T) - 2) + (n_2(T) + 1)) \\ &= \frac{1}{3}(n(T_1) + 2n_1(T_1) + n_2(T_1)) \end{aligned}$$

Again, if T is a counter-example, T_1 is a smaller counter-example. Thus, T is not a minimum counter-example.

4. **Case 4:** The last case that needs to be shown is when u has a degree greater than 3. A image depicting such a graph is presented in 4.14.

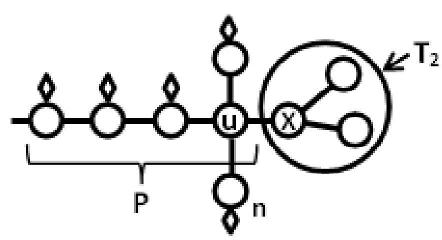

Figure 4.14: Graph T: Maximum Path with 2 Leaves where $d(u) > 3$

Once the segment to the right of u in Figure 4.14 is removed, I am left with the graph in Figure 4.15.

Then,

(a) $n(T_1) = n(T) - 3$ because 3 vertices were removed.

(b) $n_1(T_1) = n_1(T) - 2$ because two leaves were removed from T.

(c) $n_2(T_1) = n_2(T)$ because the number of vertices with degree two was unchanged after the deletion.

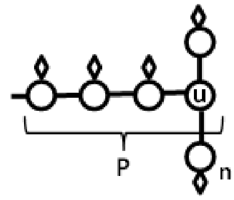

Figure 4.15: Graph T_1: Maximum Path with 2 Leaves where the degree of $d(u) > 3$ and segment removed

$$\begin{aligned}
\gamma_2(T) &\leq \gamma_2(T_1) + \gamma_2(T_2) \\
&= \gamma_2(T_1) + 2 \\
\text{Then, } \gamma_2(T_1) &\geq \gamma_2(T) - 2 \\
&> \frac{1}{3}(n(T) + 2n_1(T) + n_2(T)) - 2 \\
&= \frac{1}{3}((n(T) - 3) + 2(n_1(T) - 2) + n_2(T) + 1) \\
&> \frac{1}{3}(n(T_1) + 2n_1(T_1) + n_2(T_1)) + \frac{1}{3}
\end{aligned}$$

If T is a counter-example, then T_1 is also a counter-example, meaning that T is not the minimum counter-example. Because all of the cases above showed that there is a counter-example more simple than the provided case, I said that a maximum path with two leaves on the terminal end does not represent a minimum counter-example. Then I moved to the final consideration of a maximum length path with one leaf on the terminal vertex.

□

Claim 4. The degree of x is less than 2.

Proof. There are also many cases that need to be considered when x has 1 leaf which I called y. For each case, let us say for the sake of contradiction that T is a graph with one leaf on the second to last vertex of the maximum path, and T is the minimum counter-example. Vertices x and y are removed to generate the graph of T_1 in the following cases. Because T is the minimum counter-example, then T_1 is not a counter-example.

1. **Case 1:** The degree of u is one.

   There is only one tree which fits this condition, so it must merely be tested to see if that specific example is a counter-example.

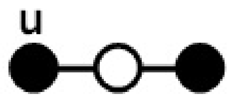

Figure 4.16: Maximum Path with 1 Leaf where the degree of u is 1

The 2-domination number of the graph in Figure 4.16 is $\gamma_2(T) = 2$. However,

$$\frac{1}{3}(n+2n_1+n_2) = \frac{1}{3}(3+2(2)+1) = 2\frac{2}{3}$$
$$2\frac{2}{3} \geq 2.$$

Therefore this is not a counte-example.

2. **Case 2:** The degree of u is 2.

   The graph of T is displayed in Figure 4.17.

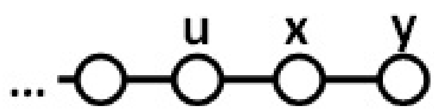

Figure 4.17: Maximum Path with 1 Leaf where the degree of u is 2

When x and y are removed, you get the the graph of T_1 displayed in Figure 4.18.

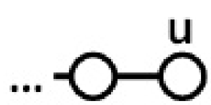

Figure 4.18: Maximum Path with 1 Leaf where the degree of u is 2 and x and y have been removed

(a) $n(T_1) = n(T) - 2$ because two vertices were removed.

(b) $n_1(T_1) = n_1(T)$ because there was no net change in the number of leaves.

(c) $n_2(T_1) = n_2(T) - 2$ because two vertices of degree 2 were removed (x was removed, and u has degree 1 in T_1.)

Any 2-dominating set of T_1 must include u, so I added the vertex y to any 2-dominating set of T_1 to get a 2-dominating set of T. Thus,

$$\begin{aligned}
\gamma_2(T) &\leq \gamma_2(T_1)+1 \\
\gamma_2(T_1) &\geq \gamma_2(T)-1 \\
&> \frac{1}{3}(n(T)+2n_1(T)+n_2(T))-1 \\
&= \frac{1}{3}((n(T)-2)+2n_1(T)+(n_2(T)-2)+1) \\
&= \frac{1}{3}(n(T_1)+2n_1(T_1)+n_2(T_1))+\frac{1}{3}
\end{aligned}$$

Thus, if T is a counter-example, T_1 is also a counter-example, making T not the minimal counter-example. This is contradiction, so T must not be the minimal counter-example.

Because a graph with u having degree of 3 or greater does not lend itself to the proof techniques previously utilized, and because there are several other possibilities for u, I tried to add the edges x and y back to the graph of T_1 in some examples instead of removing them from T. If T is the minimum counter-example, then T_1 is not a counter-example. Under the assumption that T_1 is not a counter-example, I worked to show that when x and y are added to T_1, T is no longer a counter-example. This contradiction would prove that T was never the minimum counter-example after all. This is the bases from which several of these proofs are generated.

3. **Case 3:** The degree of u is 3, and u is in some 2-domination set for the initial graph of T_1.

   Such a graph is displayed in Figure 4.19.

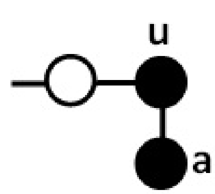

Figure 4.19: Maximum Path with 1 Leaf where the degree of u is 3, x and y are removed, and u is in set S

When x and y are added to the graph of T_1, you get the graph in Figure 4.20. A graph with more than one leaf neighboring a is not considered because previous cases of this type have already been done in Claims 1-3. Figure 4.10 in Claim 3 provides an example of such a graph with the two leaves neighboring a. Figure 4.10 just takes the perspective that sees a as being on the maximum path instead of on a dependent path.

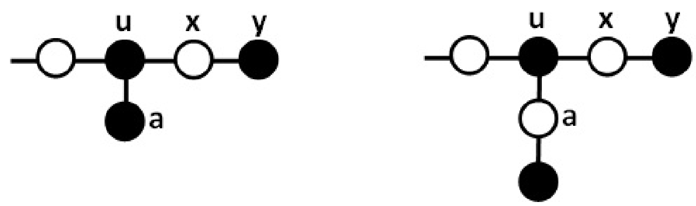

Figure 4.20: Maximum Path with 1 Leaf where the degree of u is 3 and u is in set S for T_1

Again, let T_1 be the remaining tree with x, y removed.

(a) $n(T) = n(T_1) + 2$ because two vertices were added.

(b) $n_1(T) = n_1(T_1) + 1$ because there was a net gain of one leaf.

(c) $n_2(T) = n_2(T_1)$ because the number of vertices of degree two was not altered. Vertex x had degree 2 and was removed, but u changed from degree three to degree two.

Only y must be added to set S in order to create a 2-domination set for T. So,

$$\begin{aligned}
\gamma_2(T_1) + 1 &= \gamma_2(T) \\
\gamma_2(T_1) + 1 &\leq \frac{1}{3}(n(T_1) + 2n_1(T_1) + n_2(T_1)) + 1 \\
&= \frac{1}{3}((n(T_1) + 2) + 2(n_1(T_1) + 1) + n_2(T_1) - 1) \\
\gamma(T) &\leq \frac{1}{3}(n(T) + 2n_1(T) + n_2(T))
\end{aligned}$$

This is less than the value suggested as an upper bound for the 2-domination number T. Therefore, T is not a counter-example. Because of this contradiction, T is not the minimal counter-example.

4. **Case 4:** The degree of u is 3, and u is not in any minimum 2-dominating set S of T_1. The dependent path on u has a length of 2.

   The dependent path coming from u is either of length one or two. For the first example, I considered the case where the dependent path is of length 2. An image is provided in Figure 4.21.

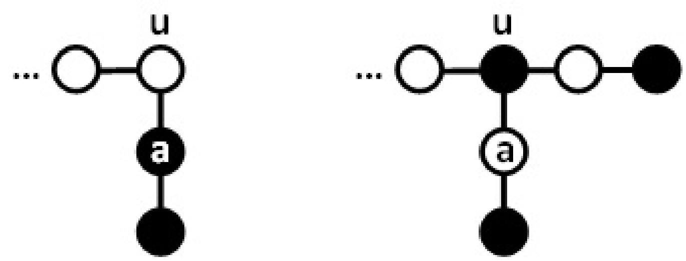

Figure 4.21: Maximum Path with 1 Leaf where the degree of u is 3 and u is not in set S for T_1. T and T_1 are both displayed. One of the dependent paths is of length 2.

For the graphs in Figure 4.21, when x and y are added back onto the graph of T_1, the vertex u is contained in the minimum 2-dominating set S_1, and a is not contained in the 2-dominating set S_1, meaning that the difference in the 2-domination number between the two is only 1.

(a) $n(T) = n(T_1) + 2$ because 2 vertices were added.

(b) $n_1(T) = n_1(T_1) + 1$ because the net total of leaves increased by one.

(c) $n_2(T) = n_2(T_1)$ because the number of vertices with degree two was unchanged.

$$\begin{aligned}
\gamma_2(T_1) + 1 &= \gamma_2(T) \\
&\leq \frac{1}{3}(n(T_1) + 2n_1(T_1) + n_2(T_1)) + 1 \\
&= \frac{1}{3}((n(T_1) + 2) + 2(n_1(T_1) + 1) + n_2(T_1) - 1) \\
&= \frac{1}{3}(n(T) + 2n_1(T) + n_2(T)) - \frac{1}{3} \\
So, \gamma_2(T) &< \frac{1}{3}(n(T) + 2n_1(T) + n_2(T))
\end{aligned}$$

This is less than the conjectured upper bound of the 2-domination number, and therefore, this is a contradiction because T is no longer a counter-example. Thus, T is not the minimum counter-example.

5. **Case 5:** The degree of u is 3, and u is not in the minimum 2-dominating set S of T. The dependent path is a leaf.

   This is depicted in Figure 4.22.

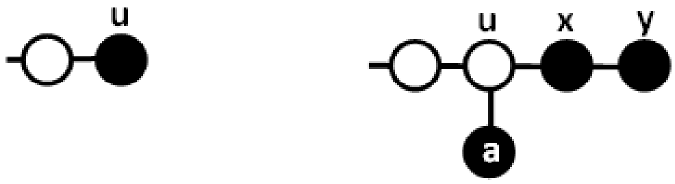

Figure 4.22: Maximum Path with 1 Leaf where the degree of u is 3 and u is not in set S for T_1. T and T_1 are both displayed. The dependent path is a leaf.

I solved this example using the initial form of proof in which edges are deleted from T. I showed that if T is a counter-example, T_1 is also a counter-example, making T not the minimum counter-example.

(a) $n(T) = n(T_1) + 3$ because three vertices were added.

(b) $n_1(T) = n_1(T_1) + 1$ because there was a net gain of one leaf.

(c) $n_2(T) = n_2(T_1) + 1$ because there was a net gain of one vertex of degree 2.

$$
\begin{aligned}
\gamma_2(T) &\leq \gamma_2(T_1) + 2 \\
\gamma_2(T_1) &\geq \gamma_2(T) - 2 \\
&> \frac{1}{3}(n(T) + 2n_1(T) + n_2(T)) - 2 \\
&= \frac{1}{3}((n(T) - 3) + 2(n_1(T) - 1) + (n_2(T) - 1)) \\
&= \frac{1}{3}(n(T_1) + 2n_1(T_1) + n_2(T_1))
\end{aligned}
$$

Because T_1 is also a counter-example, T is not the minimum counter-example. This is a contradiction to the original suggestion, therefore T is not the minimum counter-example.

6. **Case 6:** The degree of u is greater than 3, and u is in some 2-domination set for T_1. T_1 is the graph with x and y removed.

   For u to be in the 2-domination set for T_1, there is at most one leaf adjacent to the u. Otherwise, u would not necessarily be in the 2-domination set. The other cases have a dependent path of length 2. Example graphs are depicted in Figure 4.23.

   If x and y were removed, the graph of T_1 would look like the graph in Figure 4.24.

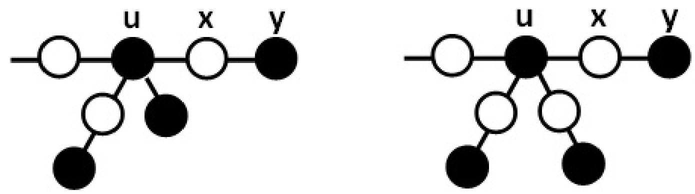

Figure 4.23: Maximum Path with 1 Leaf where the degree of u is greater than 3 and u is in set S for T_1

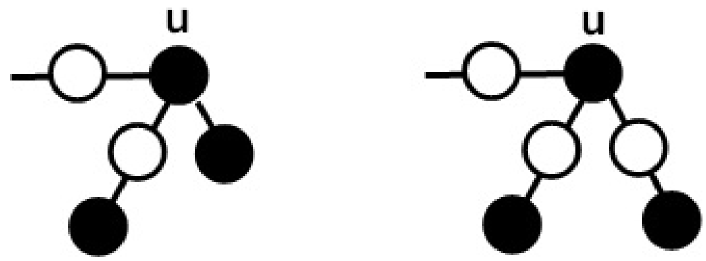

Figure 4.24: Maximum Path with 1 Leaf where the degree of u is greater than 3 and u is in set S for T_1. x and y are removed.

As before, if T is the minimum counter-example, T_1 is not a counter-example. So if I add x and y, T should remain a counter-example.

(a) $n(T) = n(T_1) + 2$ because 2 vertices were added.

(b) $n_1(T) = n_1(T_1) + 1$ because the net amount of leaves increased by one.

(c) $n_2(T) = n_2(T_1) + 1$ because the net amount of vertices with degree two increased by one.

$$\begin{aligned}
\gamma_2(T_1) + 1 &= \gamma_2(T) \\
&\leq \frac{1}{3}(n(T_1) + 2n_1(T_1) + n_2(T_1)) + 1 \\
&= \frac{1}{3}((n(T_1) + 2) + 2(n_1(T_1) + 1) + (n_2(T_1) + 1) - 2) \\
&\leq \frac{1}{3}(n(T) + 2n_1(T) + n_2(T)) - \frac{2}{3} \\
So, \gamma_2(T) &< \frac{1}{3}(n(T_1) + 2n_1(T_1) + n_2(T_1))
\end{aligned}$$

Therefore, T is not a counter-example which is a contradiction. Because of this contradiction, T is not be the minimum counter-example.

7. **Case 7:** The degree of u is greater than 3, and all of the dependent paths are leaves. The maximum path has a leaf on the terminal vertex.

   For this example I assumed that T is the minimum counter-example and that T_1 is not a counter-example. I then recreated T by adding back in the vertex removed to make T_1 and show that T is no longer a counter-example. An image of such graphs is displayed in Figure 4.25.

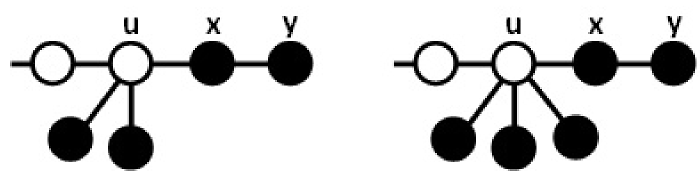

Figure 4.25: Maximum path with 1 Leaf where the degree of u is greater than 3 and u is not in set S for T_1. T and T_1 are both displayed. The dependent paths are all leaves.

   For this case I only removed the vertex y.

   (a) $n(T) = n(T_1) + 1$ because one vertex was added.

   (b) $n_1(T) = n_1(T_1) + 1$ because there was a net gain of one leaf.

   (c) $n_2(T) = n_2(T_1)$ because there was no net change to the number of vertices with degree 2.

$$
\begin{aligned}
\gamma_2(T_1) + 1 &\leq \gamma_2(T_1) \\
&\leq \frac{1}{3}(n(T_1) + 2n_1(T_1) + n_2(T_1)) + 1 \\
&= \frac{1}{3}((n(T_1) + 1) + 2(n_1(T_1) + 1) + (n_2(T_1))) \\
&= \frac{1}{3}(n(T) + 2n_1(T) + n_2(T))
\end{aligned}
$$

   This means that T is no longer a counter-example, which would make a contradiction. Therefore, by contradiction, T is not the minimum counter-example.

8. **Case 8:** The degree of u is greater than 3, and u is not in any 2-domination set of T_1.

   For such a graph, you have at most one dependent path of length 2 adjacent to u (not including the terminal path with x and y) because having 2 dependent paths would require that u be in the 2-domination set to get the minimum number of vertices in set S. I addressed the specific example where there is one dependent path of length 2 and one or more leaves adjacent to u. An image of such a graph is depicted in Figure 4.26.

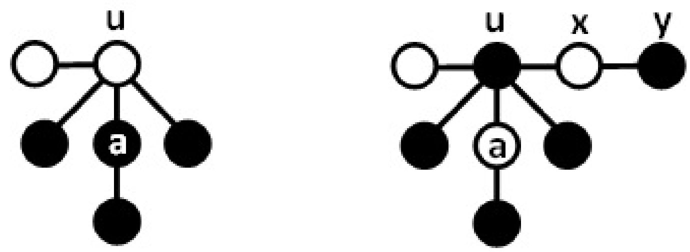

Figure 4.26: Maximum Path with 1 Leaf where the degree of u is greater than 3 and u is not in set S for T_1. T and T_1 are both displayed. One of the dependent paths is of length 2.

In this specific example with the dependent path on u having a length of 2, the vertex marked as a is no longer in set S when x and y are added back on. This means that the 2-domination number of graphs T and T_1 only differ by 1.

(a) $n(T) = n(T_1) + 2$ because 2 vertices were added.

(b) $n_1(T) = n_1(T_1) + 1$ because the net total of leaves increased by 1.

(c) $n_2(T) = n_2(T_1) + 1$ because the net total of vertices with degree two increased by 1.

$$\begin{aligned}
\gamma_2(T_1) + 1 &= \gamma_2(T) \\
&\leq \frac{1}{3}(n(T_1) + 2n_1(T_1) + n_2(T_1)) + 1 \\
&= \frac{1}{3}((n(T_1) + 2) + 2(n_1(T_1) + 1) + (n_2(T_1) + 1) - 2) \\
&= \frac{1}{3}(n(T) + 2n_1(T) + n_2(T)) - \frac{2}{3} \\
So, \gamma_2(T) &< \frac{1}{3}(n(T) + 2n_1(T) + n_2(T))
\end{aligned}$$

This value is less than the suggested estimate for the upper bound of the 2-domination number, which presents a contradiction because T is no longer a counter-example. Therefore, T is not the minimum counter-example.

Because the minimum counter-example must be a tree with a vertex of degree less than one on the maximum path, the minimum counter-example does not exist. Therefore, the suggested boundary on the 2-domination number of a tree is true for all trees.

□

4.5 Comparing Formulas: $\gamma_2(T) \leq \frac{n+n_1}{2}$ and $\gamma_2(T) \leq \frac{n+2n_1+n_2}{3}$

In an effort to derive a more exact bound on the 2-domination number of a graph, the second formula in discussion was generated in Section 4.2. This section describes when each bound provides a tighter bound for the 2-domination number. Suppose that,

$$\frac{n+n_1}{2} > \frac{n+2n_1+n_2}{3}$$

Then,

$$\frac{n}{2} > \frac{n_1}{2} + n_2$$

Observation 1. The bound in Theorem 5 provides a tighter bound than the bound in Theorem 1 when $n > n_1 + 2n_2$.

This means that when the total number of vertices is greater than the number of leaves plus two times the number of vertices with degree two, the conjectured bound $\gamma_2(T) \leq \frac{n+2n_1+n_2}{3}$ gives the better estimate. Otherwise, $\gamma_2(T) \leq \frac{n+n_1}{2}$ gives a better estimate. Some examples are provided.

Example 1: (See Figure 4.27)

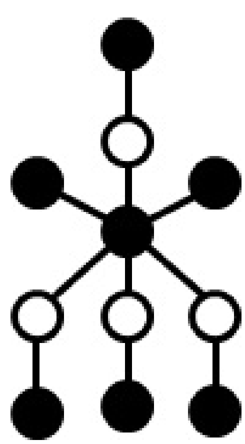

Figure 4.27: Sample Tree T_1 with $n = 11$, $n_1 = 6$, $n_2 = 4$, and $\gamma_2(T) = 7$

When information from the graph in Figure 4.27 is considered, $n < n_1 2n_2$ (that is $11 < 6+2(3)$). Because the total number of vertices is not greater than the number of leaves plus two times the number of vertices with degree two, formula 1 should give the better estimate.

Comparing Estimates:

$$\gamma_2(T_1) = 7 \leq \frac{11+6}{2} = 8.5 < 9 = \frac{11+6(2)+4}{3}$$

As you can see, the estimate for the first formula is closer to the actual 2-domination number of the graph, which is what is expected from the preliminary calculations.

Example 2: (See Figure 4.28)

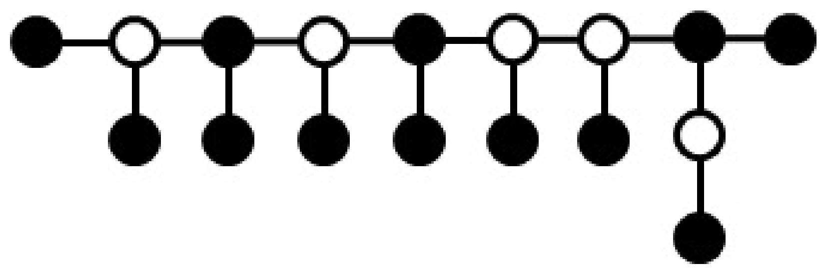

Figure 4.28: Sample Tree T_2 with $n = 17$, $n_1 = 9$, $n_2 = 1$, and $\gamma_2(T) = 12$

Note that when considering information about the group in Figure 4.28, $n > n_1 + 2n_2$ (that is $17 > 9+2(1)$).

Because the total number of vertices in the graph is greater than or equal to the number of leaves plus two times the number of vertices with degree one, formula two should provide the better estimate.

Comparing Estimates:

$$\gamma_2(T_2) = 12 \leq \frac{17+9}{2} = 13 > 12 = \frac{17+9(2)+1}{3}$$

As expected, the second formula provided a better estimate for the minimum 2-domination number for 4.28.

Chapter 5

Discussion

Though the project strayed from the initial goal of analyzing the relationship between the 2-domination number and the annihilation number, I made a lot of progress by generating new bounds for the 2-domination number. The original relation, $\gamma_2(G) \leq a(G)+1$, proposed by DeLaVina and Pepper about information collected from *Graffiti.pc* provided the initial idea and framework for this project even though it was not the main focus. After many cases were considered and many different graphs were discussed, the upper bound for the 2-domination number of trees, $\gamma_2(T) \leq \frac{1}{3}(n+2n_1+n_2)$, was proven. I utilized proof by contradiction to prove the non-existence of the smallest counter-example to our claim. Though this method was tedious it proved effective. A case was considered for all possibilities by examining graphs with varying numbers of leaves on the maximum paths. From there, more specifications were required based on other dependent paths. As can be seen from Section 4.5, there are examples of graphs who have a better upper bound from the original formula $\frac{n+n_1}{2}$, but there are also many cases that could benefit from our new bound. Our bound would be particularly useful with trees having many vertices with degrees greater than 2. The inequality designating which formula will yield a better bound was calculated by setting the two bounds equal to one another and solving for n. Any graph that satisfies $n > n_1+2n_2$ would receive a significantly better bound by the formula proved in this project. I also proved that $\gamma_2(T) \leq \gamma_2(T_1)+\gamma_2(T_2)$, which is a useful tool for working with 2-domination sets. This relation was used to compare the potential counter-examples to their sub-graphs in order to show that a minimum counter-example did not exist.

Much more research can be done in this area of graph theory. To begin with, the bound $\gamma_2(T) \leq \frac{1}{3}(n+2n_1+n_2)$ could be generalized for any graph as opposed to just trees, or a new bound for a larger class of graphs could be suggested. There could even be a formula so specific as to yield the exact value for the 2-domination number of a particular class of graphs. The initially proposed idea of a relation between the annihilation number and the 2-domination number has yet to be completely analyzed. Also, the formula $p(k,G)$ used to generate the bound could be tested for other values of $k > 3$ in order to potentially yield new upper bounds. If, for example, $k = 4$ for the formula $p(k,G)$, then a new bound would be $\gamma_2(T) \leq \frac{1}{4}(n+3n_1+2n_2+n_3)$. The tightness of each bound could be compared to see which was better. There can also be more work done with the k-domination number for values greater than 2. An example of such bounds can be found in a paper written by Y. Caro and Y. Roditty titled *A note on the k-domination number of a graph*. They provide the inequality $\gamma_k(G) \leq \frac{kp}{k+1}$ when $\gamma(G) > k$ and $p = |G|$ [Caro and Roddity, 1990]. For $k = 3$, this would translate to $\gamma_3 \leq \frac{3p}{4}$.

While working with variables for so long, it is easy to forget the practical applications of progress. Many interesting examples of graph theory applications and the domination number appear in advertising and computer programming. One example to consider is advertising on social media. Let us say that you are advertising your product on Facebook, and Facebook charges you for each post on someones time-line. You only want to post on the time-lines of those who either already like your page or on the time-lines of those who are friends with two people who like your page. The graph of such a scenario would have the Facebook users as vertices connected by edges representing friendships. Bounding the 2-domination number of your followers would give perspective on the costs of the advertising endeavor. In another example, let us say that you are a new sandwich store owner moving to a town and want to advertise your business by sending out free meal gift certificates (good for two people) to a nearby college campus. In order to be as thrifty as possible, you want to distribute these meals in such a way that everyone in each class leading up to lunch either gets a free meal or has a class with two other people who got a free meal. This way, those who did not get a free meal can hear about where these people are going to eat or maybe even be the recipient of the second free meal on the card. Essentially, you are finding the 2-domination number for the graph whose vertices are the college's morning class enrollment, such that two vertices are adjacent if they correspond to students in the same class. A bound on such a number would provide you with the information necessary to decide how many free gift certificates you want to give out. The broad spectrum of uses for graph theory extends much further than these examples though, and every additional tool added to the arsenal of formulas, bounds, and other analytical strategies is important.

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