Summer 2017

# Eignefunctions for Partial Differential Equations on Two-Dimensional Domains With Piecewise Constant Coefficients

Abdullah Muheel Momit Aurko
*University of Southern Mississippi*

# EIGENFUNCTIONS FOR PARTIAL DIFFERENTIAL EQUATIONS ON TWO-DIMENSIONAL DOMAINS WITH PIECEWISE CONSTANT COEFFICIENTS

by

Abdullah Muheel Momit Aurko

A Thesis
Submitted to the Graduate School,
the College of Science and Technology,
and the Department of Mathematics
of The University of Southern Mississippi
in Partial Fulfillment of the Requirements
for the Degree of Master of Science

August 2017

EIGENFUNCTIONS FOR PARTIAL DIFFERENTIAL EQUATIONS ON

TWO-DIMENSIONAL DOMAINS WITH PIECEWISE CONSTANT COEFFICIENTS

by Abdullah Muheel Momit Aurko

August 2017

Approved by:

---

Dr. James V. Lambers, Committee Chair
Associate Professor, Mathematics

---

Dr. Haiyan Tian, Committee Member
Associate Professor, Mathematics

---

Dr. Huiqing Zhu, Committee Member
Associate Professor, Mathematics

---

Dr. Bernd Schroeder
Chair, Department of Mathematics

---

Dr. Karen S. Coats
Dean of the Graduate School

ABSTRACT

EIGENFUNCTIONS FOR PARTIAL DIFFERENTIAL EQUATIONS ON TWO-DIMENSIONAL DOMAINS WITH PIECEWISE CONSTANT COEFFICIENTS

by Abdullah Muheel Momit Aurko

August 2017

In this thesis, we develop a highly accurate and efficient algorithm for computing the solution of a partial differential equation defined on a two-dimensional domain with discontinuous coefficients. An example of such a problem is for modeling the diffusion of heat energy in two space dimensions, in the case where the spatial domain represents a medium consisting of two different but homogeneous materials, with periodic boundary conditions.

Since diffusivity changes based on the material, it will be represented using a piecewise constant function, and this results in the formation of a complicated mathematical model. Such a model is impossible to solve analytically, and is very difficult to solve using existing numerical methods, thus the implementation of an alternative approach.

In this thesis, we take an approach that represents the solution as a linear combination of functions which change frequencies at the interfaces between different materials. Unlike previous work in the one-dimensional case, these functions are not all wave functions, like sine and cosine since we also have sinh and cosh functions. It will be demonstrated that by computing the eigenvalues and eigenfunctions to construct a basis of such functions- both independently and simultaneously, in conjunction with the secant method- a mathematical model for heat diffusion through different materials in two space dimensions can be solved much more efficiently and accurately than using conventional time-stepping methods.

# ACKNOWLEDGMENTS

I would like to take this opportunity to thank my advisor, Dr. James Lambers, who provided the inspiration for my research, and was its cornerstone. A big thank you to all my friends and roommates, who dealt with me on the bad days and encouraged me to soldier on. And last but certainly not least, I express my gratitude to my entire family, especially Ammu and Titir. I love you all.

# TABLE OF CONTENTS

# LIST OF ILLUSTRATIONS

**Figure**

# LIST OF TABLES

**Table**

# LIST OF ABBREVIATIONS

**PDE** - Partial Differential Equation
**CFL** - Courant-Friedrichs-Lewy

# NOTATION AND GLOSSARY

## General Usage and Terminology

The notation used in this text represents fairly standard mathematical and computational usage. In many cases these fields tend to use different preferred notation to indicate the same concept, and these have been reconciled to the extent possible, given the interdisciplinary nature of the material. In particular, the notation for partial derivatives varies extensively, and the notation used is chosen for stylistic convenience based on the application. While it would be convenient to utilize a standard nomenclature for this important symbol, the many alternatives currently in the published literature will continue to be utilized.

The blackboard fonts are used to denote standard sets of numbers: $\mathbb{R}$ for the field of real numbers, $\mathbb{C}$ for the complex field, $\mathbb{Z}$ for the integers, and $\mathbb{Q}$ for the rationals. The capital letters, $A, B, \cdots$ are used to denote matrices, including capital greek letters, e.g., $\Lambda$ for a diagnonal matrix. Functions which are denoted in boldface type typically represent vector valued functions, and real valued functions usually are set in lower case roman or greek letters. Caligraphic letters, e.g., $\mathcal{V}$, are used to denote spaces such as $\mathcal{V}$ denoting a vector space, $\mathcal{H}$ denoting a Hilbert space, or $\mathcal{F}$ denoting a general function space. Lower case letters such as $i, j, k, l, m, n$ and sometimes $p$ and $d$ are used to denote indices.

Vectors are typset in square brackets, e.g., $[\cdot]$, and matrices are typeset in parenthesese, e.g., $(\cdot)$. In general the norms are typeset using double pairs of lines, e.g., $||\cdot||$, and the abolute value of numbers is denoted using a single pairs of lines, e.g., $|\cdot|$. Single pairs of lines around matrices indicates the determinant of the matrix.

# Chapter 1

# INTRODUCTION

The heat equation

$$u_t = \nabla \cdot (\alpha \nabla u) + f$$

describes the diffusion of heat energy within any medium, but of particular interest is when the heat flux is across two or more different materials. The coefficient $\alpha$ is discontinuous across the interfaces between those different materials, which results in a jump in the normal derivative, $u_n$ [13]. Therefore, in this thesis, the actual PDE we are solving has the form:

$$u_t = \alpha^2 \Delta u = \alpha^2 (u_{xx} + u_{yy})$$

which has the domain $[0, 2\pi] \times [0, 2\pi]$ and where $\alpha$ is the piecewise constant coefficient with

$$\alpha = \begin{cases} \alpha_1 & \text{for} \quad 0 \leq x < 2\pi\rho \\ \alpha_2 & \text{for} \quad 2\pi\rho \leq x < 2\pi \end{cases}$$

with $0 < \rho < 1$.

Much like the above problem, discontinuities occur in a lot of physical phenomena, such as large water waves, because at the interface between water and air, the behavior of water waves is very erratic [10]. Interface problems with discontinuity are prevalent in many physical applications [17] such as electromagnetic wave propagation [19, 16], fluid mechanics [12], materials science [9], and biological science- especially in the study of blood flow patterns around valves in the heart [18] and tumor growth in the body [14].

When we try to analytically solve a linear, homogeneous PDE with a constant coefficient on a finite interval, the solution is represented in a series of sines and/or cosines since they are eigenfunctions of $\alpha^2(u_{xx} + u_{yy})$, the spatial differential operator in two dimensions. However, this is not a viable approach in cases where the coefficients are not constant, because the eigenfunctions are not known, except in special cases [1].

The discontinuity is encountered because the diffusivity changes based on the material. Unlike the heat equation with a constant coefficient, it is impossible to analytically solve the heat equation with a discontinuous coefficient. Common methods of solving PDEs, such as separation of variables, cannot be applied in this case. It is very difficult to solve such problems even numerically because discontinuous functions cannot be accurately represented using linear combinations of sines and cosines of fixed frequencies, such as those used to represent the solution in terms of a Fourier series [6].

When numerical methods are used to solve PDEs with discontinuous coefficients, it leads to discontinuities in the computed solution and/or its derivatives, due to the discontinuities in the coefficient $\alpha^2(x,y)$. Generally, the more continuous derivatives a function has, the more rapidly its Fourier series converges [8], but in this case, rapid convergence does not occur due to the discontinuities in the coefficient. This results in the solutions having non-negligible high frequency components [8]. It leads to the troublesome Gibbs phenomenon [8], which causes solutions to exhibit nonphysical oscillations. There is literature [7] devoted to finding and describing methods to remove such oscillations, but such methods become impractical when we look to implement them in the setting of a time-stepping method for solving a PDE. When using an implicit time-stepping method that causes damping of high-frequency components, it will inevitably progress toward a smooth steady-state solution, as any discontinuities will fade over time. Before then, numerical methods are bound to run into difficulty due to stiffness. Stiffness occurs when solutions have both low and high-frequency components that are coupled together and cannot be computed independently of one another, because they cannot be separated [2]. Due to stiffness, the highest frequency component in the solution forces the time step used in numerical methods to be very small, despite making a negligible contribution to the solution. This results in a drastic increase in computational effort and time. This time step constraint is due to the CFL condition, which indicates how small the time step must be relative to the grid spacing [8]. To get around these problems encountered by numerical methods to solve this complicated yet important problem, we consider an alternative approach.

We can alleviate the above-mentioned problems if approximate eigenfunctions of the spatial differential operator could be obtained in the case of a discontinuous coefficient. Unlike the one-dimensional case, this cannot be accomplished by representing the solution as a linear combination of wave functions that change frequencies at the interfaces between different materials [6] since our solution will contain sinh and cosh functions. A similar eigenproblem involving a discontinuous coefficient, but with Dirichlet boundary conditions, was described and solved in [4]; however, it did not include a practical numerical method for solving the equations that characterize the eigenfunctions.

The goal in this thesis is to circumvent these difficulties, and solve these complicated yet important problems easily and efficiently. It is the continuation of the work done by Garon and Lambers for computing the solution of a partial differential equation with discontinuous coefficients in one dimension [6], where they use the SAK principle, derived from the Uncertainty Principle by Fefferman [3], to develop an algorithm for computing the eigenfunctions of the spatial differential operator.

In this thesis, we compute eigenfunctions of a PDE that's defined on a two-dimensional domain, with discontinuous coefficients, which occur at the interface between the different materials. Our problem is different from the one-dimensional case since our spatial differential operator is non self-adjoint as opposed to the self-adjoint case considered in one dimension. Following the approach implemented by Min and Gottlieb in their paper [15], we impose continuity at the interface and we also impose periodicity on the functions at the boundaries. The first derivative is also continuous and periodic [15].

To find the eigenfunctions, we were essentially looking for eigenfunctions of second differentiation since the problem becomes a constant coefficient problem on either side of the discontinuity due to the above conditions. This enables us to treat it as an eigenvalue problem for the second derivative operator. We know that examples of such functions comprise sin, cos and exponential functions. Since the sin, cos functions produced all the eigenfunctions for the one-dimensional case [6], we tried using sin and cos functions, which are periodic functions with uniform frequency, but as we analyzed the behavior of the eigenfunctions, it was clear that we did not find all the 50 smallest eigenvalues generated by the MATLAB *eigs* function. Thus, we tried sinh and cosh functions and found that having sinh and cosh functions for the second piece, i.e. the $\sinh\cosh$ case gave all the rest of the eigenvalues which were not computed using the $\sin\cos$ case.

The outline of the thesis is as follows: Chapter 2 will present the discontinuous eigenvalue problem and describe how it will be solved. Chapter 3 will present the details of obtaining a practical and efficient algorithm for computing eigenfunctions of the spatial differential operator and describe how these eigenfunctions are used to solve the PDE. Chapter

4 will present numerical results, and Chapter 5 will have the conclusion and directions for future work.

# Chapter 2

# THE DISCONTINUOUS EIGENVALUE PROBLEM

Consider the following non self-adjoint eigenvalue problem with a piecewise constant coefficient:

$$-\frac{d^2u}{dx^2} = \lambda\varepsilon(x)u \quad \text{for} \quad x \in (-\pi,\pi), \tag{2.1}$$

where $\varepsilon(x) = 1$ for $x \in (-\pi,0)$ and $\varepsilon(x) = \beta^2$ for $x \in [0,\pi)$, $\beta \neq 1$. The $H_p^2[-\pi,\pi]$ eigenfunction $u_l(x)$ (the $p$ stands for periodic) is given by

$$u_l(x) = \begin{cases} C\cos(\sqrt{\lambda_l}x) + \beta D\sin(\sqrt{\lambda_l}x), & -\pi \le x \le 0 \\ C\cos(\beta\sqrt{\lambda_l}x) + D\sin(\beta\sqrt{\lambda_l}x), & 0 \le x \le \pi \end{cases} \tag{2.2}$$

where the constants $C$, $D$ and the eigenvalue $\lambda_l$ are determined by the demand that the system

$$\begin{aligned} C(\cos\sqrt{\lambda}\pi - \cos\beta\sqrt{\lambda}\pi) + D(-\beta\sin\sqrt{\lambda}\pi - \sin\beta\sqrt{\lambda}\pi) &= 0 \quad (2.3) \\ C(\sin\sqrt{\lambda}\pi + \beta\sin\beta\sqrt{\lambda}\pi) + D(\beta\cos\sqrt{\lambda}\pi - \beta\cos\beta\sqrt{\lambda}\pi) &= 0 \quad (2.4) \end{aligned}$$

has a nontrivial solution [15].

The two-dimensional eigenvalue problem is:

$$\alpha^2(u_{xx} + u_{yy}) = \lambda u.$$

The proposed forms of the eigenfunctions are as follows:
For the $\sin\cos$ case:

$$V_j(x,y) = \begin{cases} V_{j1}(x,y) & 0 \le x < 2\pi\rho,\ 0 \le y < 2\pi \\ V_{j2}(x,y) & 2\pi\rho \le x < 2\pi,\ 0 \le y < 2\pi \end{cases}$$

with $0 < \rho < 1$, where

$$V_{j1}(x,y) = (A_1\cos(\omega_1 x) + B_1\sin(\omega_1 x))(C_1\cos(\eta_1 y) + D_1\sin(\eta_1 y)) \tag{2.5}$$

and

$$V_{j2}(x,y) = (A_2\cos(\omega_2 x) + B_2\sin(\omega_2 x))(C_2\cos(\eta_2 y) + D_2\sin(\eta_2 y)) \tag{2.6}$$

For the sinh cosh case:

$$V_j(x,y) = \begin{cases} V_{j1}(x,y) & 0 \le x < 2\pi\rho,\ 0 \le y < 2\pi \\ V_{j2}(x,y) & 2\pi\rho \le x < 2\pi,\ 0 \le y < 2\pi \end{cases}$$

with $0 < \rho < 1$, where

$$V_{j1}(x,y) = (A_1 \cos(\omega_1 x) + B_1 \sin(\omega_1 x))(C_1 \cos(\eta_1 y) + D_1 \sin(\eta_1 y)) \tag{2.7}$$

and

$$V_{j2}(x,y) = (A_2 \cosh(\omega_2 x) + B_2 \sinh(\omega_2 x))(C_2 \cos(\eta_2 y) + D_2 \sin(\eta_2 y)) \tag{2.8}$$

In both cases above, $V_{j1}(x,y)$ corresponds to the value of the eigenfunction before the discontinuity, and $V_{j2}(x,y)$ corresponds to the value of the eigenfunction after the discontinuity. We had chosen the discontinuity to be only at $x = \pi$, as $x$ goes from 0 to $2\pi$, in the $x$-direction. So, the formula for the $y$-part remains the same in both (2.5-2.6) and (2.7-2.8). This implies that $\eta_1 = \eta_2$, where it is equal to some integer, since $y$ goes from 0 to $2\pi$ and the $y$-part is $2\pi$-periodic. This results in the coefficients $C_1, C_2, D_1$ and $D_2$ being assigned arbitrary values, and therefore enabling us to easily eliminate them.

It leaves us to deal with the following equations and variables:

For the sin cos case:
$V_{j1}(x,y) = A_1 \cos(\omega_1 x) + B_1 \sin(\omega_1 x)$
$V_{j2}(x,y) = A_2 \cos(\omega_2 x) + B_2 \sin(\omega_2 x)$

For the sinh cosh case:
$V_{j1}(x,y) = A_1 \cos(\omega_1 x) + B_1 \sin(\omega_1 x)$
$V_{j2}(x,y) = A_2 \cosh(\omega_2 x) + B_2 \sinh(\omega_2 x)$

$\sqrt{A^2 + B^2}$ in each piece represents the amplitude of the function and the angle between $A$ and $B$ in each piece indicates the phase shift. $\omega_1$ and $\omega_2$ are the frequencies in the $x$-direction of $V_{j1}(x,y)$ and $V_{j2}(x,y)$ respectively, in both cases, while the common value of $\eta_1$ and $\eta_2$ is the frequency in the $y$-direction in both pieces.

While the first set of expressions had only cos and sin as eigenfunctions in both pieces, the second set of expressions had cos and sin as eigenfunctions for its first piece, and had cosh and sinh as eigenfunctions for its second piece.

## 2.1 Eliminating more variables

Due to the periodicity of the functions and the boundary conditions, we know that for each set of eigenfunctions, the value and the partial derivative with respect to $x$ of the first piece at $x=0$ must be equal to the value and the partial derivative with respect to $x$ of the second piece at $x=2\pi$, respectively. Additionally, due to continuity across the interface, the same applies at $x=\pi$ for both the eigenfunction and its first partial derivative with respect to $x$. These properties enabled us to eliminate a lot of the variables in the above expressions for the eigenfunctions.

### 2.1.1 The sin cos case

- Since the value of the first piece at $x=0$ must be equal to the value and of the second piece at $x=2\pi$, we have

$$A_1 = A_2\cos(\omega_2 2\pi) + B_2\sin(\omega_2 2\pi). \tag{2.9}$$

- Since the partial derivative with respect to $x$ of the first piece at $x=0$ must be equal to the partial derivative with respect to $x$ of the second piece at $x=2\pi$, we have

$$B_1 = -\left(\frac{\omega_2}{\omega_1}\right)A_2\sin(\omega_2 2\pi) + \left(\frac{\omega_2}{\omega_1}\right)B_2\cos(\omega_2 2\pi) \tag{2.10}$$

  For the $x$-part, the coefficients $A_1$ and $B_1$ are expressed in terms of the remaining variables and functions by equations (2.9) and (2.10), respectively.

- Since the value of the first piece at $x=\pi$ must be equal to the value of the second piece at $x=\pi$, we have

$$A_1\cos(\omega_1\pi) + B_1\sin(\omega_1\pi) = A_2\cos(\omega_2\pi) + B_2\sin(\omega_2\pi). \tag{2.11}$$

- Since the partial derivative with respect to $x$ of the first piece at $x=\pi$ must be equal to the partial derivative with respect to $x$ of the second piece at $x=\pi$, we have

$$-\omega_1 A_1\sin(\omega_1\pi) + \omega_1 B_1\cos(\omega_1\pi) = -\omega_2 A_2\sin(\omega_2\pi) + \omega_2 B_2\cos(\omega_2\pi). \tag{2.12}$$

We substitute the value of $A_1$ and $B_1$ into equations (2.11) and (2.12) from equations (2.9) and (2.10). Then, we solve equations (2.11) and (2.12) in terms of the variables $A_2$ and $B_2$, by forming a $2 \times 2$ homogeneous system, whose entries are the coefficients of the equations that characterize the eigenfunctions. We do this for each of the two cases and find when its determinant is equal to 0, to obtain a non-trivial solution.

The $2 \times 2$ matrix $F$ that characterizes the eigenfunctions for the $\sin\cos$ case is

$$F = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \tag{2.13}$$

where

$$f_{11} = \cos(\omega_2 2\pi)\cos(\omega_1\pi) - \left(\frac{\omega_2}{\omega_1}\right)\sin(\omega_2 2\pi)\sin(\omega_1\pi) - \cos(\omega_2\pi)$$
$$f_{12} = \sin(\omega_2 2\pi)\cos(\omega_1\pi) + \left(\frac{\omega_2}{\omega_1}\right)\cos(\omega_2 2\pi)\sin(\omega_1\pi) - \sin(\omega_2\pi)$$
$$f_{21} = -\omega_1\cos(\omega_2 2\pi)\sin(\omega_1\pi) - \omega_2\sin(\omega_2 2\pi)\cos(\omega_1\pi) + \omega_2\sin(\omega_2\pi)$$
$$f_{22} = -\omega_1\sin(\omega_2 2\pi)\sin(\omega_1\pi) + \omega_2\cos(\omega_2 2\pi)\cos(\omega_1\pi) - \omega_2\cos(\omega_2\pi)$$

The value of $\omega_1$ for the $\sin\cos$ case is found by:

$$\omega_1 = \sqrt{\frac{\omega_2^2\alpha_2^2}{\alpha_1^2} + \frac{\eta_1^2\alpha_2^2}{\alpha_1^2} - \eta_1^2} \tag{2.14}$$

Equation 2.14 is derived by:

- using the fact that $\eta_1 = \eta_2$, where it is equal to a fixed integer.
- $\alpha_1$ and $\alpha_2$ are the values of the piecewise constant coefficient before and after the discontinuity, respectively.
- $\Delta \cdot V_{j1}(x,y) \cdot \alpha_1^2 = \lambda \cdot V_{j1}(x,y)$
- $\Delta \cdot V_{j2}(x,y) \cdot \alpha_2^2 = \lambda \cdot V_{j2}(x,y)$

  where $\Delta$ is the Laplacian and $\lambda$ is the eigenvalue which is the same in both cases. This enables us to equate the above two statements for $\lambda$ and find the value of $\omega_1$.

### 2.1.2 The sinh cosh case

Due to the same properties of periodicity and continuity at the interface, for the $\sinh\cosh$ case, we have:

$$A_1 = A_2\cosh(\omega_2 2\pi) + B_2\sinh(\omega_2 2\pi)), \tag{2.15}$$

$$B_1 = \left(\frac{\omega_2}{\omega_1}\right) A_2 \sinh(\omega_2 2\pi) + \left(\frac{\omega_2}{\omega_1}\right) B_2 \cosh(\omega_2 2\pi), \tag{2.16}$$

$$A_1 \cos(\omega_1 \pi) + B_1 \sin(\omega_1 \pi) = A_2 \cosh(\omega_2 \pi) + B_2 \sinh(\omega_2 \pi), \tag{2.17}$$

$$-\omega_1 A_1 \sin(\omega_1 \pi) + \omega_1 B_1 \cos(\omega_1 \pi) = \omega_2 A_2 \sinh(\omega_2 \pi) + \omega_2 B_2 \cosh(\omega_2 \pi). \tag{2.18}$$

The $2 \times 2$ matrix G that characterizes the eigenfunctions for the sinh cosh case is

$$G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \tag{2.19}$$

where
$g_{11} = \cosh(\omega_2 2\pi)\cos(\omega_1 \pi) + \left(\frac{\omega_2}{\omega_1}\right) \sinh(\omega_2 2\pi)\sin(\omega_1 \pi) - \cosh(\omega_2 \pi)$
$g_{12} = \sinh(\omega_2 2\pi)\cos(\omega_1 \pi) + \left(\frac{\omega_2}{\omega_1}\right) \cosh(\omega_2 2\pi)\sin(\omega_1 \pi) - \sinh(\omega_2 \pi)$
$g_{21} = -\omega_1 \cosh(\omega_2 2\pi)\sin(\omega_1 \pi) + \omega_2 \sinh(\omega_2 2\pi)\cos(\omega_1 \pi) - \omega_2 \sinh(\omega_2 \pi)$
$g_{22} = -\omega_1 \sinh(\omega_2 2\pi)\sin(\omega_1 \pi) + \omega_2 \cosh(\omega_2 2\pi)\cos(\omega_1 \pi) - \omega_2 \cosh(\omega_2 \pi)$

The value of $\omega_1$ for the sinh cosh case is found by:

$$\omega_1 = \sqrt{-\frac{\omega_2^2 \alpha_2^2}{\alpha_1^2} + \frac{\eta_1^2 \alpha_2^2}{\alpha_1^2} - \eta_1^2} \tag{2.20}$$

Equation 2.20 is derived by:

- using the fact that $\eta_1 = \eta_2$, where it is equal to a fixed integer
- $\alpha_1$ and $\alpha_2$ are the values of the piecewise constant coefficient before and after the discontinuity, respectively.
- $\Delta \cdot V_{j1}(x,y) \cdot \alpha_1^2 = \lambda \cdot V_{j1}(x,y)$
- $\Delta \cdot V_{j2}(x,y) \cdot \alpha_2^2 = \lambda \cdot V_{j2}(x,y)$

  where $\Delta$ is the Laplacian and $\lambda$ is the eigenvalue, which is the same in both cases. This enables us to equate the above two statements for $\lambda$ and find the value of $\omega_1$.

## 2.2 Examining the behavior of the eigenfunctions

To illustrate the behavior of the eigenfunctions, we consider the discontinuity to be at $x = 2\pi\rho$, where $0 < \rho < 1$, for three different $\rho$- values, showing the frequency for both the sin cos and sinh cosh cases at two different $\alpha$- values per $\rho$- value.

$\rho = 0.1, \alpha = 1.1, \sin\cos$ **case**

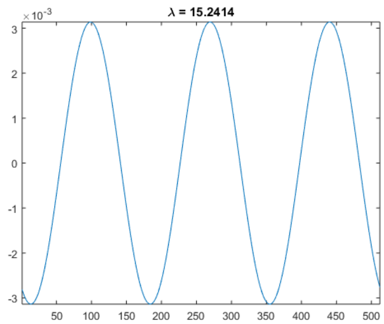


*Figure 2.1*: Index 37, $\eta = 3$, Restriction of $V_{37}(x,y)$ to the line $x = 0.4909$

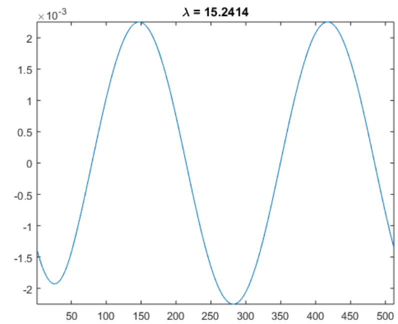


*Figure 2.2*: Index 37, $\eta = 3$, Restriction of $V_{37}(x,y)$ to the line $y = 0.4909$

$\rho = 0.1, \alpha = 1.1, \sinh\cosh$ **case**

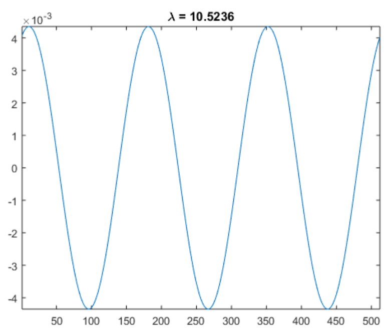


*Figure 2.3*: Index 25, $\eta = 3$, Restriction of $V_{25}(x,y)$ to the line $x = 0.4909$

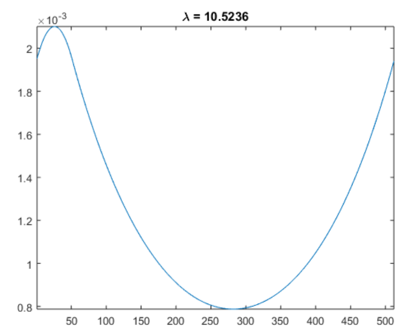


*Figure 2.4*: Index 25, $\eta = 3$, Restriction of $V_{25}(x,y)$ to the line $y = 0.4909$

In Figures (2.1)-(2.24), we examine one-dimensional slices of the eigenfunctions. For each case, the left side figure displays the restriction of the eigenfunction to the line $x = \frac{2\pi}{n} \cdot 40 = 0.4909$ and the right side figure shows the restriction of the eigenfunction to the line $y = \frac{2\pi}{n} \cdot 40 = 0.4909$, where $n = 512$ is the number of grid points, and 40 is because we selected the 40th row and column of the eigenfunction to be examined. $V_j(x,y)$ denotes the eigenfunction at the j-th index.

To find the eigenvalues, we tried using sin and cos functions, which are periodic functions with uniform frequency, but based on the figures above, it was clear that this wasn't the case, since we did not find all the eigenvalues generated by the MATLAB *eigs* function.

$\rho = 0.1, \alpha = 1.6, \sin\cos$ **case**

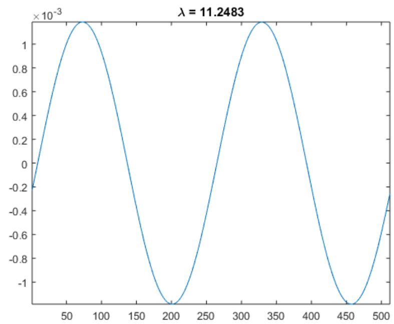


*Figure 2.5*: Index 15, $\eta = 2$, Restriction of $V_{15}(x,y)$ to the line $x = 0.4909$

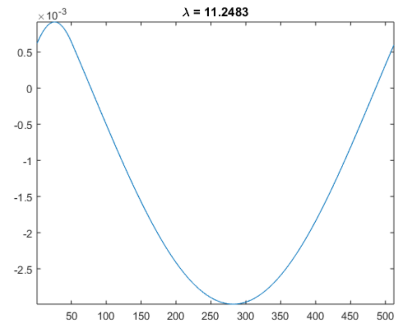


*Figure 2.6*: Index 15, $\eta = 2$, Restriction of $V_{15}(x,y)$ to the line $y = 0.4909$

$\rho = 0.1, \alpha = 1.6, \sinh\cosh$ **case**

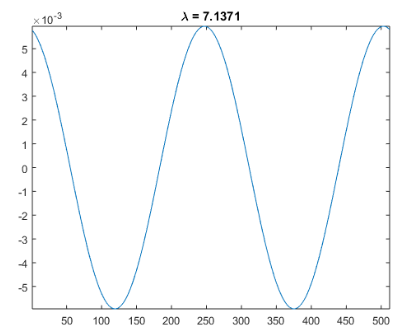


*Figure 2.7*: Index 10, $\eta = 2$, Restriction of $V_{10}(x,y)$ to the line $x = 0.4909$

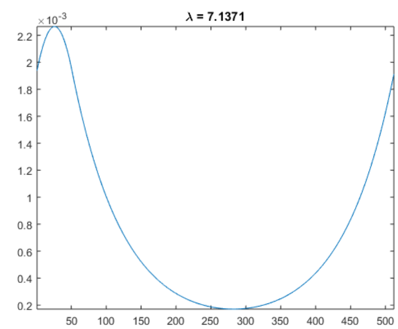


*Figure 2.8*: Index 10, $\eta = 2$, Restriction of $V_{10}(x,y)$ to the line $y = 0.4909$

Thus, we tried sinh and cosh functions. We tried all permutations of those, i.e. all pieces having sinh and cosh, and piece by piece with only the $\omega_1$ piece having sinh and cosh, followed by the $\omega_2$ piece having sinh and cosh. We found that the $\omega_2$ $\sinh\cosh$ case gave all of the remaining eigenvalues which were not computed using the $\sin\cos$ case.

$\rho = 0.5, \alpha = 1.1, \sin\cos$ **case**

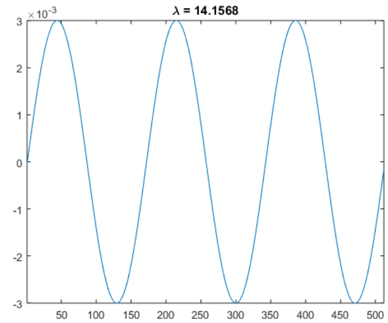


*Figure 2.9*: Index 37, $\eta = 3$, Restriction of $V_{37}(x,y)$ to the line $x = 0.4909$

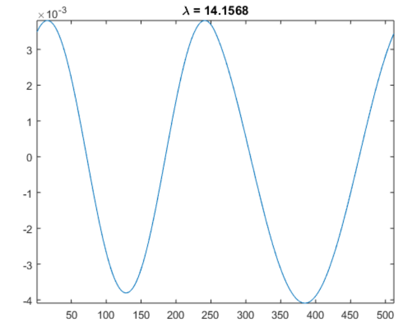


*Figure 2.10*: Index 37, $\eta = 3$, Restriction of $V_{37}(x,y)$ to the line $y = 0.4909$

$\rho = 0.5, \alpha = 1.1, \sinh\cosh$ **case**

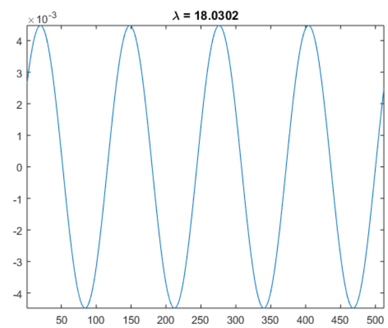


*Figure 2.11*: Index 49, $\eta = 4$, Restriction of $V_{49}(x,y)$ to the line $x = 0.4909$

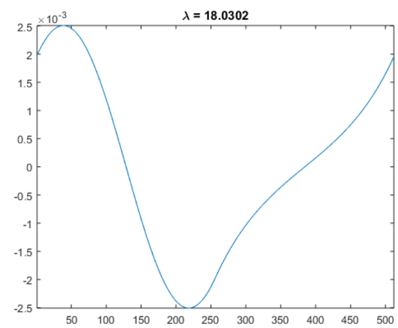


*Figure 2.12*: Index 49, $\eta = 4$, Restriction of $V_{49}(x,y)$ to the line $y = 0.4909$

$\rho = 0.5, \alpha = 1.5, \sin\cos$ **case**

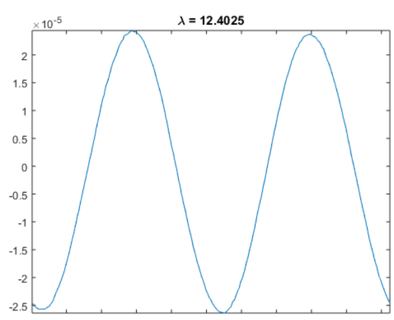


*Figure 2.13*: Index 27, $\eta = 2$, Restriction of $V_{27}(x,y)$ to the line $x = 0.4909$

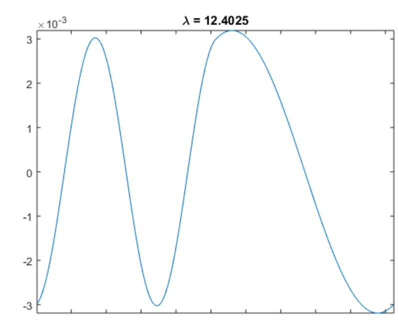


*Figure 2.14*: Index 27, $\eta = 2$, Restriction of $V_{27}(x,y)$ to the line $y = 0.4909$

$\rho = 0.5, \alpha = 1.5, \sinh\cosh$ **case**

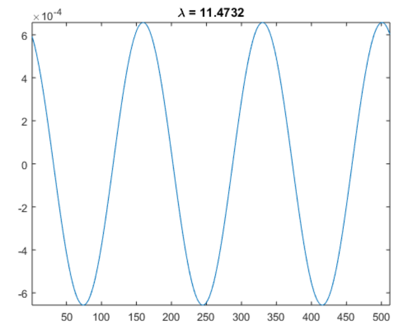


*Figure 2.15*: Index 24, $\eta = 3$, Restriction of $V_{24}(x,y)$ to the line $x = 0.4909$

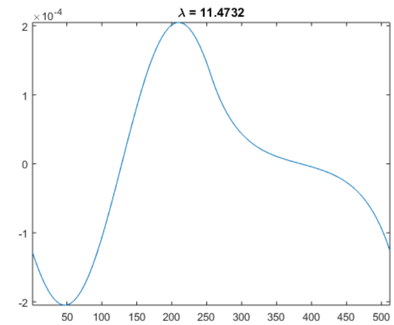


*Figure 2.16*: Index 24, $\eta = 3$, Restriction of $V_{24}(x,y)$ to the line $y = 0.4909$

$\rho = \ln 2, \alpha = 1.2, \sin\cos$ **case**

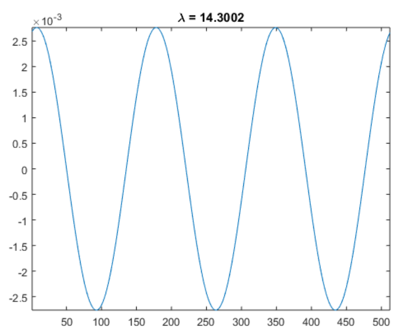


*Figure 2.17*: Index 40, $\eta = 3$, Restriction of $V_{40}(x,y)$ to the line $x = 0.4909$

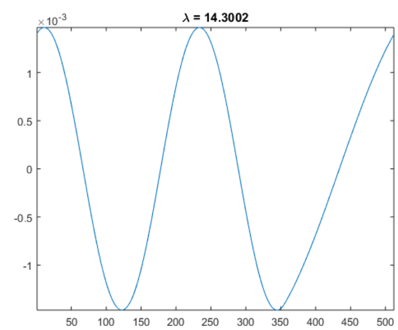


*Figure 2.18*: Index 40, $\eta = 3$, Restriction of $V_{40}(x,y)$ to the line $y = 0.4909$

$\rho = \ln 2, \alpha = 1.2, \sinh\cosh$ **case**

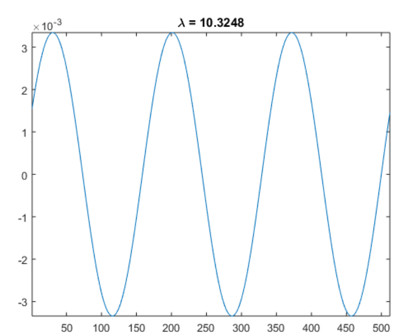


*Figure 2.19*: Index 29, $\eta = 3$, Restriction of $V_{29}(x,y)$ to the line $x = 0.4909$

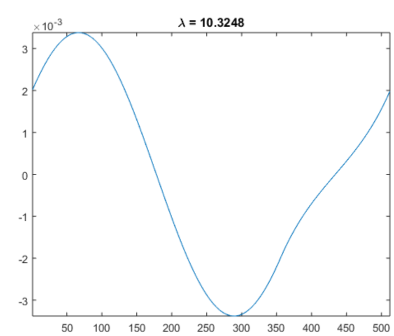


*Figure 2.20*: Index 29, $\eta = 3$, Restriction of $V_{29}(x,y)$ to the line $y = 0.4909$

$\rho = \ln 2, \alpha = 2, \sin\cos$ **case**

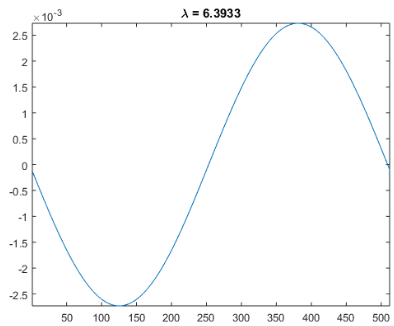


*Figure 2.21*: Index 15, $\eta = 1$, Restriction of $V_{15}(x,y)$ to the line $x = 0.4909$

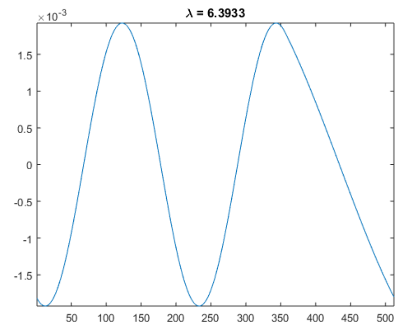


*Figure 2.22*: Index 15, $\eta = 1$, Restriction of $V_{15}(x,y)$ to the line $y = 0.4909$

$\rho = \ln 2, \alpha = 2, \sinh\cosh$ **case**

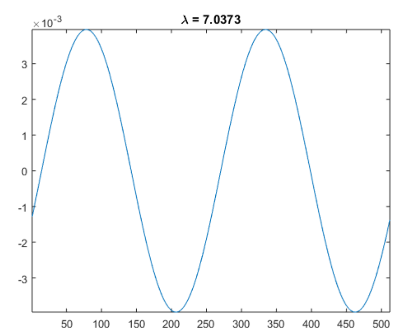


*Figure 2.23*: Index 18, $\eta = 2$, Restriction of $V_{18}(x,y)$ to the line $x = 0.4909$

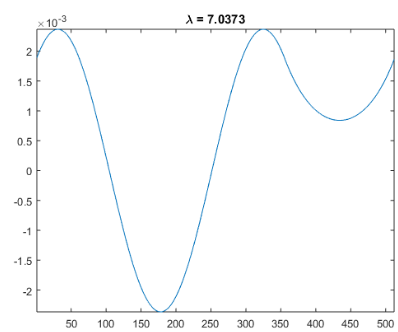


*Figure 2.24*: Index 18, $\eta = 2$, Restriction of $V_{18}(x,y)$ to the line $y = 0.4909$

# Chapter 3

# METHODOLOGY

## 3.1 Finding the correct $\omega, \eta$ pairs to compute the Eigenvalues

We solve

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \cdot \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.1}$$

for $A_2$ and $B_2$. The determinant of the matrix (2.13), which characterizes the eigenfunctions for the $\sin\cos$ case, and the secant method, are used to compute the eigenvalues of the operator.

In order for this system to have a nontrivial solution, the determinant of the matrix must be zero; therefore, the equation $f_{11}f_{22} - f_{12}f_{21} = 0$ is solved iteratively using the secant method to to obtain a proper value of $\omega_2$ for the $\sin\cos$ case.

Similar to (3.1), we solve

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \cdot \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.2}$$

for $A_2$ and $B_2$. The determinant of the matrix (2.19), which characterizes the eigenfunctions for the $\sinh\cosh$ case, and the secant method, are used to compute the eigenvalues of the operator. We use the secant method to iteratively solve the equation $g_{11}g_{22} - g_{12}g_{21} = 0$ to get a proper value of $\omega_2$ for the $\sinh\cosh$ case.

The secant method is used because it has a rapid rate of convergence [2], but unlike the more rapidly converging Newton's method, it does not require evaluating a derivative which would be very complex for this determinant [6]. The secant method uses initial guesses from the constant coefficient case since they are known analytically, where the constant would be the weighted harmonic average of $\alpha$.

Pseudo code for computing the eigenvalues:

$\varepsilon = 10^{-6}$;
$\rho = 0.5$;
$\omega_0 = \frac{1}{2(\alpha_1+\alpha_2)}$
$rad = \sqrt{\frac{-\ln\varepsilon}{t\alpha_2^2}}$

**for** $\eta = 0, \ldots, rad$
  **for** $\omega = \omega_0 : \omega_0 : rad$
    1. sin cos case
    **if** $\omega^2 + \eta^2 < 1.5 \times rad^2$
      $x_0 = \omega$
      $x_1 = x_0 + 0.01$
      $[s, niter] = secant(@(\omega_2)F\omega_2(\omega_2, \alpha_2, \eta), x_0, x_1)$
      **if** $abs(imag(s)) < 10^{-8}\ \&\ niter < 20$
        $\lambda = \alpha_2^2(s^2 + \eta^2)$
        **if** $\lambda < rad\ \&\ s > 10^{-10}$
          fill in vectors for $\lambda, \omega$, and $\eta$ for the sin cos case.
        **end**
      **end**
    **end**
    2. sinh cosh case
    **if** $\omega \leq \eta$
      $x_0 = \omega$
      $x_1 = x_0 + 0.01$
      $[s, niter] = secant(@(\omega_2)F\omega_3(\omega_2, \alpha_2, \eta), x_0, x_1)$
      $\omega_1 = \sqrt{-\frac{s^2\alpha_2^2}{\alpha_1^2} + \frac{\eta^2\alpha_2^2}{\alpha_1^2} - \eta^2}$
      **if** $abs(imag(s)) < 10^{-8}\ \&\ abs(imag(\omega_1)) < 10^{-8}\ \&\ niter < 20$
        $\lambda = \alpha_2^2(-s^2 + \eta^2)$
        **if** $\lambda < rad\ \&\ s > 10^{-10}$
          fill in vectors for $\lambda, \omega$, and $\eta$ for the sinh cosh case.
        **end**
      **end**
    **end**
  **end**
**end**

In the pseudo-code above, we assign the variable *rad* its value since we wish to compute the smallest possible eigenvalues from within that value of *rad*. This is because the solution operator is a decaying exponential, i.e. as $t \to \infty, e^{-\lambda t} \to 0$, and the larger the eigenvalue, the more negligible its contribution is to the solution. So, we decide to not include those large eigenvalues by choosing this specific *rad* value. The variable $\omega_0$ is equal to the weighted harmonic average of $\alpha$ divided by 4, which we use as an initial starting point for the $\omega$'s, as it goes through to *rad*. The output variable $s$ is the root found using the secant method and $niter$ is equal to the number of iterations needed for the secant method to converge. We choose the maximum value of $niter$ to be less than 20 because if secant method were to converge, it should converge in less than 20 iterations. We create vectors for storing the eigenvalue $\lambda$ and the corresponding $\omega$ and $\eta$ values for each eigenvalue. This enables us to find the correct $\omega$, $\eta$ pairs that correspond to the eigenvalues. $F_{\omega 2}$ and $F_{\omega 3}$ contain the matrix entries of the $2 \times 2$ homogeneous system (3.1) and (3.2) that we find the zeros of, for the $\sin\cos$ case and the $\sinh\cosh$ case, respectively.

### 3.2 Finding the Eigenfunctions

Having found the eigenvalues, we use each $\omega$, $\eta$ pair corresponding to each eigenvalue-$\lambda_j$ to construct the eigenfunctions piece by piece which will enable us to solve the partial differential equation.

When both $\omega$ and $\eta = 0$, we assign both $A_1$ and $A_2$ to be equal to 1, and both $B_1$ and $B_2$ to be equal to 0. In other cases, we express $A_1$ and $B_1$ in terms of $A_2$ and $B_2$ which we derived in Chapter 2. We express $A_2$ and $B_2$ in terms of the entries of the matrix which characterizes the eigenfunctions, in each case. $A_2 = \frac{-f_{12}}{\sqrt{f_{11}^2+f_{12}^2}}$ and $B_2 = \frac{f_{11}}{\sqrt{f_{11}^2+f_{12}^2}}$ for the $\sin\cos$ case and $A_2 = \frac{-g_{12}}{\sqrt{g_{11}^2+g_{12}^2}}$, $B_2 = \frac{g_{11}}{\sqrt{g_{11}^2+g_{12}^2}}$ for the $\sinh\cosh$ case. This is reflected in the pseudo-code for constructing the eigenfunctions and solving the PDE in the following section.

### 3.3 Solving the PDE

Pseudo code for constructing the eigenfunctions and solving the PDE:

**for** sin cos case

$\omega_1 = \sqrt{\frac{\omega_2^2 \alpha_2^2}{\alpha_1^2} + \frac{\eta_1^2 \alpha_2^2}{\alpha_1^2} - \eta_1^2}$

**if** $\omega_2 = 0$ and $\eta = 0$

$A_2 = 1; B_2 = 0;$

$A_1 = 1; B_1 = 0;$

**else**

$[A_2, B_2] = F\omega_4(\omega_2, \alpha_2, \eta)$

where $A_2 = \frac{-f_{12}}{\sqrt{f_{11}^2 + f_{12}^2}}$

and $B_2 = \frac{f_{11}}{\sqrt{f_{11}^2 + f_{12}^2}}$

$A_1 = A_2 \cos(\omega_2 2\pi) + B_2 \sin(\omega_2 2\pi)$

$B_1 = \frac{-\omega_2 A_2 \sin(\omega_2 2\pi) + \omega_2 B_2 \cos(\omega_2 2\pi)}{\omega_1}$

**end**

$\lambda = \alpha_2^2(\omega_2^2 + \eta^2)$

$W_j = \frac{V_j}{\alpha^2}$

$u_1 = \sum_{j=1} e^{-\lambda_j t} \frac{\langle W_j, F \rangle}{\langle V_j, W_j \rangle}$

**end**

**for** $\omega_2$ sinh cosh case

$\omega_1 = \sqrt{\frac{-\omega_2^2 \alpha_2^2}{\alpha_1^2} + \frac{\eta_1^2 \alpha_2^2}{\alpha_1^2} - \eta_1^2}$

$[A_2, B_2] = F\omega_5(\omega_2, \alpha_2, \eta)$

where $A_2 = \frac{-g_{12}}{\sqrt{g_{11}^2 + g_{12}^2}}$

and $B_2 = \frac{g_{11}}{\sqrt{g_{11}^2 + g_{12}^2}}$

$A_1 = A_2 \cosh(\omega_2 2\pi) + B_2 \sinh(\omega_2 2\pi)$

$B_1 = \frac{\omega_2 A_2 \sinh(\omega_2 2\pi) + \omega_2 B_2 \cosh(\omega_2 2\pi)}{\omega_1}$

$\lambda = \alpha_2^2(-\omega_2^2 + \eta^2)$

$W_j = \frac{V_j}{\alpha^2}$

$u_2 = \sum_{j=1} e^{-\lambda_j t} \frac{\langle W_j, F \rangle}{\langle V_j, W_j \rangle}$

The final solution $u$ is given by:

$u = u_1 + u_2$

**end**

In the pseudo-code above, both $F_{\omega 4}$ and $F_{\omega 5}$, the matrices that characterizes the eigenfunctions for the $\sin\cos$ case and the $\sinh\cosh$ case respectively, are assumed to be singular for the given value of $\omega$ and from there we compute the solution of the homogeneous system. By adding the solutions from each case above, we get our final solution of the partial differential equation.

Numerical results are obtained by computing each eigenvalue using our approach and comparing those with the list of the 50 smallest eigenvalues generated by the *eigs* function in MATLAB. *eigs* is applied to a matrix using the finite difference representation of the operator. The absolute error is computed to ascertain the accuracy of the method and the number of iterations taken for the secant method to converge in each case confirms the method's efficiency.

# Chapter 4

# NUMERICAL RESULTS

## 4.1 Computing the Eigenvalues and Finding the Eigenfunctions

In the following tables, $\lambda_j(exact)$ is the list of eigenvalues computed using the methods described in Chapter 3. $\lambda_j(eig)$ is the list of the 50 smallest eigenvalues generated by the *eigs* function in MATLAB, and $j$ is the index number of the eigenvalues found by $\lambda_j(eig)$.

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 3 | 1.09256 | 1.09255 | $1.4039E-05$ | 15 |
| 4 | 1.10248 | 1.10247 | $1.4624E-05$ | 11 |
| 5 | 2.18027 | 2.18024 | $2.8199E-05$ | 12 |
| 6 | 2.18027 | 2.18024 | $2.6718E-05$ | 12 |
| 7 | 2.21960 | 2.21957 | $2.6520E-05$ | 11 |
| 8 | 2.21960 | 2.21958 | $1.7860E-05$ | 11 |
| 11 | 4.37047 | 4.37024 | $2.3312E-04$ | 14 |
| 12 | 4.40970 | 4.40948 | $2.2074E-04$ | 9 |
| 13 | 5.41559 | 5.41535 | $2.4240E-04$ | 8 |
| 14 | 5.41559 | 5.41536 | $2.3237E-04$ | 8 |
| 15 | 5.46030 | 5.46006 | $2.4374E-04$ | 13 |
| 16 | 5.46030 | 5.46006 | $2.4150E-04$ | 13 |
| 17 | 5.52114 | 5.52091 | $2.2967E-04$ | 10 |
| 18 | 5.52114 | 5.52093 | $2.0832E-04$ | 10 |
| 19 | 5.64191 | 5.64166 | $2.4595E-04$ | 9 |
| 20 | 5.64191 | 5.64167 | $2.3405E-04$ | 9 |
| 21 | 8.72460 | 8.72413 | $4.7486E-04$ | 11 |
| 22 | 8.72460 | 8.72415 | $4.5354E-04$ | 11 |
| 23 | 8.87493 | 8.87449 | $4.3928E-04$ | 11 |
| 24 | 8.87493 | 8.87450 | $4.3262E-04$ | 11 |
| 27 | 9.83438 | 9.83325 | $1.1270E-03$ | 13 |
| 28 | 9.92100 | 9.91996 | $1.0385E-03$ | 8 |
| 31 | 10.92491 | 10.92369 | $1.2172E-03$ | 13 |
| 32 | 10.92491 | 10.92376 | $1.1511E-03$ | 13 |
| 33 | 11.03105 | 11.02992 | $1.1280E-03$ | 9 |
| 34 | 11.03105 | 11.02994 | $1.1033E-03$ | 9 |
| 35 | 11.45690 | 11.45571 | $1.1955E-03$ | 7 |
| 36 | 11.45690 | 11.45576 | $1.1466E-03$ | 7 |
| 37 | 14.15818 | 14.15682 | $1.3652E-03$ | 9 |
| 38 | 14.15818 | 14.15683 | $1.3557E-03$ | 9 |
| 39 | 14.19489 | 14.19352 | $1.3726E-03$ | 12 |
| 40 | 14.19489 | 14.19352 | $1.3726E-03$ | 12 |
| 41 | 14.36930 | 14.36799 | $1.3100E-03$ | 10 |
| 42 | 14.36930 | 14.36803 | $1.2696E-03$ | 10 |
| 43 | 14.51596 | 14.51458 | $1.3808E-03$ | 10 |
| 44 | 14.51596 | 14.51460 | $1.3582E-03$ | 10 |
| 47 | 17.48533 | 17.48176 | $3.5692E-03$ | 13 |
| 48 | 17.63529 | 17.63175 | $3.5397E-03$ | 8 |

*Table 4.1*: $\alpha_2 = 1.1, \sin\cos$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 1 | 1.08706 | 1.08705 | $1.5895E-05$ | 10 |
| 2 | 1.08706 | 1.08705 | $1.3778E-05$ | 10 |
| 9 | 4.26744 | 4.26722 | $2.1166E-04$ | 7 |
| 10 | 4.26744 | 4.26722 | $2.1166E-04$ | 7 |
| 25 | 9.41663 | 9.41553 | $1.0950E-03$ | 9 |
| 26 | 9.41663 | 9.41558 | $1.0461E-03$ | 9 |
| 29 | 10.72627 | 10.72516 | $1.1096E-03$ | 9 |
| 30 | 10.72627 | 10.72517 | $1.0996E-03$ | 9 |
| 45 | 16.51472 | 16.51146 | $3.2612E-03$ | 10 |
| 46 | 16.51472 | 16.51151 | $3.2069E-03$ | 10 |
| 49 | 18.03365 | 18.03018 | $3.4720E-03$ | 9 |
| 50 | 18.03365 | 18.03025 | $3.3947E-03$ | 9 |

*Table 4.2*: $\alpha_2 = 1.1, \omega_2 \sinh\cosh$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 3 | 1.17091 | 1.17089 | $1.3513E-05$ | 12 |
| 4 | 1.20972 | 1.20975 | $2.6527E-05$ | 8 |
| 5 | 2.32380 | 2.32375 | $4.9283E-05$ | 9 |
| 6 | 2.32380 | 2.32376 | $3.5746E-05$ | 9 |
| 7 | 2.47437 | 2.47433 | $3.8917E-05$ | 9 |
| 8 | 2.47437 | 2.47433 | $3.8917E-05$ | 9 |
| 11 | 4.68663 | 4.68638 | $2.4831E-04$ | 12 |
| 12 | 4.83565 | 4.83539 | $2.5844E-04$ | 8 |
| 15 | 5.85041 | 5.85012 | $2.9022E-04$ | 11 |
| 16 | 5.85041 | 5.85013 | $2.8222E-04$ | 11 |
| 17 | 6.07690 | 6.07661 | $2.8925E-04$ | 9 |
| 18 | 6.07690 | 6.07667 | $2.2577E-04$ | 9 |
| 19 | 6.43993 | 6.43966 | $2.7102E-04$ | 6 |
| 20 | 6.43993 | 6.43967 | $2.6248E-04$ | 6 |
| 21 | 9.34468 | 9.34412 | $5.6223E-04$ | 7 |
| 22 | 9.34468 | 9.34416 | $5.2656E-04$ | 7 |
| 25 | 9.85096 | 9.85046 | $4.9951E-04$ | 10 |
| 26 | 9.85096 | 9.85047 | $4.8785E-04$ | 10 |
| 27 | 10.55587 | 10.55463 | $1.2433E-03$ | 11 |
| 28 | 10.86845 | 10.86719 | $1.2570E-03$ | 9 |
| 31 | 11.72574 | 11.72438 | $1.3559E-03$ | 9 |
| 32 | 11.72574 | 11.72445 | $1.2927E-03$ | 9 |
| 33 | 12.10075 | 12.09953 | $1.2210E-03$ | 7 |
| 34 | 12.10075 | 12.09958 | $1.1698E-03$ | 7 |
| 35 | 12.97664 | 12.97522 | $1.4165E-03$ | 11 |
| 36 | 12.97664 | 12.97528 | $1.3604E-03$ | 11 |
| 37 | 15.24511 | 15.24353 | $1.5770E-03$ | 9 |
| 38 | 15.24511 | 15.24359 | $1.5228E-03$ | 9 |
| 39 | 15.34162 | 15.33997 | $1.6474E-03$ | 8 |
| 40 | 15.34162 | 15.34008 | $1.5371E-03$ | 8 |
| 41 | 15.81278 | 15.81129 | $1.4926E-03$ | 7 |
| 42 | 15.81278 | 15.81133 | $1.4537E-03$ | 7 |
| 43 | 16.18082 | 16.17926 | $1.5558E-03$ | 8 |
| 44 | 16.18082 | 16.17926 | $1.5558E-03$ | 8 |

*Table 4.3*: $\alpha_2 = 1.2, \sin\cos$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 1 | 1.15099 | 1.15098 | $1.5404E-05$ | 7 |
| 2 | 1.15099 | 1.15099 | $4.7918E-06$ | 7 |
| 9 | 4.37908 | 4.37886 | $2.1854E-04$ | 10 |
| 10 | 4.37908 | 4.37888 | $2.0310E-04$ | 10 |
| 13 | 5.68967 | 5.68941 | $2.6190E-04$ | 7 |
| 14 | 5.68967 | 5.68949 | $1.8012E-04$ | 7 |
| 23 | 9.51708 | 9.51601 | $1.0681E-03$ | 8 |
| 24 | 9.51708 | 9.51601 | $1.0681E-03$ | 8 |
| 29 | 11.08313 | 11.08197 | $1.1581E-03$ | 6 |
| 30 | 11.08313 | 11.08197 | $1.1581E-03$ | 6 |

*Table 4.4*: $\alpha_2 = 1.2, \omega_2 \sinh\cosh$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 3 | 1.23658 | 1.23657 | $1.3117E-05$ | 11 |
| 4 | 1.32121 | 1.32084 | $3.7290E-04$ | 8 |
| 5 | 2.43663 | 2.43654 | $9.2730E-05$ | 6 |
| 6 | 2.43663 | 2.43665 | $1.7830E-05$ | 6 |
| 7 | 2.75531 | 2.75517 | $1.3421E-04$ | 9 |
| 8 | 2.75531 | 2.75540 | $9.2019E-05$ | 9 |
| 11 | 4.95915 | 4.95849 | $6.5656E-04$ | 10 |
| 12 | 5.26990 | 5.26949 | $4.0765E-04$ | 10 |
| 15 | 6.19085 | 6.19037 | $4.8819E-04$ | 9 |
| 16 | 6.19085 | 6.19074 | $1.1204E-04$ | 9 |
| 17 | 6.64846 | 6.64756 | $8.9932E-04$ | 8 |
| 18 | 6.64846 | 6.64835 | $1.1041E-04$ | 8 |
| 19 | 7.23741 | 7.23697 | $4.3578E-04$ | 8 |
| 20 | 7.23741 | 7.23697 | $4.3578E-04$ | 8 |
| 23 | 9.96898 | 9.96842 | $5.6270E-04$ | 8 |
| 24 | 9.96898 | 9.96842 | $5.6270E-04$ | 8 |
| 25 | 10.82492 | 10.82305 | $1.8759E-03$ | 7 |
| 26 | 10.82492 | 10.82435 | $5.7679E-04$ | 7 |
| 27 | 11.20393 | 11.20205 | $1.8813E-03$ | 10 |
| 30 | 11.80535 | 11.80229 | $3.0553E-03$ | 10 |
| 31 | 12.45799 | 12.45621 | $1.7813E-03$ | 9 |
| 32 | 12.45799 | 12.45711 | $8.8561E-04$ | 9 |
| 33 | 13.15449 | 13.15266 | $1.8311E-03$ | 7 |
| 34 | 13.15449 | 13.15343 | $1.0550E-03$ | 7 |

*Table 4.5*: $\alpha_2 = 1.3, \sin\cos$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 1 | 1.19737 | 1.19725 | $1.2215E-04$ | 10 |
| 2 | 1.19737 | 1.19734 | $3.1434E-05$ | 10 |
| 9 | 4.43545 | 4.43513 | $3.1803E-04$ | 11 |
| 10 | 4.43545 | 4.43543 | $2.3760E-05$ | 11 |
| 13 | 5.87034 | 5.87022 | $1.2539E-04$ | 9 |
| 14 | 5.87034 | 5.87022 | $1.2539E-04$ | 9 |
| 21 | 9.56330 | 9.56245 | $8.4211E-04$ | 11 |
| 22 | 9.56330 | 9.56274 | $5.5536E-04$ | 11 |
| 28 | 11.27649 | 11.27532 | $1.1665E-03$ | 6 |
| 29 | 11.27649 | 11.27532 | $1.1665E-03$ | 6 |
| 35 | 13.90318 | 13.90172 | $1.4545E-03$ | 5 |
| 36 | 13.90318 | 13.90245 | $7.3175E-04$ | 5 |

*Table 4.6*: $\alpha_2 = 1.3, \omega_2 \sinh\cosh$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 3 | 1.29144 | 1.29142 | $2.1299E-05$ | 10 |
| 4 | 1.43619 | 1.43622 | $2.5163E-05$ | 10 |
| 5 | 2.52534 | 2.52529 | $4.6019E-05$ | 7 |
| 6 | 2.52534 | 2.52532 | $1.7320E-05$ | 7 |
| 7 | 3.05202 | 3.05196 | $5.1515E-05$ | 8 |
| 8 | 3.05202 | 3.05198 | $3.3053E-05$ | 8 |
| 11 | 5.19978 | 5.19940 | $3.8052E-04$ | 9 |
| 12 | 5.70193 | 5.70179 | $1.3744E-04$ | 10 |
| 15 | 6.50418 | 6.50382 | $3.5831E-04$ | 8 |
| 16 | 6.50418 | 6.50391 | $2.6431E-04$ | 8 |
| 17 | 7.21215 | 7.21176 | $3.8368E-04$ | 8 |
| 18 | 7.21215 | 7.21192 | $2.2928E-04$ | 8 |
| 19 | 7.91766 | 7.91729 | $3.7205E-04$ | 7 |
| 20 | 7.91766 | 7.91729 | $3.7205E-04$ | 7 |
| 23 | 10.72405 | 10.72338 | $6.6827E-04$ | 7 |
| 24 | 10.72405 | 10.72353 | $5.1689E-04$ | 7 |
| 27 | 11.69578 | 11.69508 | $7.0546E-04$ | 8 |
| 28 | 11.69578 | 11.69511 | $6.7370E-04$ | 8 |
| 29 | 11.81875 | 11.81727 | $1.4822E-03$ | 9 |
| 30 | 12.68898 | 12.68683 | $2.1479E-03$ | 10 |
| 31 | 13.17747 | 13.17594 | $1.5375E-03$ | 9 |
| 32 | 13.17747 | 13.17596 | $1.5118E-03$ | 9 |
| 33 | 14.13240 | 14.13061 | $1.7896E-03$ | 7 |
| 34 | 14.13240 | 14.13079 | $1.6129E-03$ | 7 |

*Table 4.7*: $\alpha_2 = 1.4, \sin\cos$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 1 | 1.23138 | 1.23136 | $2.5823E-05$ | 7 |
| 2 | 1.23138 | 1.23137 | $1.7013E-05$ | 7 |
| 9 | 4.46921 | 4.46900 | $2.1748E-04$ | 11 |
| 10 | 4.46921 | 4.46900 | $2.1748E-04$ | 11 |
| 13 | 5.99376 | 5.99347 | $2.9289E-04$ | 8 |
| 14 | 5.99376 | 5.99351 | $2.4772E-04$ | 8 |
| 21 | 9.59062 | 9.58935 | $1.2721E-03$ | 18 |
| 22 | 9.59062 | 9.58955 | $1.0657E-03$ | 18 |
| 25 | 11.39502 | 11.39378 | $1.2431E-03$ | 8 |
| 26 | 11.39502 | 11.39390 | $1.1292E-03$ | 8 |

*Table 4.8*: $\alpha_2 = 1.4, \omega_2 \sinh\cosh$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 3 | 1.33732 | 1.33730 | $1.7732E-05$ | 9 |
| 4 | 1.55379 | 1.55369 | $9.4777E-05$ | 11 |
| 5 | 2.59558 | 2.59551 | $7.8649E-05$ | 6 |
| 6 | 2.59558 | 2.59553 | $5.6126E-05$ | 6 |
| 7 | 3.35402 | 3.35396 | $6.1408E-05$ | 8 |
| 8 | 3.35402 | 3.35401 | $1.0251E-05$ | 8 |
| 11 | 5.41893 | 5.41865 | $2.7980E-04$ | 9 |
| 14 | 6.12056 | 6.12028 | $2.8312E-04$ | 9 |
| 15 | 6.81089 | 6.81051 | $3.7184E-04$ | 7 |
| 16 | 6.81089 | 6.81051 | $3.7184E-04$ | 7 |
| 17 | 7.74459 | 7.74419 | $3.9765E-04$ | 6 |
| 18 | 7.74459 | 7.74430 | $2.9101E-04$ | 6 |
| 25 | 11.68774 | 11.68706 | $6.7757E-04$ | 8 |
| 26 | 11.68774 | 11.68746 | $2.7893E-04$ | 8 |
| 27 | 12.40337 | 12.40250 | $8.7190E-04$ | 8 |
| 28 | 12.40337 | 12.40263 | $7.3772E-04$ | 8 |
| 29 | 12.43284 | 12.43145 | $1.3865E-03$ | 9 |

*Table 4.9*: $\alpha_2 = 1.5, \sin\cos$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 1 | 1.25685 | 1.25683 | $2.0206E-05$ | 10 |
| 2 | 1.25685 | 1.25684 | $1.4502E-05$ | 10 |
| 9 | 4.49170 | 4.49137 | $3.3305E-04$ | 7 |
| 10 | 4.49170 | 4.49148 | $2.2037E-04$ | 7 |
| 12 | 6.08168 | 6.08136 | $3.2398E-04$ | 8 |
| 13 | 6.08168 | 6.08139 | $2.8716E-04$ | 8 |
| 19 | 8.41654 | 8.41597 | $5.6566E-04$ | 8 |
| 20 | 8.41654 | 8.41608 | $4.6109E-04$ | 8 |
| 21 | 9.60881 | 9.60775 | $1.0559E-03$ | 19 |
| 22 | 9.60881 | 9.60777 | $1.0383E-03$ | 19 |
| 23 | 11.47449 | 11.47324 | $1.2530E-03$ | 8 |
| 24 | 11.47449 | 11.47336 | $1.1282E-03$ | 8 |

*Table 4.10*: $\alpha_2 = 1.5, \omega_2 \sinh\cosh$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 3 | 1.37581 | 1.37578 | $2.3658E-05$ | 8 |
| 4 | 1.67300 | 1.67300 | $5.7693E-06$ | 10 |
| 5 | 2.65182 | 2.65177 | $4.5371E-05$ | 6 |
| 6 | 2.65182 | 2.65177 | $4.5371E-05$ | 6 |
| 7 | 3.65108 | 3.65103 | $5.0379E-05$ | 8 |
| 8 | 3.65108 | 3.65109 | $1.8324E-05$ | 8 |
| 11 | 5.62499 | 5.62462 | $3.7497E-04$ | 8 |
| 14 | 6.51555 | 6.51523 | $3.2785E-04$ | 9 |
| 15 | 7.12850 | 7.12795 | $5.4947E-04$ | 7 |
| 16 | 7.12850 | 7.12805 | $4.4921E-04$ | 7 |
| 17 | 8.22729 | 8.22685 | $4.4429E-04$ | 8 |
| 18 | 8.22729 | 8.22685 | $4.4429E-04$ | 8 |

*Table 4.11*: $\alpha_2 = 1.6, \sin\cos$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 1 | 1.27636 | 1.27633 | $2.0605E-05$ | 9 |
| 2 | 1.27636 | 1.27633 | $2.0605E-05$ | 9 |
| 9 | 4.50774 | 4.50752 | $2.2163E-04$ | 9 |
| 10 | 4.50774 | 4.50759 | $1.5271E-04$ | 9 |
| 12 | 6.14672 | 6.14640 | $3.2228E-04$ | 8 |
| 13 | 6.14672 | 6.14640 | $3.2228E-04$ | 8 |
| 19 | 8.75366 | 8.75305 | $6.0354E-04$ | 6 |
| 20 | 8.75366 | 8.75315 | $5.0089E-04$ | 6 |
| 21 | 9.62179 | 9.62072 | $1.0675E-03$ | 1 |
| 22 | 9.62179 | 9.62072 | $1.0675E-03$ | 1 |

*Table 4.12*: $\alpha_2 = 1.6, \omega_2 \sinh\cosh$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 3 | 1.40827 | 1.40822 | $4.0560E-05$ | 8 |
| 4 | 1.79277 | 1.79277 | $8.0357E-07$ | 9 |
| 7 | 3.93371 | 3.93364 | $7.1541E-05$ | 7 |
| 8 | 3.93371 | 3.93368 | $3.3108E-05$ | 7 |
| 11 | 5.82451 | 5.82407 | $4.4068E-04$ | 8 |
| 14 | 6.87904 | 6.87870 | $3.3610E-04$ | 9 |
| 15 | 7.47138 | 7.47099 | $3.8834E-04$ | 7 |
| 16 | 7.47138 | 7.47099 | $3.8834E-04$ | 7 |
| 17 | 8.64971 | 8.64920 | $5.0822E-04$ | 8 |
| 18 | 8.64971 | 8.64927 | $4.3538E-04$ | 8 |

*Table 4.13*: $\alpha_2 = 1.7, \sin\cos$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 1 | 1.29162 | 1.29160 | $1.6863E-05$ | 6 |
| 2 | 1.29162 | 1.29160 | $1.6863E-05$ | 6 |
| 5 | 2.69737 | 2.69731 | $5.9081E-05$ | 9 |
| 6 | 2.69737 | 2.69732 | $5.4596E-05$ | 9 |
| 9 | 4.51973 | 4.51949 | $2.3465E-04$ | 11 |
| 10 | 4.51973 | 4.51951 | $2.1349E-04$ | 11 |
| 12 | 6.19638 | 6.19602 | $3.5477E-04$ | 9 |
| 13 | 6.19638 | 6.19605 | $3.3169E-04$ | 9 |
| 19 | 8.98238 | 8.98179 | $5.9009E-04$ | 6 |
| 20 | 8.98238 | 8.98179 | $5.9009E-04$ | 6 |

*Table 4.14*: $\alpha_2 = 1.7, \omega_2 \sinh\cosh$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 3 | 1.43580 | 1.43583 | $2.3381E-05$ | 7 |
| 4 | 1.91205 | 1.91182 | $2.2482E-04$ | 9 |
| 7 | 4.19407 | 4.19400 | $6.8867E-05$ | 5 |
| 8 | 4.19407 | 4.19416 | $8.7367E-05$ | 5 |
| 11 | 6.02252 | 6.02203 | $4.8819E-04$ | 8 |
| 14 | 7.20629 | 7.20585 | $4.3535E-04$ | 9 |
| 15 | 7.85026 | 7.84957 | $6.8876E-04$ | 8 |
| 16 | 7.85026 | 7.84983 | $4.3431E-04$ | 8 |
| 17 | 9.00968 | 9.00902 | $6.5989E-04$ | 9 |
| 18 | 9.00968 | 9.00915 | $5.2812E-04$ | 9 |

*Table 4.15*: $\alpha_2 = 1.8, \sin\cos$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 1 | 1.30380 | 1.30378 | $3.9622E-05$ | 11 |
| 2 | 1.30380 | 1.30378 | $1.9849E-05$ | 11 |
| 5 | 2.73472 | 2.73467 | $5.7523E-05$ | 9 |
| 6 | 2.73472 | 2.73476 | $3.1047E-05$ | 9 |
| 9 | 4.52900 | 4.52876 | $2.3711E-04$ | 6 |
| 10 | 4.52900 | 4.52892 | $7.1886E-05$ | 6 |
| 12 | 6.23529 | 6.23497 | $3.2480E-04$ | 6 |
| 13 | 6.23529 | 6.23497 | $3.2480E-04$ | 6 |
| 19 | 9.14411 | 9.14345 | $6.6327E-04$ | 7 |
| 20 | 9.14411 | 9.14358 | $5.3354E-04$ | 7 |

*Table 4.16*: $\alpha_2 = 1.8, \omega_2 \sinh\cosh$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 3 | 1.45931 | 1.45925 | $5.4837E-05$ | 7 |
| 4 | 2.02976 | 2.02978 | $1.7692E-05$ | 9 |
| 7 | 4.42695 | 4.42686 | $9.7273E-05$ | 7 |
| 8 | 4.42695 | 4.42686 | $9.7273E-05$ | 7 |
| 11 | 6.22286 | 6.22254 | $3.1574E-04$ | 8 |
| 14 | 7.49580 | 7.49543 | $3.7613E-04$ | 9 |
| 15 | 8.27173 | 8.27112 | $6.0616E-04$ | 8 |
| 16 | 8.27173 | 8.27140 | $3.2442E-04$ | 8 |
| 19 | 9.31137 | 9.31071 | $6.5933E-04$ | 9 |
| 20 | 9.31137 | 9.31087 | $4.9987E-04$ | 9 |

*Table 4.17*: $\alpha_2 = 1.9, \sin\cos$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 1 | 1.31369 | 1.31367 | $1.9160E-05$ | 10 |
| 2 | 1.31369 | 1.31369 | $8.9760E-07$ | 10 |
| 5 | 2.76570 | 2.76556 | $1.4880E-04$ | 7 |
| 6 | 2.76570 | 2.76564 | $6.6125E-05$ | 7 |
| 9 | 4.53636 | 4.53612 | $2.3840E-04$ | 11 |
| 10 | 4.53636 | 4.53618 | $1.8142E-04$ | 11 |
| 12 | 6.26645 | 6.26612 | $3.3159E-04$ | 7 |
| 13 | 6.26645 | 6.26612 | $3.3159E-04$ | 7 |
| 17 | 9.26364 | 9.26292 | $7.1833E-04$ | 6 |
| 18 | 9.26364 | 9.26302 | $6.2553E-04$ | 6 |

*Table 4.18*: $\alpha_2 = 1.9, \omega_2 \sinh\cosh$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 3 | 1.47950 | 1.47947 | $2.9088E-05$ | 7 |
| 4 | 2.14493 | 2.14487 | $6.2681E-05$ | 9 |
| 9 | 4.63023 | 4.63010 | $1.2761E-04$ | 8 |
| 10 | 4.63023 | 4.63012 | $1.1104E-04$ | 8 |
| 13 | 6.42846 | 6.42828 | $1.8439E-04$ | 8 |
| 14 | 7.74872 | 7.74777 | $9.5580E-04$ | 9 |
| 15 | 8.73796 | 8.73746 | $4.9938E-04$ | 8 |
| 16 | 8.73796 | 8.73749 | $4.6907E-04$ | 8 |
| 19 | 9.56227 | 9.56154 | $7.3289E-04$ | 6 |
| 20 | 9.56227 | 9.56196 | $3.0879E-04$ | 6 |

*Table 4.19*: $\alpha_2 = 2, \sin\cos$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 1 | 1.32183 | 1.32181 | $2.1159E-05$ | 8 |
| 2 | 1.32183 | 1.32181 | $1.8662E-05$ | 8 |
| 5 | 2.79167 | 2.79153 | $1.4608E-04$ | 8 |
| 6 | 2.79167 | 2.79160 | $7.2281E-05$ | 8 |
| 7 | 4.54233 | 4.54191 | $4.2021E-04$ | 9 |
| 8 | 4.54233 | 4.54211 | $2.1737E-04$ | 9 |
| 11 | 6.29185 | 6.29149 | $3.5636E-04$ | 9 |
| 12 | 6.29185 | 6.29157 | $2.8266E-04$ | 9 |
| 17 | 9.35531 | 9.35453 | $7.8379E-04$ | 7 |
| 18 | 9.35531 | 9.35462 | $6.9329E-04$ | 7 |

*Table 4.20*: $\alpha_2 = 2, \omega_2 \sinh\cosh$ case

In Tables 4.1-4.20, the discontinuity was at $x = \pi$, and $x = 2\pi\rho$, so $\rho = 0.5$. But, since the discontinuity can be anywhere in the 2-D domain $[0,2\pi] \times [0,2\pi]$, we test 4 values of $\rho$, with $0 < \rho < 1$, using 3 distinct $\alpha_2$- values for each $\rho$- value, with $1 < \alpha_2 < 2$. The results are presented below in Tables 4.21-4.44.

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 1 | 1.16287 | 1.16302 | $1.4959E-04$ | 9 |
| 4 | 1.20835 | 1.20836 | $4.3223E-06$ | 13 |
| 5 | 2.32619 | 2.32647 | $2.8780E-04$ | 6 |
| 6 | 2.32619 | 2.32648 | $2.8961E-04$ | 6 |
| 7 | 2.41668 | 2.41669 | $8.3305E-06$ | 12 |
| 8 | 2.41668 | 2.41669 | $8.7524E-06$ | 12 |
| 9 | 4.67394 | 4.67414 | $2.0122E-04$ | 10 |
| 12 | 4.81480 | 4.81486 | $5.7419E-05$ | 12 |
| 13 | 5.82178 | 5.82223 | $4.5150E-04$ | 9 |
| 14 | 5.82178 | 5.82224 | $4.5280E-04$ | 9 |
| 15 | 5.84449 | 5.84474 | $2.5141E-04$ | 9 |
| 16 | 5.84449 | 5.84474 | $2.5793E-04$ | 9 |
| 17 | 6.01830 | 6.01840 | $9.9135E-05$ | 11 |
| 18 | 6.01830 | 6.01842 | $1.1612E-04$ | 11 |
| 19 | 6.04152 | 6.04135 | $1.6897E-04$ | 10 |
| 20 | 6.04152 | 6.04136 | $1.5819E-04$ | 10 |
| 21 | 9.36152 | 9.36172 | $2.0222E-04$ | 8 |
| 22 | 9.36152 | 9.36178 | $2.6144E-04$ | 8 |
| 23 | 9.62832 | 9.62843 | $1.1417E-04$ | 11 |
| 24 | 9.62832 | 9.62844 | $1.1765E-04$ | 11 |
| 27 | 10.58170 | 10.58095 | $7.4440E-04$ | 11 |
| 28 | 10.77288 | 10.77289 | $1.2573E-05$ | 11 |
| 29 | 11.68314 | 11.68290 | $2.3893E-04$ | 7 |
| 30 | 11.68314 | 11.68290 | $2.3893E-04$ | 7 |
| 31 | 11.76051 | 11.75972 | $7.9367E-04$ | 11 |
| 32 | 11.76051 | 11.75972 | $7.9367E-04$ | 11 |
| 33 | 11.96923 | 11.96937 | $1.4111E-04$ | 10 |
| 34 | 11.96923 | 11.96937 | $1.4111E-04$ | 10 |
| 35 | 12.08241 | 12.08138 | $1.0343E-03$ | 8 |
| 36 | 12.08241 | 12.08140 | $1.0107E-03$ | 8 |
| 37 | 15.24192 | 15.24141 | $5.1211E-04$ | 7 |
| 38 | 15.24192 | 15.24142 | $5.0196E-04$ | 7 |
| 39 | 15.30052 | 15.29955 | $9.6496E-04$ | 9 |
| 40 | 15.30052 | 15.29956 | $9.5417E-04$ | 9 |
| 41 | 15.55749 | 15.55785 | $3.6010E-04$ | 8 |
| 42 | 15.55749 | 15.55786 | $3.7237E-04$ | 8 |
| 43 | 15.64329 | 15.64284 | $4.5286E-04$ | 10 |
| 44 | 15.64329 | 15.64284 | $4.4832E-04$ | 10 |

*Table 4.21*: $\rho = 0.1, \alpha_2 = 1.1, \sin\cos$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 2 | 1.18373 | 1.18382 | $8.8843E-05$ | 10 |
| 3 | 1.18373 | 1.18382 | $8.9569E-05$ | 10 |
| 10 | 4.71584 | 4.71615 | $3.1371E-04$ | 10 |
| 11 | 4.71584 | 4.71616 | $3.2068E-04$ | 10 |
| 25 | 10.52294 | 10.52358 | $6.4008E-04$ | 9 |
| 26 | 10.52294 | 10.52359 | $6.5106E-04$ | 9 |

*Table 4.22*: $\rho = 0.1, \alpha_2 = 1.1, \omega_2 \sinh\cosh$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 1 | 1.49353 | 1.49410 | $5.6469E-04$ | 9 |
| 4 | 1.68234 | 1.68242 | $7.9823E-05$ | 10 |
| 5 | 2.99946 | 3.00046 | $1.0039E-03$ | 5 |
| 6 | 2.99946 | 3.00047 | $1.0162E-03$ | 5 |
| 7 | 3.36431 | 3.36443 | $1.2088E-04$ | 9 |
| 8 | 3.36431 | 3.36448 | $1.7329E-04$ | 9 |
| 11 | 6.11105 | 6.11193 | $8.8361E-04$ | 10 |
| 12 | 6.63910 | 6.64014 | $1.0436E-03$ | 8 |
| 13 | 7.66749 | 7.66847 | $9.7701E-04$ | 9 |
| 14 | 7.66749 | 7.66854 | $1.0520E-03$ | 9 |
| 15 | 7.67457 | 7.67536 | $7.8225E-04$ | 7 |
| 16 | 7.67457 | 7.67541 | $8.3866E-04$ | 7 |
| 17 | 8.29568 | 8.29711 | $1.4268E-03$ | 7 |
| 18 | 8.29568 | 8.29721 | $1.5237E-03$ | 7 |
| 19 | 8.40777 | 8.40796 | $1.8638E-04$ | 8 |
| 20 | 8.40777 | 8.40801 | $2.3129E-04$ | 8 |
| 21 | 12.40481 | 12.40549 | $6.7949E-04$ | 8 |
| 22 | 12.40481 | 12.40552 | $7.0897E-04$ | 8 |
| 25 | 13.25659 | 13.25906 | $2.4632E-03$ | 6 |
| 26 | 13.25659 | 13.25918 | $2.5854E-03$ | 6 |
| 27 | 14.08218 | 14.08192 | $2.6630E-04$ | 9 |
| 28 | 14.63300 | 14.63745 | $4.4436E-03$ | 10 |

*Table 4.23*: $\rho = 0.1, \alpha_2 = 1.3, \sin\cos$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 2 | 1.56163 | 1.56213 | $4.9720E-04$ | 9 |
| 3 | 1.56163 | 1.56213 | $4.9974E-04$ | 9 |
| 9 | 5.96200 | 5.96520 | $3.2030E-03$ | 8 |
| 10 | 5.96200 | 5.96525 | $3.2460E-03$ | 8 |
| 23 | 12.52780 | 12.53601 | $8.2120E-03$ | 8 |
| 24 | 12.52780 | 12.53608 | $8.2758E-03$ | 8 |

*Table 4.24*: $\rho = 0.1, \alpha_2 = 1.3, \omega_2 \sinh\cosh$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 1 | 1.98895 | 1.99030 | $1.3535E-03$ | 8 |
| 4 | 2.53313 | 2.53343 | $3.0126E-04$ | 8 |
| 5 | 4.09662 | 4.09838 | $1.7599E-03$ | 6 |
| 6 | 4.09662 | 4.09840 | $1.7790E-03$ | 6 |
| 7 | 5.06325 | 5.06385 | $6.0966E-04$ | 7 |
| 8 | 5.06325 | 5.06385 | $6.0966E-04$ | 7 |
| 11 | 8.52252 | 8.52359 | $1.0710E-03$ | 8 |
| 12 | 9.79185 | 9.79657 | $4.7230E-03$ | 8 |
| 13 | 10.81229 | 10.81335 | $1.0659E-03$ | 8 |
| 14 | 10.81229 | 10.81347 | $1.1797E-03$ | 8 |
| 15 | 11.24829 | 11.24838 | $9.2395E-05$ | 6 |
| 16 | 11.24829 | 11.24838 | $9.2395E-05$ | 6 |
| 17 | 12.21197 | 12.21839 | $6.4260E-03$ | 7 |
| 18 | 12.21197 | 12.21852 | $6.5569E-03$ | 7 |

*Table 4.25*: $\rho = 0.1, \alpha_2 = 1.6, \sin\cos$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 2 | 2.09387 | 2.09560 | $1.7302E-03$ | 8 |
| 3 | 2.09387 | 2.09560 | $1.7302E-03$ | 8 |
| 9 | 7.12827 | 7.13703 | $8.7652E-03$ | 8 |
| 10 | 7.12827 | 7.13709 | $8.8211E-03$ | 8 |

*Table 4.26*: $\rho = 0.1, \alpha_2 = 1.6, \omega_2 \sinh\cosh$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 1 | 1.10775 | 1.10772 | $2.4638E-05$ | 7 |
| 4 | 1.16610 | 1.16607 | $2.3629E-05$ | 13 |
| 5 | 2.22647 | 2.22643 | $3.7434E-05$ | 10 |
| 6 | 2.22647 | 2.22644 | $3.4972E-05$ | 10 |
| 7 | 2.32939 | 2.32932 | $6.8761E-05$ | 6 |
| 8 | 2.32939 | 2.32933 | $6.3734E-05$ | 6 |
| 11 | 4.48367 | 4.48338 | $2.9307E-04$ | 8 |
| 12 | 4.60790 | 4.60763 | $2.7271E-04$ | 12 |
| 13 | 5.60549 | 5.60517 | $3.1901E-04$ | 7 |
| 14 | 5.60549 | 5.60518 | $3.1267E-04$ | 7 |
| 15 | 5.65306 | 5.65279 | $2.6209E-04$ | 9 |
| 16 | 5.65306 | 5.65281 | $2.4489E-04$ | 9 |
| 17 | 5.76404 | 5.76376 | $2.7597E-04$ | 12 |
| 18 | 5.76404 | 5.76380 | $2.3778E-04$ | 12 |
| 19 | 5.80068 | 5.80034 | $3.4451E-04$ | 9 |
| 20 | 5.80068 | 5.80035 | $3.3510E-04$ | 9 |
| 21 | 8.97669 | 8.97612 | $5.6843E-04$ | 10 |
| 22 | 8.97669 | 8.97613 | $5.5462E-04$ | 10 |
| 23 | 9.23878 | 9.23825 | $5.2077E-04$ | 6 |
| 24 | 9.23878 | 9.23825 | $5.2077E-04$ | 6 |
| 27 | 10.21662 | 10.21526 | $1.3629E-03$ | 13 |
| 28 | 10.23998 | 10.23882 | $1.1605E-03$ | 16 |
| 29 | 11.34931 | 11.34788 | $1.4287E-03$ | 12 |
| 30 | 11.34931 | 11.34790 | $1.4082E-03$ | 12 |
| 31 | 11.38500 | 11.38383 | $1.1679E-03$ | 16 |
| 32 | 11.38500 | 11.38383 | $1.1639E-03$ | 16 |
| 33 | 11.51415 | 11.51274 | $1.4067E-03$ | 10 |
| 34 | 11.51415 | 11.51274 | $1.4067E-03$ | 10 |
| 35 | 11.53309 | 11.53187 | $1.2202E-03$ | 11 |
| 36 | 11.53309 | 11.53187 | $1.2202E-03$ | 11 |
| 37 | 14.62761 | 14.62609 | $1.5255E-03$ | 9 |
| 38 | 14.62761 | 14.62609 | $1.5255E-03$ | 9 |
| 39 | 14.74484 | 14.74314 | $1.7000E-03$ | 10 |
| 40 | 14.74484 | 14.74316 | $1.6837E-03$ | 10 |
| 41 | 14.82780 | 14.82640 | $1.4006E-03$ | 14 |
| 42 | 14.82780 | 14.82641 | $1.3878E-03$ | 14 |
| 43 | 15.04020 | 15.03865 | $1.5500E-03$ | 9 |
| 44 | 15.04020 | 15.03866 | $1.5398E-03$ | 9 |

*Table 4.27*: $\rho = 1/\pi, \alpha_2 = 1.1, \sin\cos$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 2 | 1.12722 | 1.12719 | $2.9509E-05$ | 7 |
| 3 | 1.12722 | 1.12719 | $2.4905E-05$ | 7 |
| 9 | 4.42496 | 4.42467 | $2.8411E-04$ | 10 |
| 10 | 4.42496 | 4.42468 | $2.7473E-04$ | 10 |
| 25 | 9.71628 | 9.71509 | $1.1954E-03$ | 7 |
| 26 | 9.71628 | 9.71510 | $1.1847E-03$ | 7 |
| 45 | 16.93780 | 16.93423 | $3.5689E-03$ | 12 |
| 46 | 16.93780 | 16.93430 | $3.4995E-03$ | 12 |

*Table 4.28*: $\rho = 1/\pi, \alpha_2 = 1.1, \omega_2 \sinh\cosh$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 3 | 1.56735 | 1.56725 | $9.5837E-05$ | 8 |
| 4 | 1.79231 | 1.79218 | $1.3275E-04$ | 9 |
| 5 | 3.46175 | 3.46171 | $4.6483E-05$ | 18 |
| 6 | 3.46175 | 3.46171 | $4.6483E-05$ | 18 |
| 7 | 3.46401 | 3.46363 | $3.7923E-04$ | 18 |
| 8 | 3.46401 | 3.46363 | $3.7923E-04$ | 18 |
| 11 | 6.32632 | 6.32587 | $4.4946E-04$ | 8 |
| 12 | 7.09336 | 7.09226 | $1.1042E-03$ | 8 |
| 15 | 8.07985 | 8.07926 | $5.8940E-04$ | 5 |
| 16 | 8.07985 | 8.07933 | $5.1861E-04$ | 5 |
| 17 | 8.84625 | 8.84513 | $1.1229E-03$ | 9 |
| 18 | 8.84625 | 8.84523 | $1.0253E-03$ | 9 |
| 19 | 9.82467 | 9.82408 | $5.9215E-04$ | 5 |
| 20 | 9.82467 | 9.82408 | $5.9215E-04$ | 5 |

*Table 4.29*: $\rho = 1/\pi, \alpha_2 = 1.5, \sin\cos$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 1 | 1.43800 | 1.43790 | $9.9049E-05$ | 6 |
| 2 | 1.43800 | 1.43791 | $9.0913E-05$ | 6 |
| 9 | 4.90782 | 4.90737 | $4.4644E-04$ | 9 |
| 10 | 4.90782 | 4.90739 | $4.2977E-04$ | 9 |
| 13 | 7.82364 | 7.82234 | $1.3002E-03$ | 8 |
| 14 | 7.82364 | 7.82243 | $1.2027E-03$ | 8 |
| 21 | 10.19546 | 10.19406 | $1.4036E-03$ | 14 |
| 22 | 10.19546 | 10.19406 | $1.4036E-03$ | 14 |

*Table 4.30*: $\rho = 1/\pi, \alpha_2 = 1.5, \omega_2 \sinh\cosh$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 3 | 2.25008 | 2.25028 | $1.9699E-04$ | 11 |
| 4 | 2.35485 | 2.35446 | $3.8941E-04$ | 12 |
| 5 | 4.24432 | 4.24352 | $7.9690E-04$ | 8 |
| 6 | 4.24432 | 4.24355 | $7.6232E-04$ | 8 |
| 9 | 5.65058 | 5.65018 | $3.9676E-04$ | 7 |
| 10 | 5.65058 | 5.65048 | $1.0030E-04$ | 7 |
| 13 | 9.00136 | 9.00076 | $5.9382E-04$ | 6 |
| 14 | 9.41825 | 9.41628 | $1.9709E-03$ | 11 |

*Table 4.31*: $\rho = 1/\pi, \alpha_2 = 2, \sin\cos$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 1 | 1.57887 | 1.57873 | $1.4631E-04$ | 9 |
| 2 | 1.57887 | 1.57874 | $1.3004E-04$ | 9 |
| 7 | 5.03736 | 5.03689 | $4.7580E-04$ | 7 |
| 8 | 5.03736 | 5.03697 | $3.9444E-04$ | 7 |
| 11 | 8.58039 | 8.57874 | $1.6513E-03$ | 7 |
| 12 | 8.58039 | 8.57902 | $1.3703E-03$ | 7 |

*Table 4.32*: $\rho = 1/\pi, \alpha_2 = 2, \omega_2 \sinh\cosh$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 1 | 1.02737 | 1.02730 | $6.6315E-05$ | 11 |
| 4 | 1.08982 | 1.08976 | $5.5174E-05$ | 7 |
| 5 | 2.05324 | 2.05310 | $1.3173E-04$ | 9 |
| 6 | 2.05324 | 2.05311 | $1.2595E-04$ | 9 |
| 7 | 2.18624 | 2.18613 | $1.0646E-04$ | 9 |
| 8 | 2.18624 | 2.18614 | $1.0025E-04$ | 9 |
| 11 | 4.20364 | 4.20342 | $2.1809E-04$ | 13 |
| 12 | 4.26114 | 4.26061 | $5.2575E-04$ | 11 |
| 13 | 5.12285 | 5.12240 | $4.5353E-04$ | 5 |
| 14 | 5.12285 | 5.12241 | $4.4282E-04$ | 5 |
| 15 | 5.26136 | 5.26112 | $2.4465E-04$ | 13 |
| 16 | 5.26136 | 5.26113 | $2.3051E-04$ | 13 |
| 17 | 5.32453 | 5.32387 | $6.5687E-04$ | 12 |
| 18 | 5.32453 | 5.32391 | $6.2312E-04$ | 12 |
| 19 | 5.49655 | 5.49602 | $5.2487E-04$ | 7 |
| 20 | 5.49655 | 5.49604 | $5.0446E-04$ | 7 |
| 21 | 8.45474 | 8.45428 | $4.5770E-04$ | 11 |
| 22 | 8.45474 | 8.45430 | $4.3304E-04$ | 11 |
| 23 | 8.50879 | 8.50771 | $1.0861E-03$ | 14 |
| 24 | 8.50879 | 8.50772 | $1.0761E-03$ | 14 |
| 27 | 9.45121 | 9.44950 | $1.7154E-03$ | 12 |
| 28 | 9.59888 | 9.59769 | $1.1957E-03$ | 9 |
| 31 | 10.50045 | 10.49867 | $1.7779E-03$ | 12 |
| 32 | 10.50045 | 10.49868 | $1.7695E-03$ | 12 |
| 33 | 10.67098 | 10.66971 | $1.2686E-03$ | 11 |
| 34 | 10.67098 | 10.66971 | $1.2686E-03$ | 11 |
| 35 | 10.99876 | 10.99695 | $1.8158E-03$ | 9 |
| 36 | 10.99876 | 10.99696 | $1.8009E-03$ | 9 |
| 37 | 13.64803 | 13.64588 | $2.1488E-03$ | 7 |
| 38 | 13.64803 | 13.64589 | $2.1411E-03$ | 7 |
| 39 | 13.79298 | 13.79058 | $2.4022E-03$ | 12 |
| 40 | 13.79298 | 13.79062 | $2.3658E-03$ | 12 |
| 41 | 13.85629 | 13.85497 | $1.3216E-03$ | 8 |
| 42 | 13.85629 | 13.85497 | $1.3194E-03$ | 8 |
| 43 | 13.89295 | 13.89134 | $1.6029E-03$ | 11 |
| 44 | 13.89295 | 13.89137 | $1.5763E-03$ | 11 |
| 47 | 16.82755 | 16.82383 | $3.7205E-03$ | 12 |
| 48 | 17.03317 | 17.02873 | $4.4369E-03$ | 11 |

*Table 4.33*: $\rho = \ln 2, \alpha_2 = 1.1, \sin\cos$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 2 | 1.05124 | 1.05119 | $4.9996E-05$ | 10 |
| 3 | 1.05124 | 1.05119 | $4.9931E-05$ | 10 |
| 9 | 4.15756 | 4.15725 | $3.1060E-04$ | 8 |
| 10 | 4.15756 | 4.15725 | $3.0320E-04$ | 8 |
| 25 | 9.24801 | 9.24683 | $1.1792E-03$ | 8 |
| 26 | 9.24801 | 9.24685 | $1.1606E-03$ | 8 |
| 29 | 10.21712 | 10.21566 | $1.4584E-03$ | 7 |
| 30 | 10.21712 | 10.21569 | $1.4364E-03$ | 7 |
| 45 | 16.30640 | 16.30298 | $3.4174E-03$ | 16 |
| 46 | 16.30640 | 16.30299 | $3.4090E-03$ | 16 |
| 49 | 17.31632 | 17.31250 | $3.8158E-03$ | 12 |
| 50 | 17.31632 | 17.31253 | $3.7870E-03$ | 12 |

*Table 4.34*: $\rho = \ln 2, \alpha_2 = 1.1, \omega_2 \sinh\cosh$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 1 | 1.04810 | 1.04800 | $1.0259E-04$ | 10 |
| 4 | 1.17579 | 1.17573 | $6.1469E-05$ | 7 |
| 5 | 2.09156 | 2.09136 | $1.9881E-04$ | 7 |
| 6 | 2.09156 | 2.09137 | $1.9047E-04$ | 7 |
| 7 | 2.37076 | 2.37052 | $2.3456E-04$ | 7 |
| 8 | 2.37076 | 2.37053 | $2.3072E-04$ | 7 |
| 11 | 4.41037 | 4.41027 | $9.4365E-05$ | 13 |
| 12 | 4.47282 | 4.47197 | $8.4996E-04$ | 12 |
| 15 | 5.54273 | 5.54250 | $2.3545E-04$ | 13 |
| 16 | 5.54273 | 5.54250 | $2.2905E-04$ | 13 |
| 17 | 5.58242 | 5.58136 | $1.0579E-03$ | 10 |
| 18 | 5.58242 | 5.58138 | $1.0392E-03$ | 10 |
| 19 | 5.95475 | 5.95383 | $9.2304E-04$ | 8 |
| 20 | 5.95475 | 5.95387 | $8.7939E-04$ | 8 |
| 21 | 8.88552 | 8.88374 | $1.7809E-03$ | 11 |
| 22 | 8.88552 | 8.88377 | $1.7511E-03$ | 11 |
| 23 | 9.03411 | 9.03363 | $4.8666E-04$ | 10 |
| 24 | 9.03411 | 9.03365 | $4.6228E-04$ | 10 |
| 27 | 9.81973 | 9.81764 | $2.0954E-03$ | 10 |
| 28 | 10.17996 | 10.17832 | $1.6478E-03$ | 9 |
| 31 | 10.91155 | 10.90932 | $2.2286E-03$ | 9 |
| 32 | 10.91155 | 10.90932 | $2.2286E-03$ | 9 |
| 33 | 11.32679 | 11.32495 | $1.8413E-03$ | 8 |
| 34 | 11.32679 | 11.32504 | $1.7471E-03$ | 8 |
| 37 | 14.19749 | 14.19486 | $2.6283E-03$ | 9 |
| 38 | 14.19749 | 14.19486 | $2.6283E-03$ | 9 |
| 39 | 14.30355 | 14.30020 | $3.3500E-03$ | 7 |
| 40 | 14.30355 | 14.30023 | $3.3210E-03$ | 7 |
| 41 | 14.77512 | 14.77257 | $2.5501E-03$ | 6 |
| 42 | 14.77512 | 14.77262 | $2.5036E-03$ | 6 |
| 43 | 15.14063 | 15.13859 | $2.0402E-03$ | 8 |
| 44 | 15.14063 | 15.13865 | $1.9722E-03$ | 8 |

*Table 4.35*: $\rho = \ln 2, \alpha_2 = 1.2, \sin\cos$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 2 | 1.08678 | 1.08670 | $7.9163E-05$ | 8 |
| 3 | 1.08678 | 1.08670 | $7.5799E-05$ | 8 |
| 9 | 4.22327 | 4.22290 | $3.6800E-04$ | 10 |
| 10 | 4.22327 | 4.22292 | $3.4559E-04$ | 10 |
| 13 | 5.19932 | 5.19871 | $6.0937E-04$ | 8 |
| 14 | 5.19932 | 5.19874 | $5.8118E-04$ | 8 |
| 25 | 9.30684 | 9.30562 | $1.2232E-03$ | 6 |
| 26 | 9.30684 | 9.30565 | $1.1985E-03$ | 6 |
| 29 | 10.32646 | 10.32483 | $1.6220E-03$ | 7 |
| 30 | 10.32646 | 10.32484 | $1.6137E-03$ | 7 |
| 35 | 11.64869 | 11.64615 | $2.5359E-03$ | 7 |
| 36 | 11.64869 | 11.64619 | $2.4982E-03$ | 7 |

*Table 4.36*: $\rho = \ln 2, \alpha_2 = 1.2, \omega_2 \sinh\cosh$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 1 | 1.11657 | 1.11633 | $2.4178E-04$ | 7 |
| 4 | 1.65221 | 1.65159 | $6.1399E-04$ | 8 |
| 11 | 5.19340 | 5.19108 | $2.3162E-03$ | 8 |
| 14 | 6.11494 | 6.11438 | $5.6787E-04$ | 9 |
| 15 | 6.39594 | 6.39324 | $2.7061E-03$ | 7 |
| 16 | 6.39594 | 6.39328 | $2.6636E-03$ | 7 |
| 19 | 8.04288 | 8.04031 | $2.5673E-03$ | 6 |
| 20 | 8.04288 | 8.04043 | $2.4527E-03$ | 6 |

*Table 4.37*: $\rho = \ln 2, \alpha_2 = 2, \sin\cos$ case

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 2 | 1.18006 | 1.17992 | $1.3973E-04$ | 6 |
| 3 | 1.18006 | 1.17992 | $1.3973E-04$ | 6 |
| 5 | 2.20575 | 2.20529 | $4.6410E-04$ | 6 |
| 6 | 2.20575 | 2.20533 | $4.2056E-04$ | 6 |
| 7 | 3.23734 | 3.23606 | $1.2841E-03$ | 6 |
| 8 | 3.23734 | 3.23610 | $1.2435E-03$ | 6 |
| 9 | 4.31856 | 4.31812 | $4.3210E-04$ | 18 |
| 10 | 4.31856 | 4.31817 | $3.8781E-04$ | 18 |
| 12 | 5.37936 | 5.37848 | $8.7524E-04$ | 9 |
| 13 | 5.37936 | 5.37851 | $8.5056E-04$ | 9 |
| 17 | 7.03936 | 7.03723 | $2.1284E-03$ | 6 |
| 18 | 7.03936 | 7.03727 | $2.0924E-03$ | 6 |
| 23 | 9.84532 | 9.84171 | $3.6142E-03$ | 7 |
| 24 | 9.84532 | 9.84171 | $3.6142E-03$ | 7 |

*Table 4.38*: $\rho = \ln 2, \alpha_2 = 2, \omega_2 \sinh\cosh$ case

For Tables 4.37 and 4.38, eigenvalues 9.37433 and 9.37441, index 21 and 22 respectively of $\lambda_j(eig)$, were not found using our approach: $\lambda_j(exact)$. For Tables 4.39 and 4.40, the eigenvalue 16.02172, index 46 and 47 of $\lambda_j(eig)$, was not found using our approach: $\lambda_j(exact)$. For Tables 4.41 and 4.42, the eigenvalue 16.03808, index 46 and 47 of $\lambda_j(eig)$, was not found using our approach: $\lambda_j(exact)$. For Tables 4.43 and 4.44, the eigenvalue 9.03679, index 26 and 27 of $\lambda_j(eig)$, was not found using our approach: $\lambda_j(exact)$.

The possible reasons for a couple of the eigenvalues not found using our approach in the Tables 4.37-4.44 are:

- The initial guesses had to be very precise. When $\rho$ has a very high value along with a

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 1 | 1.00000 | 0.99999 | $1.2616E-05$ | 17 |
| 4 | 1.00348 | 1.00339 | $9.4318E-05$ | 15 |
| 5 | 2.00000 | 1.99998 | $2.5403E-05$ | 14 |
| 6 | 2.00000 | 1.99998 | $2.5403E-05$ | 14 |
| 7 | 2.00695 | 2.00676 | $1.8830E-04$ | 16 |
| 8 | 2.00695 | 2.00676 | $1.8830E-04$ | 16 |
| 9 | 4.00002 | 3.99982 | $2.0243E-04$ | 16 |
| 12 | 4.01391 | 4.01338 | $5.2695E-04$ | 15 |
| 13 | 5.00001 | 4.99979 | $2.1452E-04$ | 7 |
| 14 | 5.00001 | 4.99979 | $2.1300E-04$ | 7 |
| 15 | 5.00002 | 4.99981 | $2.1550E-04$ | 13 |
| 16 | 5.00002 | 4.99981 | $2.1494E-04$ | 13 |
| 17 | 5.01734 | 5.01672 | $6.2066E-04$ | 13 |
| 18 | 5.01734 | 5.01672 | $6.2066E-04$ | 13 |
| 19 | 5.01738 | 5.01676 | $6.2123E-04$ | 15 |
| 20 | 5.01738 | 5.01676 | $6.2101E-04$ | 15 |
| 21 | 8.00004 | 7.99963 | $4.0561E-04$ | 14 |
| 22 | 8.00004 | 7.99963 | $4.0561E-04$ | 14 |
| 23 | 8.02780 | 8.02675 | $1.0533E-03$ | 12 |
| 24 | 8.02780 | 8.02675 | $1.0506E-03$ | 12 |
| 25 | 9.00009 | 8.99907 | $1.0265E-03$ | 14 |
| 28 | 9.03126 | 9.02951 | $1.7478E-03$ | 15 |
| 29 | 10.00010 | 9.99898 | $1.1257E-03$ | 12 |
| 30 | 10.00010 | 9.99898 | $1.1219E-03$ | 12 |
| 35 | 10.03473 | 10.03289 | $1.8433E-03$ | 11 |
| 36 | 10.03473 | 10.03289 | $1.8413E-03$ | 11 |
| 37 | 13.00006 | 12.99884 | $1.2232E-03$ | 12 |
| 38 | 13.00006 | 12.99884 | $1.2213E-03$ | 12 |
| 39 | 13.00013 | 12.99890 | $1.2317E-03$ | 14 |
| 40 | 13.00013 | 12.99890 | $1.2289E-03$ | 14 |
| 41 | 13.04514 | 13.04286 | $2.2808E-03$ | 11 |
| 42 | 13.04514 | 13.04286 | $2.2785E-03$ | 11 |
| 43 | 13.04515 | 13.04287 | $2.2770E-03$ | 11 |
| 44 | 13.04515 | 13.04287 | $2.2752E-03$ | 11 |
| 45 | 16.00029 | 15.99705 | $3.2459E-03$ | 13 |
| 48 | 16.05548 | 16.05095 | $4.5307E-03$ | 15 |

*Table 4.39*: $\rho = 0.99, \alpha_2 = 1.1, \sin\cos$ case

high value for $\alpha$, the discontinuity is at the edge.

- The radius (rad) from which the eigenvalues are calculated becomes smaller. As the $\rho$

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 2 | 1.00173 | 1.00168 | $5.2969E-05$ | 9 |
| 3 | 1.00173 | 1.00168 | $5.2910E-05$ | 9 |
| 10 | 4.00680 | 4.00644 | $3.5761E-04$ | 10 |
| 11 | 4.00680 | 4.00644 | $3.5725E-04$ | 10 |
| 26 | 9.01489 | 9.01353 | $1.3527E-03$ | 11 |
| 27 | 9.01489 | 9.01354 | $1.3509E-03$ | 11 |
| 31 | 10.00001 | 9.99906 | $9.4932E-04$ | 13 |
| 32 | 10.00001 | 9.99906 | $9.4891E-04$ | 13 |
| 33 | 10.03453 | 10.03269 | $1.8392E-03$ | 11 |
| 34 | 10.03453 | 10.03269 | $1.8384E-03$ | 11 |
| 49 | 17.00002 | 16.99679 | $3.2257E-03$ | 13 |
| 50 | 17.00002 | 16.99679 | $3.2257E-03$ | 13 |

*Table 4.40*: $\rho = 0.99, \alpha_2 = 1.1, \omega_2 \sinh\cosh$ case

value went close to 1, it resulted in the eigenvalues generated to be clustered together, prompting the need for even better initial guesses for the secant method than those that were used to compute eigenvalues using our approach.

The fact that from over a 1000 eigenvalues generated, all of those were computed using our approach except 4, confirms the success of this method. The lowest absolute error was in the $10^{-7}$ range while the highest absolute error was within $10^{-3}$. This shows the accuracy of our method.

## 4.2 Results from Solving the PDE

We now apply the approach of Section 3 to compute the solution of the diffusion problem $u_t + Lu = 0$ on $(0, 2\pi)$, with periodic boundary conditions and initial condition $u(x,0) = f(x)$. Here, $L$ denotes the differential operator. For the following experiments, we use $\alpha_1 = 1$, $\alpha_2 = 1.1$, and $\rho = 0.5$ [6].

We conduct two experiments, each with different functions as initial data. In the first experiment, the initial data is the periodic function $u(x,0) = \cos(x)\sin(y)$. Its results are shown in Figures 4.1-4.4 . As the final time is increased, the solution becomes more smooth, as expected. Figures 4.5-4.8 illustrate the solution computed using the initial data $u(x,0) = (|x-\pi| < 0.5)(|y-\pi| < 0.5)$, a product of characteristic functions. As time

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 1 | 1.00000 | 0.99999 | $1.2953E-05$ | 18 |
| 4 | 1.00614 | 1.00598 | $1.5740E-04$ | 15 |
| 5 | 2.00000 | 1.99998 | $2.5805E-05$ | 13 |
| 6 | 2.00000 | 1.99998 | $2.5150E-05$ | 13 |
| 7 | 2.01226 | 2.01195 | $3.1405E-04$ | 13 |
| 8 | 2.01226 | 2.01195 | $3.1343E-04$ | 13 |
| 9 | 4.00003 | 3.99983 | $2.0362E-04$ | 13 |
| 12 | 4.02454 | 4.02376 | $7.8020E-04$ | 15 |
| 13 | 5.00004 | 4.99979 | $2.4718E-04$ | 14 |
| 14 | 5.00004 | 4.99980 | $2.4405E-04$ | 14 |
| 19 | 5.03067 | 5.02973 | $9.3590E-04$ | 10 |
| 20 | 5.03067 | 5.02973 | $9.3590E-04$ | 10 |
| 21 | 8.00006 | 7.99965 | $4.0978E-04$ | 12 |
| 22 | 8.00006 | 7.99966 | $4.0848E-04$ | 12 |
| 23 | 8.04903 | 8.04747 | $1.5575E-03$ | 11 |
| 24 | 8.04903 | 8.04748 | $1.5502E-03$ | 11 |
| 25 | 9.00016 | 8.99913 | $1.0315E-03$ | 8 |
| 28 | 9.05517 | 9.05284 | $2.3325E-03$ | 15 |
| 29 | 10.00018 | 9.99899 | $1.1917E-03$ | 14 |
| 30 | 10.00018 | 9.99899 | $1.1867E-03$ | 14 |
| 31 | 10.00018 | 9.99913 | $1.0487E-03$ | 17 |
| 32 | 10.00018 | 9.99913 | $1.0469E-03$ | 17 |
| 35 | 10.06130 | 10.05883 | $2.4749E-03$ | 12 |
| 36 | 10.06130 | 10.05883 | $2.4731E-03$ | 12 |
| 37 | 13.00023 | 12.99888 | $1.3570E-03$ | 13 |
| 38 | 13.00023 | 12.99888 | $1.3518E-03$ | 13 |
| 39 | 13.00023 | 12.99899 | $1.2432E-03$ | 13 |
| 40 | 13.00023 | 12.99899 | $1.2411E-03$ | 13 |
| 41 | 13.07954 | 13.07644 | $3.0964E-03$ | 12 |
| 42 | 13.07954 | 13.07645 | $3.0931E-03$ | 12 |
| 43 | 13.07968 | 13.07659 | $3.0942E-03$ | 10 |
| 44 | 13.07968 | 13.07659 | $3.0942E-03$ | 10 |
| 45 | 16.00051 | 15.99724 | $3.2672E-03$ | 13 |
| 48 | 16.09798 | 16.09243 | $5.5475E-03$ | 12 |

*Table 4.41*: $\rho = 0.99, \alpha_2 = 1.2, \sin\cos$ case

increases, this solution also behaves as expected and becomes more smooth, much like the one-dimensional case [6].

For both these problems, we compare the efficiency of our approach to the Crank-

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 2 | 1.00303 | 1.00295 | $8.3268E-05$ | 10 |
| 3 | 1.00303 | 1.00295 | $8.3231E-05$ | 10 |
| 10 | 4.01179 | 4.01132 | $4.6828E-04$ | 8 |
| 11 | 4.01179 | 4.01132 | $4.6828E-04$ | 8 |
| 15 | 5.00001 | 4.99982 | $1.8771E-04$ | 15 |
| 16 | 5.00001 | 4.99983 | $1.8456E-04$ | 15 |
| 17 | 5.03051 | 5.02958 | $9.3024E-04$ | 9 |
| 18 | 5.03051 | 5.02958 | $9.2922E-04$ | 9 |
| 26 | 9.02529 | 9.02372 | $1.5675E-03$ | 19 |
| 27 | 9.02529 | 9.02372 | $1.5675E-03$ | 19 |
| 33 | 10.06052 | 10.05807 | $2.4432E-03$ | 11 |
| 34 | 10.06052 | 10.05807 | $2.4422E-03$ | 11 |

*Table 4.42*: $\rho = 0.99, \alpha_2 = 1.2, \omega_2 \sinh\cosh$ case

Nicholson method, which is second-order accurate and unconditionally stable [8]. We choose this method since it is a standard numerical method to solve PDEs. Although we can choose a large time step while still ensuring stability, if it is chosen to be too large, then the solution will exhibit high-frequency oscillations due to the Gibbs' phenomenon. In Tables 4.45 and 4.46, we list the computational time needed to compute a solution using the Crank-Nicholson method, and compare it to the computational time needed to compute the solution at the same final time using the methods of Chapter 3 to compute an eigenfunction expansion, for $f(x) = \cos(x)\sin(y)$ and $f(x) = (|x-\pi| < 0.5)(|y-\pi| < 0.5)$, respectively.

In Tables 4.45 and 4.46, it can be observed that as the final time $T$ increases, using an eigenfunction expansion requires fewer terms. This happens due to the exponential decay of the coefficients in the expansion. Furthermore, as $T$ increases, the eigenfunction expansion yields the solution in far less computational time, due to the need for fewer terms while Crank-Nicholson requires more time steps [6].

Figures 4.9-4.14 and 4.15-4.20 exhibit a comparison in the behavior of the solution computed using our approach- the eigenfunction expansion, and Crank-Nicholson for $f(x) = \cos(x)\sin(y)$ and $f(x) = (|x-\pi| < 0.5)(|y-\pi| < 0.5)$ respectively.

When $f(x) = \cos(x)\sin(y)$, the solution behaves very similarly for both the methods since $f(x)$ is a smooth and periodic function. However, when $f(x) = (|x-\pi| < 0.5)(|y-\pi| < 0.5)$, there is a discrepancy in the behavior of the final solution as we compare our

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 1 | 1.00000 | 0.99999 | $1.2535E-05$ | 14 |
| 4 | 1.00987 | 1.00962 | $2.4607E-04$ | 16 |
| 5 | 2.00001 | 1.99998 | $2.6141E-05$ | 11 |
| 6 | 2.00001 | 1.99998 | $2.5570E-05$ | 11 |
| 7 | 2.01969 | 2.01920 | $4.8943E-04$ | 10 |
| 8 | 2.01969 | 2.01920 | $4.8927E-04$ | 10 |
| 9 | 4.00005 | 3.99984 | $2.0666E-04$ | 14 |
| 12 | 4.03946 | 4.03833 | $1.1251E-03$ | 11 |
| 15 | 5.00006 | 4.99984 | $2.2031E-04$ | 13 |
| 16 | 5.00006 | 4.99984 | $2.1993E-04$ | 13 |
| 19 | 5.04930 | 5.04792 | $1.3786E-03$ | 9 |
| 20 | 5.04930 | 5.04792 | $1.3783E-03$ | 9 |
| 21 | 8.00010 | 7.99969 | $4.1583E-04$ | 8 |
| 22 | 8.00010 | 7.99969 | $4.1161E-04$ | 8 |
| 23 | 8.07875 | 8.07649 | $2.2651E-03$ | 10 |
| 24 | 8.07875 | 8.07649 | $2.2616E-03$ | 10 |
| 25 | 9.00026 | 8.99921 | $1.0477E-03$ | 12 |
| 28 | 9.08875 | 9.08563 | $3.1203E-03$ | 13 |
| 29 | 10.00029 | 9.99900 | $1.2905E-03$ | 13 |
| 30 | 10.00029 | 9.99900 | $1.2878E-03$ | 13 |
| 31 | 10.00029 | 9.99923 | $1.0612E-03$ | 11 |
| 32 | 10.00029 | 9.99923 | $1.0565E-03$ | 11 |
| 33 | 10.09859 | 10.09299 | $5.6089E-03$ | 11 |
| 34 | 10.09859 | 10.09299 | $5.6012E-03$ | 11 |
| 35 | 10.09859 | 10.09523 | $3.3645E-03$ | 13 |
| 36 | 10.09859 | 10.09523 | $3.3645E-03$ | 13 |
| 37 | 13.00038 | 12.99892 | $1.4604E-03$ | 12 |
| 38 | 13.00038 | 12.99892 | $1.4300E-03$ | 12 |
| 39 | 13.00038 | 12.99912 | $1.2530E-03$ | 12 |
| 40 | 13.00038 | 12.99912 | $1.2530E-03$ | 12 |
| 43 | 13.12812 | 13.12386 | $4.2597E-03$ | 10 |
| 44 | 13.12812 | 13.12387 | $4.2496E-03$ | 10 |

*Table 4.43*: $\rho = 0.99, \alpha_2 = 1.4, \sin\cos$ case

approach with the Crank-Nicholson method, since the Crank-Nicholson method exhibits high frequency oscillation as time increases. The expected qualitative behavior in the final solution is observed in the figures using our approach and not those of the Crank-Nicholson method.

| $j$ | $\lambda_j(exact)$ | $\lambda_j(eig)$ | error | no. of iterations |
|---|---|---|---|---|
| 2 | 1.00484 | 1.00472 | $1.2504E-04$ | 11 |
| 3 | 1.00484 | 1.00472 | $1.2468E-04$ | 11 |
| 10 | 4.01849 | 4.01788 | $6.1205E-04$ | 10 |
| 11 | 4.01849 | 4.01788 | $6.1089E-04$ | 10 |
| 13 | 5.00002 | 4.99980 | $2.1564E-04$ | 12 |
| 14 | 5.00002 | 4.99980 | $2.1492E-04$ | 12 |
| 17 | 5.04884 | 5.04749 | $1.3591E-03$ | 9 |
| 18 | 5.04884 | 5.04749 | $1.3591E-03$ | 9 |
| 41 | 13.12754 | 13.12330 | $4.2390E-03$ | 10 |
| 42 | 13.12754 | 13.12331 | $4.2345E-03$ | 10 |

*Table 4.44*: $\rho = 0.99, \alpha_2 = 1.4, \omega_2$ sinh cosh case

Solution with $\alpha_1 = 1$, $\alpha_2 = 1.1$, $\rho = 0.5$ and $u(x,0) = \cos(x)\sin(y)$

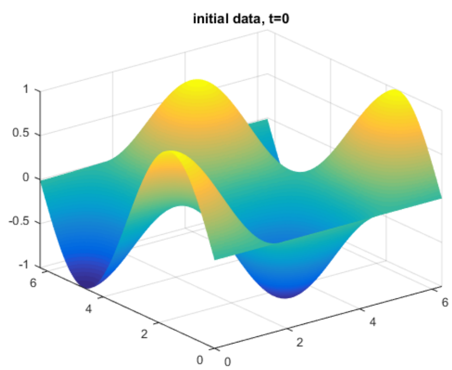


*Figure 4.1*: $t = 0$

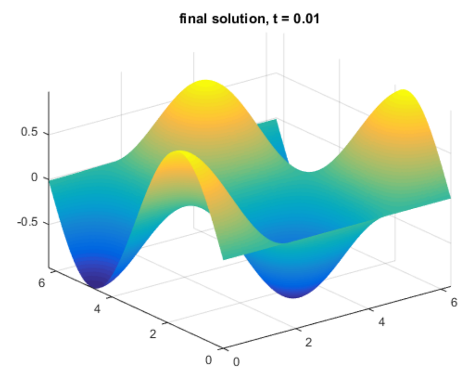


*Figure 4.2*: $t = 0.01$

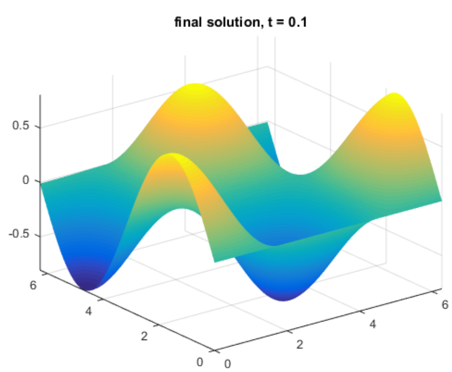


*Figure 4.3*: $t = 0.1$

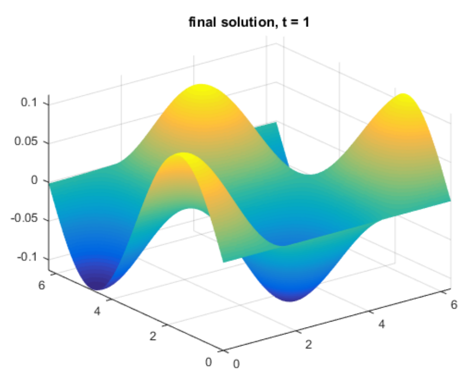


*Figure 4.4*: $t = 1$

Solution with $\alpha_1 = 1$, $\alpha_2 = 1.1$, $\rho = 0.5$ and $u(x,0) = (|x-\pi| < 0.5)(|y-\pi| < 0.5)$

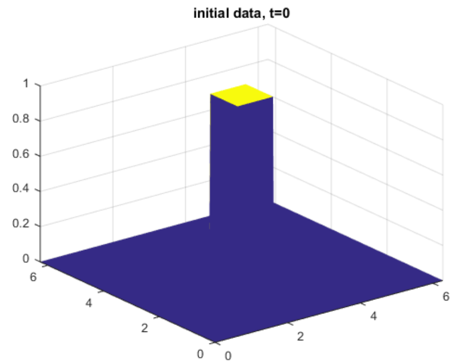


*Figure 4.5*: $t = 0$

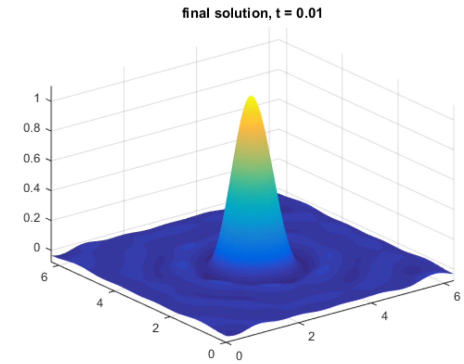


*Figure 4.6*: $t = 0.01$

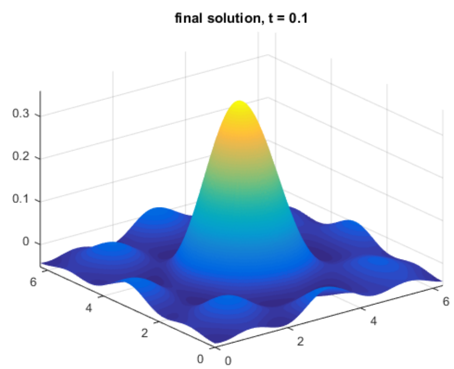


*Figure 4.7*: $t = 0.1$

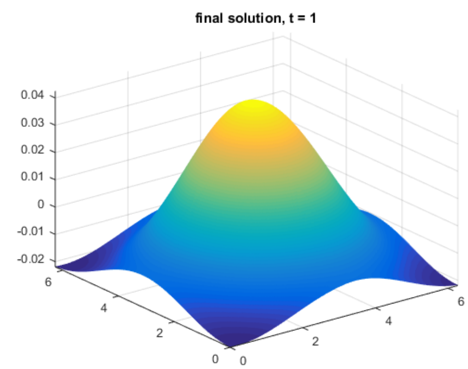


*Figure 4.8*: $t = 1$

| | Crank-Nicholson | | Eigenfunction expansion | |
|---|---|---|---|---|
| $T$ | time | $\triangle t$ | time | $m$ |
| 0.01 | 62.2519 | 0.001 | 29.5853 | 53 |
| 0.1 | 65.8706 | 0.01 | 3.5680 | 17 |
| 1 | 67.5999 | 0.1 | 0.7395 | 5 |

*Table 4.45*: Computational time, in seconds, using Crank-Nicholson with time step $\triangle t$, and eigenfunction expansion with $m$ terms, for $f(x) = \cos(x)\sin(y)$

| | Crank-Nicholson | | Eigenfunction expansion | |
|---|---|---|---|---|
| $T$ | time | $\triangle t$ | time | $m$ |
| 0.01 | 64.0805 | 0.001 | 26.7623 | 53 |
| 0.1 | 66.9893 | 0.01 | 3.8016 | 17 |
| 1 | 67.2349 | 0.1 | 0.6975 | 5 |

*Table 4.46*: Computational time, in seconds, using Crank-Nicholson with time step $\triangle t$, and eigenfunction expansion with $m$ terms, for $f(x) = (|x-\pi| < 0.5)(|y-\pi| < 0.5)$

Final solution for $f(x) = \cos(x)\sin(y)$

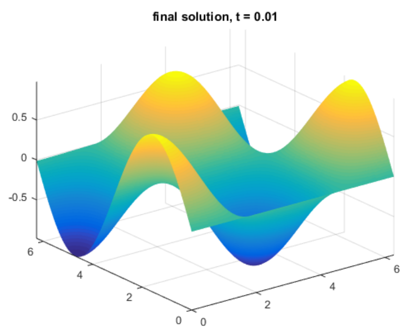


*Figure 4.9*: Eigenfunction expansion, $t = 0.01$

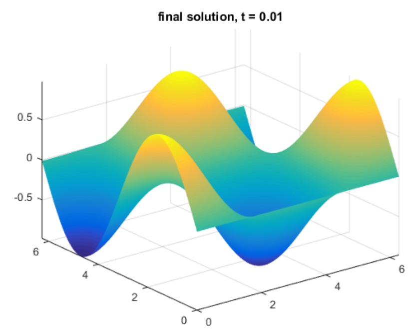


*Figure 4.10*: Crank- Nicholson, $t = 0.01$

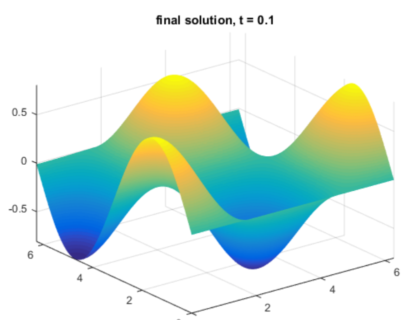


*Figure 4.11*: Eigenfunction expansion, $t = 0.1$

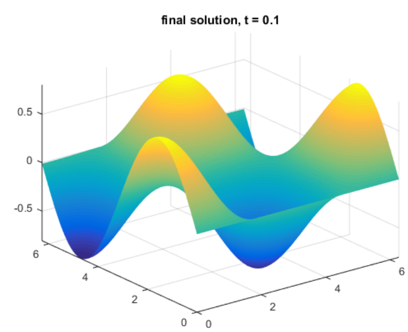


*Figure 4.12*: Crank- Nicholson, $t = 0.1$

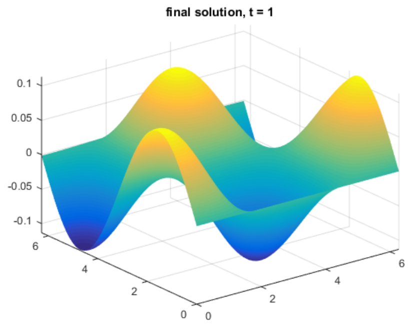


*Figure 4.13*: Eigenfunction expansion, $t = 1$

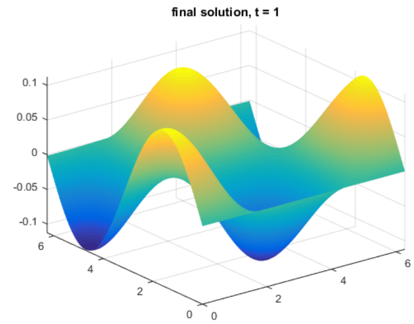


*Figure 4.14*: Crank- Nicholson, $t = 1$

Final solution for $f(x) = (|x-\pi| < 0.5)(|y-\pi| < 0.5)$

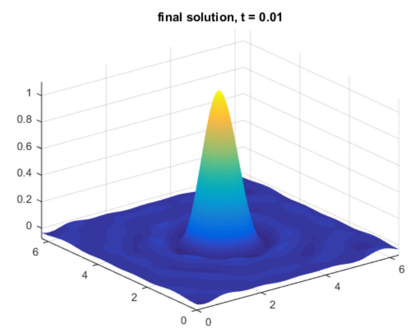


*Figure 4.15*: Eigenfunction expansion, $t = 0.01$

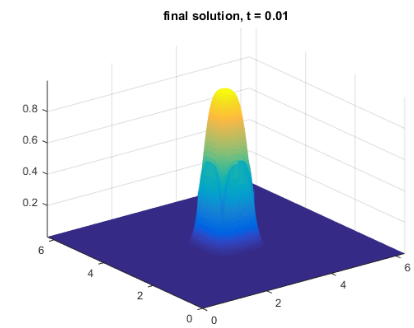


*Figure 4.16*: Crank- Nicholson, $t = 0.01$

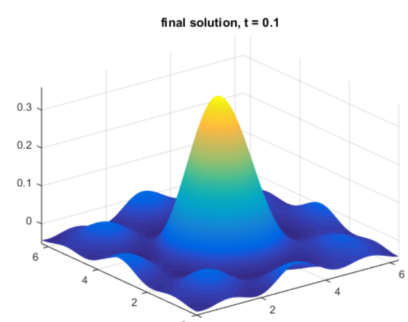


*Figure 4.17*: Eigenfunction expansion, $t = 0.1$

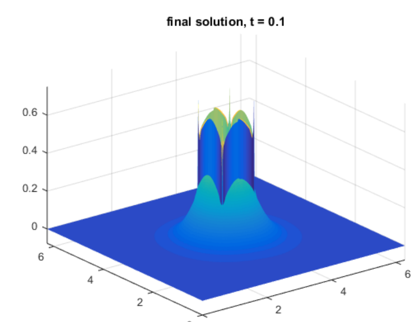


*Figure 4.18*: Crank- Nicholson, $t = 0.1$

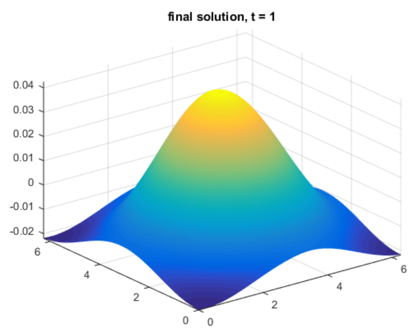


*Figure 4.19*: Eigenfunction expansion, $t = 1$

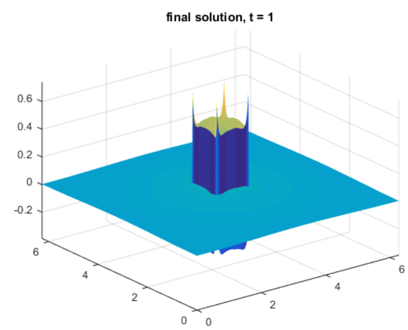


*Figure 4.20*: Crank- Nicholson, $t = 1$

# Chapter 5
# CONCLUSION

The outcome of this research was a success. An accurate and efficient algorithm for computing eigenfunctions of differential operators- used to model the diffusion of heat energy in two space dimensions through two different homogeneous materials- was designed and implemented. This method substantially simplifies the solution of such models with discontinuous coefficients and bypasses the limitations of analytical and numerical methods.

Firstly, solving the PDEs using this approach greatly reduces the number of variables of the PDE and therefore enables it to be solved more easily and with higher accuracy. Secondly, and more importantly, with this approach we can compute the eigenvalues independently and simultaneously which is not possible using existing numerical methods to solve these kinds of problems such as the Crank-Nicholson method, and this makes our approach much faster than existing numerical methods.

This higher speed and efficiency obtained was the main goal of this paper because with today's powerful computers, we are required to perform precise real time simulations of physical phenomena and that too in very high clarity. Our approach will enable that, whereas, for other numerical methods, it was observed that it would present difficulties that would make fast and efficient real-time simulation impossible. Our research solves complex yet important physical problems with discontinuities more easily and with greater efficiency and speed. It gives us the platform to gain more precision and strive for further improvement in this field.

To improve this method, a way for gaining more precision by improving the algorithm should be explored. For continuation of this work, it is worthwhile to investigate the case comprising more than two homogeneous materials, as similar conditions would apply to the eigenfunctions. This could improve our understanding of the qualitative behavior of solutions of different types of equations with discontinuous coefficients. The natural extension of this research is to study the behavior of these equations in three-dimensional models, as our approach to compute the eigenvalues to solve PDEs is not limited to two or lower space dimensions.

# Appendix A

# Code

### A.1 *TwoDPDE* (TwoDPDE.m)

```
n=512;
dx=2*pi/n;



% 2nd derivative matrix, periodic BC
e=ones(n,1);
D2=((1/dx)^2 *spdiags([e -2*e e], [-1 0 1], n, n));
D2(1,n)=1/dx^2;
D2(n,1)=1/dx^2;

% forward, backward and centered 1st derivative matrices, periodic BCs
Dp= (1/dx)*spdiags([-e e], [0 1], n, n);
Dp(n,1)=1/dx;
Dm= (1/dx)*spdiags([-e e], [-1 0], n, n);
Dm(1,n)=-1/dx;
Dc= (1/(2*dx))*spdiags([-e e], [-1 1], n, n);
Dc(1,n)=-1/(2*dx);
Dc(n,1)=1/(2*dx);
I=speye(n);

% 2-D first partial derivative matrices
Dpx=kron(Dp,I);
Dpy=kron(I,Dp);
Dmx=kron(Dm,I);
Dmy=kron(I,Dm);

alpha1=1;
alpha2=alpha1; %changed
```

```
alpha3=1.1;
alpha4=alpha3; %changed

% piecewise constant coefficient
v(1: n/2, 1: n/2)=(alpha1)^2;
v(1: n/2, (n/2)+1 :n ) = (alpha3)^2;
v((n/2)+1:n , 1: n/2)= (alpha2)^2;
v((n/2)+1:n, (n/2)+1:n)=(alpha4)^2;

vm=reshape(v, numel(v), 1);

D2x=kron(D2,I);
D2y=kron(I,D2);

k=spdiags(vm,0,n^2,n^2);

AN=-(k*Dpx*Dmx + k*Dpy*Dmy);

ops.tol=1e-8;
[XN,DN]= eigs(AN,51,'sm',ops);
[DN,inds]=sort(real(diag(DN)));
XN=XN(:,inds);
```

## A.2 *TwoDPDErho* (`TwoDPDErho.m`)

```
n=512;
dx=2*pi/n;

% this code is used for the four diffrent values of rho
rho=0.1;
% rho=1/pi;
% rho=log(2);
% rho=0.99;

% 2nd derivative matrix, periodic BC
e=ones(n,1);
```

```
D2=((1/dx)^2 *spdiags([e -2*e e], [-1 0 1], n, n));
D2(1,n)=1/dx^2;
D2(n,1)=1/dx^2;

% forward, backward and centered 1st derivative matrices, periodic BCs
Dp= (1/dx)*spdiags([-e e], [0 1], n, n);
Dp(n,1)=1/dx;
Dm= (1/dx)*spdiags([-e e], [-1 0], n, n);
Dm(1,n)=-1/dx;
Dc= (1/(2*dx))*spdiags([-e e], [-1 1], n, n);
Dc(1,n)=-1/(2*dx);
Dc(n,1)=1/(2*dx);
I=speye(n);

% 2-D first partial derivative matrices
Dpx=kron(Dp,I);
Dpy=kron(I,Dp);
Dmx=kron(Dm,I);
Dmy=kron(I,Dm);

alpha1=1;
alpha2=alpha1; %changed
alpha3=2;
alpha4=alpha3; %changed

% piecewise constant coefficient
v(1:round(n*rho), 1:round(n*rho))=(alpha1)^2;
v(1:round(n*rho), round(n*rho)+1:n ) = (alpha3)^2;
v(round(n*rho)+1:n,1:round(n*rho))= (alpha2)^2;
v(round(n*rho)+1:n,round(n*rho)+1:n)=(alpha4)^2;

vm=reshape(v, numel(v), 1);

D2x=kron(D2,I);
D2y=kron(I,D2);
```

```
k=spdiags(vm,0,n^2,n^2);

AN=-(k*Dpx*Dmx + k*Dpy*Dmy);

[XN,DN]= eigs(AN,51,'sm',ops);

[DN,inds]=sort(real(diag(DN)));
XN=XN(:,inds);
```

### A.3 *Visualize2D* (Visualize2D.m)

```
 for j=1:51
j
% xj = jth eigenfunction
xj=real(XN(:,j));

% d^2/dx^2(xj)
xjxx=D2x*xj;

% d^2/dy^2(xj)
xjyy=D2y*xj;

% Laplacian(xj)
xjLapl=xjxx+xjyy;

% to see if xj is an eigenfunction (at least locally) of these
% operators
vec1 = (xjxx./xj);
vec2 = (xjyy./xj);
vec3 = xjLapl./xj;

% to examine 1-D slices of eigenfunction or derivatives
xj= reshape(xj,n,n);
I2=speye(n/2);
M =blkdiag((alpha1^2)*I2,(alpha3^2)*I2);
```

```
% d/dx
Ddxcolumn= -D2*xj(:,16)./xj(:,35);
% d/dy
Ddyrow=(Dc*xj')';

figure(3)
plot(real(xj(:,40)))
title([ '\lambda = ' num2str(DN(j)) ])
axis tight
figure(4)
plot(real(xj(40,:)))
title([ '\lambda = ' num2str(DN(j)) ])
axis tight
pause

end
```

### A.4 *Fw*2 (Fw2.m)

```
%Function Y
% eta1= eta2 = fixed integer

function Y=Fw2(w2,alpha3,eta1)
alpha1=1;

% w1 for sin cos case
w1=sqrt(w2^2*alpha3^2/alpha1^2+eta1^2*alpha3^2/alpha1^2-eta1^2);

% sin cos case
f_11=cos(w2*2*pi)*cos(w1*pi)-(w2/w1)*(sin(w2*2*pi)*sin(w1*pi))-cos(w2*pi);
f_12=sin(w2*2*pi)*cos(w1*pi)+(w2/w1)*(cos(w2*2*pi)*sin(w1*pi))-sin(w2*pi);
f_21=-w1*cos(w2*2*pi)*sin(w1*pi)-w2*sin(w2*2*pi)*cos(w1*pi)+w2*sin(w2*pi);
f_22=-w1*sin(w2*2*pi)*sin(w1*pi)+w2*cos(w2*2*pi)*cos(w1*pi)-w2*cos(w2*pi);

F=[f_11,f_12;f_21,f_22];
Y=det(F);
```

### A.5 *Fw*3 (Fw3.m)

```
%Function Y
% eta1= eta2 = fixed integer

function Y=Fw3(w2,alpha3,eta1)
alpha1=1;



%w1 for sinh cosh case
w1=sqrt(-w2^2*alpha3^2/alpha1^2+eta1^2*alpha3^2/alpha1^2-eta1^2);

% sinh cosh case
f_11=cosh(w2*2*pi)*cos(w1*pi)+(w2/w1)*(sinh(w2*2*pi)*sin(w1*pi))-cosh(w2*pi);
f_12=sinh(w2*2*pi)*cos(w1*pi)+(w2/w1)*(cosh(w2*2*pi)*sin(w1*pi))-sinh(w2*pi);
f_21=-w1*cosh(w2*2*pi)*sin(w1*pi)+w2*sinh(w2*2*pi)*cos(w1*pi)-w2*sinh(w2*pi);
f_22=-w1*sinh(w2*2*pi)*sin(w1*pi)+w2*cosh(w2*2*pi)*cos(w1*pi)-w2*cosh(w2*pi);

F=[f_11,f_12;f_21,f_22];
Y=det(F);
```

### A.6 Secant Method (secant.m)

```
% Secant method
function [x,n]=secant(f,x0,x1,tolx,toly)
if nargin<4
% default value for x-tolerance
tolx=1e-10;
end
if nargin<5
% default value for y-tolerance
toly=tolx;
end
n=0;
x=x1;
for k=1:20
```

```
% main step
x2=(x1*f(x0)-x0*f(x1))/(f(x0)-f(x1));
n=n+1;
% check for convergence (in x)
if abs(x2-x0)<tolx
x=x2;
return
end
% set up next iteration
x0=x1;
x1=x2;
end
```

### A.7 Finding the Eigenvalues (`new.m`)

```
vecw=[];
vecf=[];

vec1lam=[];
vec2lam=[];
vec1w=[];
vec1eta=[];
vec2w=[];
vec2eta=[];
vec1iter=[];
vec2iter=[];
t=1;
ep=1e-6;
alpha3=1.1;
w0=1/(2*(1+alpha3));
rad=sqrt(-log(ep)/(t*alpha3^2))

for eta = 0:rad
for w=w0:w0:rad
if (w^2+eta^2)<1.5*rad^2
x0=w;
```

```
x1=x0+0.01;
[s,niter]=secant(@(w2)Fw2(w2,alpha3,eta),x0,x1);
if abs(imag(s))<1e-8 && niter<20
lam=alpha3^2*(s^2+eta^2);
if lam<rad && s>1e-10
vec1lam=[vec1lam;lam];
vec1w=[vec1w;s];
vec1eta=[vec1eta;eta];
vec1iter=[vec1iter;niter];
end
end
end
if w<=eta
x0=w;
x1=x0+0.01;
[s,niter]=secant(@(w2)Fw3(w2,alpha3,eta),x0,x1);
w1=sqrt(-s^2*alpha3^2/alpha1^2+eta^2*alpha3^2/alpha1^2-eta^2);
if abs(imag(s))<1e-8 && abs(imag(w1))<1e-8 && niter<20
lam=alpha3^2*(-s^2+eta^2);
if lam<rad && s>1e-10
vec2lam=[vec2lam;lam];
vec2w=[vec2w;s];
vec2eta=[vec2eta;eta];
vec2iter=[vec2iter;niter];
end
end
end

end
end


[vec1lam,niter]=sort(vec1lam);
vec1eta=vec1eta(niter);
vec1iter=vec1iter(niter);
vec1w=vec1w(niter);
```

```
[vec2lam,niter]=sort(vec2lam);
vec2eta=vec2eta(niter);
vec2iter=vec2iter(niter);
vec2w=vec2w(niter);
```

### A.8 Fw4 (Fw4.m)

```
function [A2,B2]=Fw4(w2,alpha3,eta1)
alpha1=1;

% w1 for sin cos case
w1=sqrt(w2^2*alpha3^2/alpha1^2+eta1^2*alpha3^2/alpha1^2-eta1^2);

% sin cos case
f_11=cos(w2*2*pi)*cos(w1*pi)-(w2/w1)*(sin(w2*2*pi)*sin(w1*pi))-cos(w2*pi);
f_12=sin(w2*2*pi)*cos(w1*pi)+(w2/w1)*(cos(w2*2*pi)*sin(w1*pi))-sin(w2*pi);
f_21=-w1*cos(w2*2*pi)*sin(w1*pi)-w2*sin(w2*2*pi)*cos(w1*pi)+w2*sin(w2*pi);
f_22=-w1*sin(w2*2*pi)*sin(w1*pi)+w2*cos(w2*2*pi)*cos(w1*pi)-w2*cos(w2*pi);


F=[f_11,f_12;f_21,f_22];

A2=-f_12/sqrt(f_11^2+f_12^2);
B2=f_11/sqrt(f_11^2+f_12^2);
```

### A.9 Fw5 (Fw5.m)

```
function [A2,B2]=Fw5(w2,alpha3,eta1)
alpha1=1;

%w1 for sinh cosh case
w1=sqrt(-w2^2*alpha3^2/alpha1^2+eta1^2*alpha3^2/alpha1^2-eta1^2);
```

```
% sinh cosh case
f_11=cosh(w2*2*pi)*cos(w1*pi)+(w2/w1)*(sinh(w2*2*pi)*sin(w1*pi))-cosh(w2*pi);
f_12=sinh(w2*2*pi)*cos(w1*pi)+(w2/w1)*(cosh(w2*2*pi)*sin(w1*pi))-sinh(w2*pi);
f_21=-w1*cosh(w2*2*pi)*sin(w1*pi)+w2*sinh(w2*2*pi)*cos(w1*pi)-w2*sinh(w2*pi);
f_22=-w1*sinh(w2*2*pi)*sin(w1*pi)+w2*cosh(w2*2*pi)*cos(w1*pi)-w2*cosh(w2*pi);


F=[f_11,f_12;f_21,f_22];

A2=-f_12/sqrt(f_11^2+f_12^2);
B2=f_11/sqrt(f_11^2+f_12^2);
```

## A.10 Solving the PDE (`ef.m`)

```
n=512;
dx=2*pi/n;
x=(0:n-1)*dx;
[xm,ym]=meshgrid(x,x);

% The two different functions used as initial data
% F=cos(xm).*sin(ym);
F=(abs(xm-pi)<0.5).*(abs(ym-pi)<0.5);

surf(x,x,F)
title('initial data, t=0')
axis tight
shading flat
pause
F=reshape(F,numel(F),1);
x1=xm(:,1:n/2);
x2=xm(:,n/2+1:end);
y1=ym(:,1:n/2);
y2=ym(:,n/2+1:end);

alpha1=1;
```

```
alpha3=1.1;

v=zeros(n);
v(1: n/2, 1: n/2)=(alpha1)^2;
v(1: n/2, (n/2)+1 :n ) = (alpha3)^2;
v((n/2)+1:n , 1: n/2)= (alpha1)^2;
v((n/2)+1:n, (n/2)+1:n)=(alpha3)^2;

vm=reshape(v, numel(v), 1);

soln=0;

vec1=[ vec1w vec1eta ];
uniquetol(vec1);

vec2=[ vec2w vec2eta ];
uniquetol(vec2);

vec1u=uniquetol(vec1,1e-8,'ByRows',true);
vec1lam=alpha3^2*(vec1u(:,1).^2+vec1u(:,2).^2);
[vec1lam,inds]=sort(vec1lam);
vec1eta=vec1u(inds,2);
vec1w=vec1u(inds,1);

vec2u=uniquetol(vec2,1e-8,'ByRows',true);
vec2lam=alpha3^2*(vec2u(:,1).^2+vec2u(:,2).^2);
[vec2lam,inds]=sort(vec2lam);
vec2eta=vec2u(inds,2);
vec2w=vec2u(inds,1);

for i=1:length(vec1w)
w2=vec1w(i);
eta=vec1eta(i);
% w1 for the sin cos case
w1=sqrt(w2^2*alpha3^2/alpha1^2+eta^2*alpha3^2/alpha1^2-eta^2);
```

```
if w2==0 && eta==0
A2=1; B2=0;
A1=1; B1=0;
else
[A2,B2]=Fw4(w2,alpha3,eta)
A1=A2*cos(w2*2*pi)+B2*sin(w2*2*pi);
B1=(-w2*A2*sin(w2*2*pi)+w2*B2*cos(w2*2*pi))/w1;
end

lambda=alpha3^2*(w2^2+eta^2);

v1=(A1*cos(w1*x1)+B1*sin(w1*x1)).*cos(eta*y1);
v2=(A2*cos(w2*x2)+B2*sin(w2*x2)).*cos(eta*y2);
V=[v1 v2];

V=reshape(V,numel(V),1);
W=V./vm;

term=exp(-lambda*t)*(W'*F)/(V'*W);
soln=soln+term*V;
if eta~=0
v1=(A1*cos(w1*x1)+B1*sin(w1*x1)).*sin(eta*y1);
v2=(A2*cos(w2*x2)+B2*sin(w2*x2)).*sin(eta*y2);
V=[v1 v2];
V=reshape(V,numel(V),1);
W=V./vm;
term=exp(-lambda*t)*(W'*F)/(V'*W);
soln=soln+term*V;
end
end

for i=1:length(vec2w)
w2=vec2w(i);
eta=vec2eta(i);
```

```
%w1 for w2 sinh cosh case
w1=sqrt(-w2^2*alpha3^2/alpha1^2+eta^2*alpha3^2/alpha1^2-eta^2);

[A2,B2]=Fw5(w2,alpha3,eta);


A1=A2*cosh(w2*2*pi)+B2*sinh(w2*2*pi);
B1=(w2*A2*sinh(w2*2*pi)+w2*B2*cosh(w2*2*pi))/w1;


v1=(A1*cos(w1*x1)+B1*sin(w1*x1)).*cos(eta*y1);
v2=(A2*cosh(w2*x2)+B2*sinh(w2*x2)).*cos(eta*y2);
V=[v1 v2];
V=reshape(V,numel(V),1);
W=V./vm;
lambda=alpha3^2*(eta^2-w2^2);
term=exp(-lambda*t)*(W'*F)/(V'*W);
soln=soln+term*V;

v1=(A1*cos(w1*x1)+B1*sin(w1*x1)).*sin(eta*y1);
v2=(A2*cosh(w2*x2)+B2*sinh(w2*x2)).*sin(eta*y2);
V=[v1 v2];

V=reshape(V,numel(V),1);
W=V./vm;
term=exp(-lambda*t)*(W'*F)/(V'*W);
soln=soln+term*V;
end

soln=reshape(soln,512,512);
surf(x,x,soln)
axis tight
shading flat
title([ 'final solution, t = ' num2str(t) ])
```

### A.11 Crank-Nicholson (cn.m)

```
N=512; % no. of grid pts.
I=speye(N^2);

n=512;
dx=2*pi/n;
x=(0:n-1)*dx;
[xm,ym]=meshgrid(x,x);

% The two different functions used as initial data
% F=cos(xm).*sin(ym);
F=(abs(xm-pi)<0.5).*(abs(ym-pi)<0.5);

surf(x,x,F)
title('initial data, t=0')
axis tight
shading flat
pause
F=reshape(F,numel(F),1);
x1=xm(:,1:n/2);
x2=xm(:,n/2+1:end);
y1=ym(:,1:n/2);
y2=ym(:,n/2+1:end);

u=F; % intitial guesses from ef
t=1;% final time
dt=0.1;%the chosen time-step
n=t/dt;


for i=1:n
u= (I + dt/2*AN)\((I - dt/2*AN)*u);
end

u=reshape(u,N,N);
```

```
surf(x,x,u)
axis tight
shading flat
title([ 'final solution, t = ' num2str(t) ])
```

# BIBLIOGRAPHY

[1] Arfken, G. B., Weber, H. J. and Harris, F. E. (2012) *Mathematical Methods for Physicists: A Comprehensive Guide*, 7th Edition. Waltham, MA: Academic Press.

[2] Burden, R. L. and Faires, J. D. (2010) *Numerical Analysis*, 9th Edition. Stamford, CT: Cengage Learning.

[3] Fefferman, C. L. (1983) The Uncertainty Principle. *Bulletin of the American Mathematical Society* **9**(2), 129-206.

[4] Filoche, M., Mayboroda, S. and Patterson, B.(2012) Localization of eigenfunctions of a one-dimensional elliptic operator. *Contemporary Mathematics* **581**, 99-116.

[5] Fix, G. J. (1973) Eigenvalue approximation by the finite element method. *Advances in Mathematics* **10**(2) , 300-316.

[6] Garon, E. M. and Lambers, J. V. (2015). Modeling the Diffusion of Heat Energy within Composites of Homogeneous Materials using the Uncertainty Principle. *Submitted*.

[7] Gelb, A. and Tanner, J. (2006) Robust reprojection methods for the resolution of the Gibbs phenomenon. *Applied and Computational Harmonic Analysis* **20**(1), 3-25.

[8] Gustafsson, B., Kreiss, H.-O. and Oliger, J. (1995) *Time Dependent Problems And Difference Methods*. Baltimore, MD: New York, NY: John Wiley & Sons, Inc.

[9] Hou, T. Y., Li, Z. L., Osher, S. and Zhao, H. K. (1997) A hybrid method for moving interface problems with application to the heleshaw flow. *Journal of Computational Physics* **134**(2), 236-252.

[10] Hur, V. M. (2012) On the formation of singularities for surface water waves. *Communications on Pure and Applied Analysis* **11**(4) , 1465-1474.

[11] Lambers, J. V. (2012)Approximate Diagonalization of Variable-Coefficient Differential Operators Through Similarity Transformations. *Computers & Mathematics with Applications* **64**(8) , 2575-2593.

[12] Layton, A. T. (2009) Using integral equations and the immersed interface method to solve immersed boundary problems with stiff forces.*Computers & Fluids* **38**(2), 266-272.

[13] Li, Z. (1994). The Immersed Interface Method- A Numerical Approach for Partial Differential Equations with Interfaces (Doctoral dissertation). *Retrieved from the University of Washington ResearchWorks Archive*.

[14] Lorenzo, G., Scott, M. A., Tew, K., Hughes, T. J. R. and Gomez, H. (2017) Hierarchically refined and coarsened splines for moving interface problems, with particular application to phase-field models of prostate tumor growth. *Computer Methods in Applied Mechanics and Engineering* **319**, 515-548.

[15] Min, M. S. and Gottlieb, D. (2002) On the Convergence of the Fourier Approximation for Eigenvalues and Eigenfunctions of Discontinuous Problems. *SIAM Journal on Numerical Analysis* **40**(6) , 2254-2269.

[16] Min, M. S. and Gottlieb, D. (2005) Domain Decomposition Spectral Approximations for an Eigenvalue Problem with a Piecewise Constant Coefficient. *SIAM Journal on Numerical Analysis* **40**(6) , 502-520.

[17] Mu, L., Wang, J., Wei, G., Ye, X. and Zhao, S. (2013) Weak Galerkin methods for second order elliptic interface problems. *Journal of Computational Physics* **250**, 106-125.

[18] Peskin, C. S. (2002) The immersed boundary method. *Acta Numerica* **11**, 479-517.

[19] Zhao, S. and Wei, G. W. (2004) High-order FDTD methods via derivative matching for Maxwell's equations with material interfaces. *Journal of Computational Physics* **200**(1), 60-103.

[20] Zhu, L., Zhang, Z. and Li, Z. (2015) An immersed finite volume element method for 2D PDEs with discontinuous coefficients and non-homogeneous jump conditions. *Computers & Mathematics with Applications* **70**(2) , 89-103.