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Abstract

We propose a piecewise linear approximation method, based on the maximum entropy principle, to approximate a nonnegative solution of a Fredholm integral equation numerically. The theoretical analysis and numerical examples show that our method has a convergence rate of order 2, and it can get more accurate approximations with more moments used without ill-condition of the classic maximum entropy approach. The method can also be applied to solve Frenholm integral equations with singular kernels.

Abbreviated title: Piecewise Linear Maximum Entropy Method

Key Words: Fredholm integral equation, maximum entropy, piecewise linear approximations

AMS Subject Classifications: 41A35, 65D07, 65J10

1 Introduction

Recently a maximum entropy method using piecewise linear functions has been developed for recovering a density function for the statistical study of deterministic dynamical system on an interval [6]. This approach has its origin in the 1957 paper [9] by Jayne and was first adopted in [5] for the computation of a stationary density of an interval mapping $S : [0, 1] \to [0, 1]$. The renewed interest in the maximum entropy scheme is due to the observation that the employment of piecewise polynomial functions, which are polynomial functions on each sub-domain from a finite partition of the whole domain, can effectively eliminate the difficulty of ill-condition of the traditional maximum entropy method that uses only polynomials as moment functions. Because of the locally nonzero property of the basis for such functions, the finite element spaces of piecewise polynomial functions are widely used in numerical partial differential equations and integral equations.

Encouraged by the efficiency of the new finite element maximum entropy method, we would like to extend the approach to compute a nonnegative solution of a Fredholm integral equation numerically under the assumption that such a solution does exist.

A maximum entropy method was first proposed in [11] for solving Fredholm integral equations. Like all other traditional maximum entropy approaches for calculating a density function numerically, the moment related functions used in [11] were polynomials. Since the polynomial involved numerical computation is highly ill-conditioned as usual, it is not surprising that no more than 10 or so moments could be used in the computation of the examples in the paper. In the last section of [11] the author raised some question of the method: "It is not clear whether or not solutions generated by a larger number of moments than used here would improve the numerical accuracy significantly." Our motivation of writing this paper is for answering the above question.

After giving some preliminaries in the next section, we present our piecewise linear maximum entropy method for the integral equation in Sections 3. A convergence analysis will be presented in Section 4. Numerical examples will be demonstrated in Section 5 and we conclude in Section 6.

2 Preliminaries

In this section we present the Fredholm integral equation and obtain an explicit formula for the solution of the maximum entropy problem.

Let a and b be two real numbers with $a < b$, let $L^1(a, b)$ be the Banach space of real-valued Lebesgue integrable functions defined on (a, b) with the L^1-norm $\|f\|_1 = \int_a^b |f(x)|dx$, and let K be a linear integral operator from

$L^1(a,b)$ into itself defined by

$$(Kf)(x) = \int_a^b k(x,y)f(y)dy,$$

where the kernel function k is measurable and bounded on its domain. Let $g \in L^1(a,b)$ be given. The following operator equation

$$(I - K)f = g \tag{1}$$

is called the *Fredholm integral equation of the second kind*. For the purpose of the applicability of the maximum entropy method for finding a solution of (1), we assume that a nonnegative solution f^* exists for the given g. A sufficient condition to make it happen is that g is a nonnegative function and K is a positive operator such that $\|K\|_1 < 1$, where $\|K\|_1$ is the operator norm of K. This follows from the Neumann series

$$(I - K)^{-1} = I + K + K^2 + \cdots$$

so that $(I - K)^{-1}$ exists and is positive.

For any $f \in L^1(a,b)$ and $h \in L^\infty(a,b)$, the Banach space of essentially bounded Lebesgue measurable functions, we write $\langle f, h \rangle = \int_a^b f(x)h(x)dx$. Denote by $K' : L^\infty(a,b) \to L^\infty(a,b)$ the *dual operator* of K, defined by

$$\langle Kf, h \rangle = \langle f, K'h \rangle$$

for all $f \in L^1(a,b)$ and $h \in L^\infty(a,b)$. Then we have

$$(K'h)(y) = \int_a^b k(x,y)h(x)dx.$$

Let $h_1, \ldots, h_s \in L^\infty(a,b)$ be a set of linearly independent functions. Multiplying each of them to the equality

$$(I - K)f^* = g$$

and integrating over $[a, b]$, we have

$$\langle f^*, (I - K')h_i \rangle = \langle g, h_i \rangle, \quad i = 1, \ldots, s.$$

We call such initially chosen functions h_i the *generating functions*.

Define $g_i = (I - K')h_i$ for each i, which are called the *moment functions*. Denote $m_i = \langle g, h_i \rangle$ for each i. Then the above equalities can be written as the moment conditions

$$\langle f^*, g_i \rangle = m_i, \quad i = 1, \ldots, s.$$

Under the condition that $f^* \in L^p(a, b)$ for some $1 < p < \infty$, if an infinite sequence of the functions $\{h_i\}$ is chosen such that the resulting sequence $\{g_i\}$ is dense in $L^q(a, b)$ with $1/p + 1/q = 1$, then the condition

$$\langle f^*, g_i \rangle = m_i, \quad i = 1, 2, \ldots$$

determines f^* uniquely. This is possible, for example, if $I - K$ is one-to-one and onto, so that $(I - K)^{-1}$ is also bounded by Banach's Inverse Mapping Theorem, and if $\{h_i\}$ is dense in $L^q(a, b)$. The convergence analysis in Section 4 even shows that the sequence of piecewise linear functions satisfies the above purpose.

Our strategy to recover the unknown solution f^* numerically is via solving the equations

$$\langle w, g_i \rangle = m_i, \quad i = 1, \ldots, s$$

for the unknown nonnegative function w with the maximum entropy principle. In the maximum entropy approximation, for the convenience of notation and analysis, we maximize a *modified Boltzmann entropy* [12], which is a nonlinear functional H defined by

$$H(f) = -\int_a^b f(x)\ln f(x)dx + \int_a^b f(x)dx, \quad \forall\, f \in L^1(a,b),\ f \geq 0 \tag{2}$$

with the moment equality constraints

$$\int_a^b f(x)g_i(x)dx = m_i, \quad i = 1, 2, \ldots, s, \tag{3}$$

where $g_1, \ldots, g_s \in L^\infty(a,b)$ and $m_1, \ldots, m_s$ are fixed constants. Here, the modified Boltzmann entropy defined above is the usual Boltzmann entropy of f [10] plus the integral of f.

Since only a linear term $\int_a^b f(x)dx$ of f is added to the original definition of the Boltzmann entropy, the modified Boltzmann entropy H in (2) shares the same properties as the original one, which are listed below and can be proved with the same argument as in [4, 10]:

(i) $H(f)$ is either finite or $-\infty$ for any nonnegative function $f \in L^1(a,b)$.

(ii) H is proper, upper-semicontinuous, and concave, and is strictly concave on its effective domain consisting of all nonnegative L^1-functions f with $H(f) > -\infty$.

(iii) The upper level sets $\{f \in L^1(a,b) : f \geq 0, H(f) \geq \alpha\}$ are weakly compact in $L^1(a,b)$ for all numbers $\alpha > -\infty$.

We need the following lemma for solving the above maximum entropy optimization problem.

Lemma 2.1 *Given any two nonnegative functions $f,\ g \in L^1(a,b)$, then*

$$\int_a^b f(x)dx - \int_a^b f(x)\ln f(x)dx \leq \int_a^b g(x)dx - \int_a^b f(x)\ln g(x)dx.$$

Proof In the Gibbs inequality

$$u - u\ln u \leq v - u\ln v, \ \ \forall\, u, v \geq 0,$$

letting $u = f(x)$ and $v = g(x)$ and then integrating both sides, we obtain the required integral inequality. Q.E.D.

Proposition 2.1 *The nonnegative function in* $L^1(a, b)$ *that maximizes the entropy functional (2) under the constraints (3) is*

$$f_s(x) = e^{\sum_{i=1}^{s}\lambda_i g_i(x)},$$

if the constants $\lambda_1, \ldots, \lambda_s$*, which are called the* Lagrange multipliers*, satisfy the given constraints*

$$\int_a^b e^{\sum_{i=1}^{s}\lambda_i g_i(x)} g_j(x)dx = m_j, \ j = 1, \ldots, s.$$

Proof Let $f \in L^1(a, b)$ be nonnegative and satisfy the moment constraints

$$\int_a^b f(x)g_j(x)dx = m_j, \ j = 1, \ldots, s.$$

Then, using Lemma 2.1, we have

$$\begin{aligned}
H(f) &= -\int_a^b f(x)\ln f(x)dx + \int_a^b f(x)dx \\
&\leq \int_a^b f_s(x)dx - \int_a^b f(x)\ln f_s(x)dx \\
&= \int_a^b f_s(x)dx - \sum_{i=1}^{s}\lambda_i \int_a^b f(x)g_i(x)dx \\
&= \int_a^b f_s(x)dx - \sum_{i=1}^{s}\lambda_i m_i \\
&= \int_a^b f_s(x)dx - \int_a^b f_s(x)\left[\sum_{i=1}^{s}\lambda_i g_i(x)\right]dx \\
&= \int_a^b f_s(x)dx - \int_a^b f_s(x)\ln f_s(x)dx = H(f_s).
\end{aligned}$$

This shows that f_s is an optimal solution. Furthermore, since H is strictly concave, the optimal solution is unique. Q.E.D.

3 The Piecewise Linear Maximum Entropy Method

Now we develop a maximum entropy approach based on piecewise linear functions for solving the Fredholm integral equation (1). Suppose the equation has a unique solution f^* that is nonnegative and integrable.

We divide the interval $I = [a, b]$ into n equal subintervals $I_i = [x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$ with $\delta = (b-a)/n$ the length of each subinterval. For each $i = 0, 1, \cdots, n$ let ϕ_i be the unique continuous piecewise linear function such that $\phi_i(x_i) = 1$ and $\phi_i(x_j) = 0$ for $j \neq i$. Then

$$\phi_i(x) = w\left(\frac{x - x_i}{\delta}\right), \quad i = 0, 1, \cdots, n,$$

where

$$w(x) = \begin{cases} 1 + x, & \text{if } -1 \leq x \leq 0 \\ 1 - x, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{if } x \notin [-1, 1] \end{cases}$$

is the standard hat function. Such functions $\phi_0, \phi_1, \cdots, \phi_n$ form the *canonical basis* for the space, denoted as Δ_n, of all continuous piecewise linear functions corresponding to the partition of $[a, b]$, and they satisfy the identity of *partition of unity*:

$$\sum_{i=0}^{n} \phi_i(x) \equiv 1. \tag{4}$$

This property was used in [6] for the elimination of ill-condition of the traditional maximum entropy method, resulting in an efficient computation of stationary densities of discrete dynamical systems. But since our f^* is not required to be a density, it does not play an important role here.

Let suppϕ_i be the *support* of ϕ_i, the closure of the set of all the points x such that $\phi_i(x) \neq 0$. Then it is clear that supp$\phi_0 = [x_0, x_1]$, supp$\phi_n = [x_{n-1}, x_n]$, and supp$\phi_i = [x_{i-1}, x_{i+1}]$ for $i = 1, 2, \ldots, n-1$.

Now we use the above basic piecewise linear functions to solve the Fredholm integral equation (1). For this purpose we denote $g_i = (I - K')\phi_i$ for each $i = 0, 1, \ldots, n$. Also for each i we calculate the ith moment

$$m_i = \int_a^b g(x)\phi_i(x)dx = \int_{\mathrm{supp}\phi_i} g(x)\phi_i(x)dx \tag{5}$$

of the known function g of the right hand side of (1) with respect to the ith generating function ϕ_i. Because of (4), we have

$$\begin{aligned}\sum_{i=0}^{n} m_i &= \sum_{i=0}^{n} \int_a^b g(x)\phi_i(x)dx = \int_a^b g(x) \sum_{i=0}^{n} \phi_i(x)dx \\ &= \int_a^b g(x)dx = \int_a^b (I-K)f^*(x)dx.\end{aligned}$$

Our numerical scheme is thus the following:

Piecewise Linear Maximum Entropy Method. Calculate the moments m_i from (5) and solve the following maximum entropy problem

$$\max\left\{H(f) : f \geq 0,\ f \in L^1(a,b),\ \int_a^b f(x)g_i(x)dx = m_i,\ i = 0, 1, \ldots, n\right\}. \tag{6}$$

The solution of the above entropy maximization problem (6), according to Proposition 2.1, is

$$f_n(x) = e^{\sum_{i=0}^{n} \lambda_i g_i(x)}, \tag{7}$$

where the numbers λ_i's satisfy

$$\int_a^b g_j(x) e^{\sum_{i=0}^n \lambda_i g_i(x)} dx = m_j, \quad j = 0, 1, \ldots, n. \tag{8}$$

The same idea can be applied to solve the *Fredholm integral equation of the first kind*

$$Kf = g, \tag{9}$$

where K is as defined in Section 2. Suppose K is injective and equation (9) has a unique nonnegative solution f^* with the given g. A sufficient condition for this to happen is that K^{-1} is a positive operator and g is nonnegative.

With a minor revision, our piecewise linear maximum entropy method for solving (9) is the following:

Step 1. Calculate the moments m_i via (5) and the moment functions $g_i = K'\phi_i$ for $i = 0, 1, \ldots, n$.

Step 2. Solve the maximization problem (6) to obtain the maximum entropy approximate solution (7) in which the Lagrange multipliers λ_i's solve (8).

4 Convergence and Convergence Rate

Now we use the general convergence theory of [4] for the moment problem to prove the convergence of our algorithm and give an error estimate. For the simplicity of presentation we only focus on the maximum entropy method for solving (1). The convergence analysis will be the same for solving (9).

The basic ingredient of the convergence theory that is adapted to our case can be summarized as follows. Let $F : L^1(a, b) \to [-\infty, \infty)$ be a nonlinear functional with weakly compact upper level sets, and let $\{D_n\}$ be a sequence of closed subsets of $L^1(a,b)$ such that $D_{n+1} \subset D_n$ for all n. Consider the sequence of maximization problems

$$\max\{F(f) : f \in D_n\} \tag{10}$$

and the "limiting problem"

$$\max\left\{F(f) : f \in \bigcap_{n=1}^{\infty} D_n\right\}. \tag{11}$$

Let f_n denote a solution of (10) for each n. Assume that f^* is a unique solution of (11) that satisfies $F(f^*) > -\infty$. Then, as proved in [4], the sequence of functions f_n converges to f^* weakly in $L^1(a,b)$ and the sequence of numbers $F(f_n)$ converges to the number $F(f^*)$. Furthermore, if F is Kadec, that is, the weak convergence of u_n to u and the convergence of $F(u_n)$ to $F(u)$ imply $u_n \to u$ in norm for any sequence u_n in $L^1(a,b)$, then $\|f_n - f^*\|_1 \to 0$ as $n \to \infty$.

The entropy functional (2), which is Kadec, satisfies all the requirements for the application of the general theory. We only need to make nested partitions of $[a,b]$, so that the feasible sets of our moment constraints (3) are monotonically decreasing with respect to the inclusion relation.

This particular partition is easy to be made, and in fact it is often used in scientific computing, which is to divide $[a,b]$ into 2^r equal subintervals in succession with $r = 1, 2, \ldots$. Now we may use the index r instead of $n = 2^r$ to denote the approximate maximum entropy solution, in other words, f_r is the computational maximum entropy solution with $2^r + 1$ basic piecewise linear generating functions.

We can put all the above conclusion as the following convergence theorem.

Theorem 4.1 *Suppose that $H(f^*) > -\infty$. Let f_r be the maximum entropy solution of the integral equation (1) with respect to the equal partition of $[a, b]$ into 2^r subintervals. Then $\lim_{r\to\infty} \|f_r - f^*\|_1 = 0$.*

According to the results of [4], the convergence rate of our method depends on the minimal distance

$$d_r = \inf \left\{ \left\| \ln f^* - \sum_{i=0}^{2^r} \alpha_i g_i \right\|_\infty : \ \forall\ \alpha_0, \alpha_1, \ldots, \alpha_{2^r} \right\}$$

of the function $\ln f^*$ to the subspace spanned by all g_i's. Since

$$\begin{aligned} \ln f^* - \sum_{i=0}^{2^r} \alpha_i g_i &= \ln f^* - \sum_{i=0}^{2^r} \alpha_i (I - K')\phi_i \\ = \quad (I - K') &\left[(I - K')^{-1} \ln f^* - \sum_{i=0}^{2^r} \alpha_i \phi_i \right], \end{aligned}$$

it follows that

$$d_r \le \|I - K'\|_\infty \inf \left\{ \left\| (I - K')^{-1} \ln f^* - \sum_{i=0}^{2^r} \alpha_i \phi_i \right\|_\infty : \ \forall\ \alpha_0, \alpha_1, \ldots, \alpha_{2^r} \right\}.$$

Here, to make our mathematical manipulation rigorous, we should assume that the operator $I - K'$ is invertible and the function $\ln f^*$ belongs to its range $R(I - K')$. Then, by a theorem of [4] and a standard pointwise approximation result for continuous piecewise linear functions, we obtain the following error estimate.

Theorem 4.2 *Suppose that $f^*(x) \ge c$ over $[a, b]$ for a positive number c and the function $(I - K')^{-1} f^*$ is twice continuously differentiable on $[a, b]$. Then*

$$\|f_r - f^*\|_1 = O(\delta^2) = O\left(\frac{1}{4^r}\right),$$

where $\delta = (b - a)/2^r$ is the length of the 2^r subintervals of $[a, b]$.

Proof From Theorem 4.7 in [4],

$$\|f_r - f^*\|_1 \leq d_r e^{\frac{d_r}{2}} = O(d_r).$$

Denote $v = (I - K')^{-1} \ln f^*$ and let $v_r \in \Delta_{2^r}$ be such that

$$v_r(x_i) = v(x_i), \ \ \forall\, i = 0, 1, \cdots, 2^r.$$

Then from the interpolation theory (see, e.g., Lemma 3.1 of [7]), $\|v - v_r\|_\infty = O(\delta^2)$. Therefore,

$$d_r \leq \|I - K'\|_\infty \|v - v_r\|_\infty = O(\delta^2).$$

The theorem follows. Q.E.D.

5 Numerical Results

In this section we present some numerical examples for solving Fredholm integral equations. In the implementation of the algorithm, the used moment functions are $g_i = (I - K')\phi_i$ for $i = 0, 1, \ldots, n$ and the moments m_i are defined by formula (5).

Example 1. The first example is the Fredholm integral equation [11]

$$f(x) - \frac{3}{2}\int_0^1 e^{|x-y|} f(y)dy = -1. \tag{12}$$

The exact solution of (12), which is a positive function on $[0, 1]$, is

$$f^*(x) = \frac{3}{4}\left[\frac{3 + e^2}{e^4 - 9}\left(e^{2x} + e^{2(1-x)}\right) - 1\right].$$

Since $g(x) \equiv -1$, to guarantee that the moments are positive, we choose the generating functions to be $-\phi_i,\ i = 0, 1, \ldots, n$. The moment functions are

n	L^1-norm error	L^∞-norm error
4	7.3×10^{-3}	2.0×10^{-2}
8	1.8×10^{-3}	5.0×10^{-3}
16	4.5×10^{-4}	1.3×10^{-3}
32	1.1×10^{-4}	3.2×10^{-4}
64	2.8×10^{-5}	8.1×10^{-5}
128	7.0×10^{-6}	2.0×10^{-5}
256	1.8×10^{-6}	5.0×10^{-6}
512	4.5×10^{-7}	1.2×10^{-6}
1024	1.2×10^{-7}	2.8×10^{-7}

Table 1: Errors for Example 1

then $g_i = -(I - K')\phi_i,\ i = 0, 1, \ldots, n$, and the moments are

$$m_i = \int_0^1 \phi_i(x)dx = \begin{cases} h, & i = 1, \ldots, n-1, \\ \frac{h}{2}, & i = 0, n. \end{cases}$$

The L^1-norm errors $\|f_n - f^*\|_1$ and the L^∞-norm ones $\|f_n - f^*\|_\infty$ of the numerical approximations are shown in Table 1.

Example 2. We consider the Wiener-Hopf integral equation

$$f(x) + 4\int_0^\infty e^{|x-y|} f(y)dy = e^{-|x|} \tag{13}$$

defined on the whole number axis. The exact solution of (13) is

$$f^*(x) = \begin{cases} \frac{1}{2}e^{-3x}, & x \geq 0, \\ \frac{1}{2}e^{x}, & x \leq 0. \end{cases}$$

As the right hand side of equation (13) only includes the part of the function $f(x)$ where $x \geq 0$, we can solve $f(x)$ for $x \geq 0$ first. The function values

n	L^1-norm error	L^∞-norm error
16	3.2×10^{-3}	2.5×10^{-2}
32	8.0×10^{-4}	8.8×10^{-3}
64	2.0×10^{-4}	2.6×10^{-3}
128	4.8×10^{-5}	7.0×10^{-4}
256	1.2×10^{-5}	1.8×10^{-4}
512	3.0×10^{-6}	4.6×10^{-5}
1024	7.5×10^{-7}	1.2×10^{-5}

Table 2: Errors for Example 2

for $x \leq 0$ can be derived from $f(x)$ for $x \geq 0$. It is a widely used technique to truncate the half line integral equation to a finite-section Wiener-Hopf equation in solving the integral equation numerically [1, 2, 8, 13]. Thus we instead solve the following approximate integral equation

$$f(x) + 4\int_0^8 e^{|x-y|} f(y)dy = e^{-x}, \quad \text{for } x \geq 0.$$

The errors of the approximate solutions are listed in Table 2.

Example 3. The third example is from [3] and has a singular kernel. The equation is

$$f(x) + \int_0^1 \left\{ \sqrt{\frac{1+y}{1+x}} \ln|x-y| + \frac{1}{x+y+2} \right\} f(y)dy = u(x), \ 0 \leq x \leq 1, \quad (14)$$

where

$$u(x) = \frac{x\ln(x) + (1-x)\ln(1-x) + 2\arctan\left(\sqrt{\frac{2}{x+1}}\right) - 2\arctan\left(\sqrt{\frac{1}{x+1}}\right)}{\sqrt{x+1}}.$$

n	L^1-norm error	L^∞-norm error
4	8.5×10^{-3}	3.9×10^{-2}
8	2.3×10^{-3}	1.0×10^{-2}
16	6.7×10^{-4}	5.7×10^{-3}
32	2.5×10^{-4}	3.3×10^{-3}
64	1.2×10^{-4}	1.8×10^{-3}
128	5.9×10^{-5}	9.4×10^{-4}
256	2.9×10^{-5}	4.8×10^{-4}
512	1.4×10^{-5}	2.5×10^{-4}
1024	7.1×10^{-6}	1.2×10^{-4}

Table 3: Errors for Example 3

The exact solution of equation (14) is

$$f^*(x) = \frac{1}{\sqrt{x+1}}.$$

As the kernel is not defined on the line $y = x$, in the numerical example we use the 3-point Gaussian quadrature in the x direction and the 4-point Gaussian quadrature in the y direction to avoid $x = y$.

The numerical results are shown in Table 3.

Example 4. The last example has a singular kernel. The equation is

$$f(x) - \int_0^\pi \left\{ e^{x-y} \ln|\cos x - \cos y| + \sin(x-y) \right\} f(y)dy = u(x), \ 0 \le x \le \pi,$$

where

$$u(x) = e^x(1 + \pi \ln 2) + \frac{1+e^\pi}{2}(\cos x + \sin x).$$

n	L^1-norm error	L^∞-norm error
4	1.1×10^0	4.1×10^0
8	2.7×10^{-1}	1.7×10^0
16	6.8×10^{-2}	6.4×10^{-1}
32	1.7×10^{-2}	2.3×10^{-1}
64	4.4×10^{-3}	7.6×10^{-2}
128	1.1×10^{-3}	2.4×10^{-2}
256	3.2×10^{-4}	7.1×10^{-3}
512	1.1×10^{-4}	2.0×10^{-3}
1024	4.2×10^{-5}	5.4×10^{-4}

Table 4: Errors for Example 4

The exact solution is

$$f^*(x) = e^x.$$

In [3], the kernel has to be dissociated into several parts to compute the numerical result. Using our method, this example can be computed without additional treating. The numerical results are shown in Table 4.

6 Conclusions

In this study, we presented a piecewise linear maximum entropy method to give approximate nonnegative solutions of the Fredholm integral equation. With more moments that are available, the method can get more accurate approximations. The numerical results show that our method is efficient even for the Fredholm integral equations with singular kernel, which are important in physics. The numerical results in Section 5 coincide with the theoretical analysis of Section 4 in the convergence rate. Furthermore, since the main numerical work is solving a system of nonlinear equations for the Lagrange multipliers without ill-conditions, the algorithm is easy to implement. Finally, our method can also be applied to two or three dimensional problems, which will be studied further in the future.

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