5-2020

Probabilistic Analysis of Revenues in Online Games

Nishchal Sapkota

The University of Southern Mississippi

Probabilistic Analysis of Revenues in Online Games

by

Nishchal Sapkota

A Thesis
Submitted to the Honors College of
The University of Southern Mississippi
in Partial Fulfillment of
the Honors Requirement

May 2020

Approved By

Bernd Schroeder, Ph.D., Thesis Adviser
Professor of Mathematics

Bernd Schroeder, Ph.D., Interim Director
School of Mathematics and Natural Sciences

Ellen Weinauer, Ph.D., Dean
Honors College

Abstract

Online games are captivating and engage users across the world. Some game formats maintain a pseudo-currency to give incentive to the players to play the game in search of rewards as set by the game provider. We model a multi-stage online game and predict how much revenue game providers obtain per game. We compare the revenues generated from different tournament formats to find the one with the maximum per-game revenue for the provider. We have also found the limiting value of the revenue as the game provider increases the number of stages.

Our methods are based on concepts of the transition matrix and its stationary form from Markov Chains. The method is scalable and can be used in complex tournament formats by adjusting the proportion of players making it to the next stage in accordance with the tournament format.

Keywords: Online games, Multi-stage Tournament, Game Provider, Markov Chain, Transition Matrix, Stationary Form, Revenue per Game.

Dedication

I want to dedicate this work to my parents Mr. Dor Pd. Sapkota, Mrs. Rekha Sapkota, and my beloved Ms. Srija Uprety.

Acknowledgement

I am thankful to my advisor and the interim director of the School of Mathematics and Natural Sciences Dr. Bernd Schroeder for guiding me through the entire process of this project. Without him, this work would not have been possible. I would also like to acknowledge the support I have received from Wright W. and Annie Rea Cross Endowed Chair in Mathematics and Undergraduate Research Dr. Zhifu Xie. It was his encouragement with which I had started my undergraduate research journey. The team of cross scholars Ms. Yumi Maharjan, Mr. Gokul Bhusal, Mr. Shiron Manandhar, Mr. Aaditya Kharel, and Mr. Hamas Tahir was also an integral part in the successful completion of this project as they provided me necessary feedbacks during the informal presentations of this project.

My work supervisor and director of the Math Zone Ms. Emileigh McCardle, my roommates Mr. Pujan Paudel, Mr. Shiva Adhikari, Mr. Prashanna Kharel and my dearest friends Mr. Anup Gautam, Mr. Keshav Sapkota, Mr Navodit Paudel, Mr. Sakar Paudel, and Mr. Sanjan Adhikari have played important roles by motivating me throughout my undergraduate career.

I am grateful to the School of Mathematics and Natural Sciences for providing me the Cross Scholarship and Honors College for providing me the Honors Keystone Scholarship to accomplish this project.

Contents

1 Introduction **1**

2 Model **2**

2.1 Transition Between Stages 3

2.2 Transition Matrix For An m Stage Tournament 3

3 Powers of Transition Matrix **5**

3.1 Lemmas on Structures of arbitrary powers of the transition matrix 5

3.2 Theorems on Powers of the Transition Matrix 9

3.2.1 General Formula For Powers Of Transition Matrix 15

4 Stationary Form **16**

4.1 Limits of the first row elements in the n^{th} power Transition Matrix 16

5 Player's Perspective **19**

6 Sample Simulation **20**

6.1 Generalization 20

6.2 Numerical Results 23

7 Conclusion **24**

List of Tables

6.1 Numerical results for revenue for different m 23

Chapter 1

Introduction

Online games are very popular among players of different age groups. Some game providers engage the players in playing for rewards. The rewards can be of different types. They could be bonus elements in the game, special privileges or something similar. The rewards could also be in some form of pseudo-currency maintained by the provider across their game network. A pseudo-currency functions like actual currency within the game. Sometimes, the players have an option to cash their reward out, in the form of real currency. The game provider collects entry fees from players who enter a game and rewards the winners. Since the rewards are only a fraction of the collected amount, the game providers collect revenues in this fashion.

Our work relates to evaluating the revenues collected by game providers in a multi-stage tournament format. The closest resemblance would be an online card game where players enter a tournament paying a certain fee and in each stage compete among the three others in the table to qualify for the next stage and ultimately win the prize money. We compute the revenues per game for different tournament formats and compare them to find the format with maximum per-game revenue for the provider.

Chapter 2

Model

Because the inspiration comes from a card game, we have the following visualization. Players entering the game are grouped in different tables with a fixed number of players per table. The winner from each table advances to the next stage. The first runner up repeats the stage and the rest are eliminated. The stage-winners from the earlier stage are again grouped in different tables of four players each and the winners from the individual tables advance to the next round. The first runner up repeats that stage and the rest are eliminated. After the completion of the allocated number of stages, winners are rewarded. If a player, at any given round, opts out, the table is back-filled with the same number of bots as place holders and the bots are treated similar to any other players.

We assume discrete time steps. That is, in our model players enter games simultaneously and all games at time t terminate before the next set of games starts at time $t+1$. We are representing the number of players at different stages in a vector $[N]$. Let r be the number of allocated stages to be completed by a player to earn a reward. $N_1^{(t)}$, $N_2^{(t)}$, . . . , $N_{r+1}^{(t)}$ are the numbers of players at stages 1, 2, . . . , $r+1$ respectively in time-phase t. Stage $r+1$ is used to collect the players who have won a tournament by successfully completing r stages. The vector is expressed as:

$$[N]^{(t)} = \begin{bmatrix} N_{r+1}^{(t)} \\ N_r^{(t)} \\ \vdots \\ N_2^{(t)} \\ N_1^{(t)} \end{bmatrix}$$

Let w be the proportion of players moving to the next stage and let p be the proportion of players repeating/staying in the same stage. The number of players at stage $1 < s \leq r$ in time-phase (t+1) can be computed as:

$$N_s^{(t+1)} = pN_s^{(t)} + wN_{s-1}^{(t)} \tag{2.1}$$

As noted, $N_{r+1}^{(t+1)}$ represents the total number of winners. Because winners are not retroactively removed, we add the winners from the earlier stage r in time phase t.

$$N_{r+1}^{(t+1)} = N_{r+1}^{(t)} + wN_r^{(t)} \tag{2.2}$$

And, finally, we consider the players who repeat stage 1:

$$N_1^{(t+1)} = pN_1^t \tag{2.3}$$

2.1 Transition Between Stages

Our work is loosely based on Markov Chains [2] and differs on the grounds that the sums of the column elements of our transition matrix (see below) do not necessarily equal 1. In our model, a transition matrix $[T]$ for a tournament with r stages is a unique matrix, that when multiplied with the vector of the numbers of players $[N]^{(t)}$ at stage s at time-phase t, gives the number of players at each stage s in time phase $t+1$.

$$[N]^{(t+1)} = [T] \times [N]^{(t)} \tag{2.4}$$

2.2 Transition Matrix For An m Stage Tournament

For the given values of w and p, the transition matrix for a tournament with r stages is an upper triangle banded $(r+1) \times (r+1)$ matrix. To satisfy equation (2.2), the first diagonal entry of the transition matrix in always 1 and every other diagonal entry is p. Elements below the diagonal are 0. In each column a w occurs before every p and all other remaining elements in the column are 0.

$$[T] = \begin{bmatrix} 1 & w & 0 & \cdots & \cdots & 0 \\ 0 & p & w & \ddots & & \vdots \\ \vdots & \ddots & p & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & p & w \\ 0 & \cdots & \cdots & \cdots & 0 & p \end{bmatrix}$$

The transition matrix $[T]$ when multiplied with $[N]^{(t)}$, yields $[N]^{(t+1)}$, as seen from equations (2.1),(2.2),(2.3). This is true because at initial state, the numbers of players entering the games are the only players in the game and they are all at stage 1. This means, only the last element in $[N]^{(0)}$ has non zero identity. In every iteration, if we multiply $[T]$ with $[N]^{(t)}$, we get $[N]^{(t+1)}$.

As the iteration continues, the vector $[N]$ attains non zero values starting from the bottom one at a time until the number of allocated stages is completed. As the tournament progresses, the entries in lower components of $[N]$ start converging to 0 because p and w make the entries smaller in every iteration. At the end, all but the top element in $[N]$ are zeros. This top element is retained by the first element of $[T]$ and it is the number of winners to be rewarded.

From equation (2.2):

$$
\begin{aligned}
&[N]^{(t+1)} = [T] \times [N]^{(t)} \\
\Rightarrow &[N]^{(t+1)} = [T] \times ([T] \times [N]^{(t-1)}) \\
\Rightarrow &[N]^{(t+1)} = [T]^2 \times [N]^{(t-1)} \\
\Rightarrow &[N]^{(t+1)} = [T]^2 \times ([T] \times [N]^{(t-2)}) \\
\Rightarrow &[N]^{(t+1)} = [T]^3 \times [N]^{(t-2)} \\
&\vdots \\
&[N]^{(t+1)} = [T]^n \times [N]^{(t-(n-1))} \qquad (2.5)
\end{aligned}
$$

From equation (2.5), it is seen that the powers of the transition matrix can be used to compute the number of players in any given stage, provided the numbers of players in each stage at an earlier time phase is known.

Chapter 3

Powers of Transition Matrix

To compute arbitrary powers of the transition matrix, we start with a number of lemmas.

3.1 Lemmas on Structures of arbitrary powers of the transition matrix

Lemma 3.1 ***Structure of the lower right block***
Let A, B be $n \times n$ matrices and let stars $*$, $**$, $***$ denote unknown entries. Then

$$\left[\begin{array}{c|c} 1 & * \\ \hline 0 & \\ \vdots & \\ \vdots & A \\ \vdots & \\ 0 & \end{array}\right]\left[\begin{array}{c|c} 1 & ** \\ \hline 0 & \\ \vdots & \\ \vdots & B \\ \vdots & \\ 0 & \end{array}\right]=\left[\begin{array}{c|c} 1 & *** \\ \hline 0 & \\ \vdots & \\ \vdots & AB \\ \vdots & \\ 0 & \end{array}\right].$$

Proof:
Let

$$\widehat{A} := \left[\begin{array}{c|c} 1 & * \\ \hline 0 & \\ \vdots & \\ \vdots & A \\ \vdots & \\ 0 & \end{array}\right] \quad \text{and let} \quad \widehat{B} := \left[\begin{array}{c|c} 1 & ** \\ \hline 0 & \\ \vdots & \\ \vdots & B \\ \vdots & \\ 0 & \end{array}\right].$$

The first row of $\widehat{A}$ is multiplied with the first column of $\widehat{B}$ to get the $(1,1)$-element of $\widehat{A}\widehat{B}$: By the rule for matrix multiplication, the first item of the first row of $\widehat{A}$ is multiplied with the first item of the first column of $\widehat{B}$, the second item from the first row is multiplied with the second item from the first column, and so on. The sum of these products is the $(1,1)$-term of $\widehat{A}\widehat{B}$. Because all the non-zero entries $(*)$ in the first row are multiplied with the zeros from the first column, the only non-zero product is 1. Thus, the total sum is just 1. Hence, the $(1,1)$-element of the matrix $\widehat{A}\widehat{B}$ is 1.

Similarly, the second row of $\widehat{A}$ times the first column of $\widehat{B}$ results in the second element in the first column of $\widehat{A}\widehat{B}$. This term is going to be zero because the zero from the first term in the row of $\widehat{A}$ is multiplied with the 1 from the first term in the first column of $\widehat{B}$ and all non zero entries from the same row are multiplied with the zeros from the first column. The total sum is therefore 0. Hence the $(2,1)$-entry of $\widehat{A}\widehat{B}$ is zero.

The same argument proves that the other elements up to the $(n+1)^{th}$ element in the first column of $\widehat{A}\widehat{B}$ are zero. Visually,

$$\left[\begin{array}{c|c} 1 & * \\ \hline 0 & \\ \vdots & \\ \vdots & A \\ \vdots & \\ 0 & \end{array}\right] \left[\begin{array}{c} 1 \\ \hline 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{array}\right] = \left[\begin{array}{c} 1 \\ \hline 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{array}\right].$$

Now let us consider the block of $\widehat{A}$ without the first row and the block of $\widehat{B}$ without the first column. The multiplication of this block results in the block of $\widehat{AB}$ without the first row and first column. The zero entry in each row of this block of $\widehat{A}$ is multiplied with the first unidentified non-zero entry in each column from this block of $\widehat{B}$ resulting in zero terms that are added to the product of A and B. This sum gives the AB elements of the block of $\widehat{AB}$. Visually,

$$\left[\begin{array}{c|c} 0 & \\ \vdots & \\ \vdots & A \\ \vdots & \\ 0 & \end{array}\right] \left[\begin{array}{c} ** \\ \hline \\ B \\ \\ \end{array}\right] = \left[\begin{array}{c} \\ AB \\ \\ \end{array}\right].$$

No argument is needed for the remaining entries of the first row of $\widehat{A}\widehat{B}$, as these are considered unknown, as of now. ■

Lemma 3.2 ***Structure of the upper left block***
Let A, B be $n \times n$ matrices and let stars $*$, $**$, $***$ denote unknown entries. Then

$$\left[\begin{array}{cccc|c} & & & & \\ & A & & & * \\ & & & & \\ \hline 0 & .. & .. & .. \; 0 & a \end{array}\right] \left[\begin{array}{cccc|c} & & & & * \\ & B & & & \\ & & & & * \\ \hline 0 & .. & .. & .. \; 0 & b \end{array}\right] = \left[\begin{array}{cccc|c} & & & & * \\ & AB & & & * \\ & & & & * \\ \hline 0 & .. & .. & .. \; 0 & ab \end{array}\right].$$

Proof:
Let

$$\widehat{A} := \left[\begin{array}{ccccc|c} & & & & & \\ & & A & & & * \\ & & & & & \\ \hline 0 & .. & .. & .. & 0 & a \end{array}\right] \quad \text{and let} \quad \widehat{B} := \left[\begin{array}{ccccc|c} & & & & & * \\ & & B & & & * \\ & & & & & \\ \hline 0 & .. & .. & .. & 0 & b \end{array}\right].$$

The last row of $\widehat{A}$ is multiplied with the last column of $\widehat{B}$ to get the (n,n)-element of $\widehat{A}\widehat{B}$: By the rule for matrix multiplication, the first item of the last row of $\widehat{A}$ is multiplied with the first item of the last column of $\widehat{B}$, the second item from the last row is multiplied with the second item from the last column, and so on. The sum of these products is the $(n+1,n+1)$-term of $\widehat{A}\widehat{B}$. Because all the zero entries in the last row are multiplied with the non-zero entries $(**)$ from the last column, the only non-zero product is ab. Thus, the total sum is ab. Hence, the $(n+1,n+1)$-element of the matrix $\widehat{A}\widehat{B}$ is ab. No argument is needed for the remaining entries of the last column of $\widehat{A}\widehat{B}$, as these are considered unknown.

$$\left[\begin{array}{ccccc|c} & & & & & \\ & & A & & & * \\ & & & & & \\ \hline 0 & .. & .. & .. & 0 & a \end{array}\right] \left[\begin{array}{c} * \\ \\ * \\ \hline b \end{array}\right] = \left[\begin{array}{c} * \\ * \\ * \\ \hline ab \end{array}\right].$$

Now, let us consider the last row of $\widehat{A}$ and the block of $\widehat{B}$ without the last column. The multiplication of these two blocks results in the last row of $\widehat{A}\widehat{B}$. All the zero entries from the last row of $\widehat{A}$ are multiplied with the non zero entries from the block of $\widehat{B}$ and get added to the product of the nonzero entry (a) from the last row of $\widehat{A}$ and the zero from the last row of the block $\widehat{B}$. Thus, the total sum is 0. Therefore all the entries in this last row of $\widehat{A}\widehat{B}$ are 0.Visually,

$$\left[\begin{array}{ccccc|c} 0 & .. & .. & .. & 0 & a \end{array}\right] \left[\begin{array}{ccccc} & & & & \\ & & B & & \\ & & & & \\ \hline 0 & .. & .. & .. & 0 \end{array}\right] = \left[\begin{array}{ccccc} 0 & .. & .. & .. & 0 \end{array}\right].$$

Now let us consider the block of $\widehat{A}$ without the last row and the block of $\widehat{B}$ without the last column. The multiplication of these blocks results in the block of $\widehat{A}\widehat{B}$ without the last row and last column. The entries from the rows of A are multiplied with entries from the columns of B and the total sum results in the elements of AB. Since the unknown entries in the last column of $\widehat{A}$ are multiplied with the zeros in each last row of the block of $\widehat{B}$, the total sum is AB.

$$\left[\begin{array}{c|c} \quad A \quad & * \end{array}\right] \left[\begin{array}{cccc} & B & & \\ \hline 0 & .. & .. \; .. & 0 \end{array}\right] = \left[\begin{array}{c} \quad AB \quad \end{array}\right].$$

■

Lemma 3.3 ***Banded Structures***

$$\text{Let} \quad \widehat{A} := \begin{bmatrix} a_1 & a_2 & \cdots & a_m \\ 0 & a_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_2 \\ 0 & \cdots & 0 & a_1 \end{bmatrix} \quad \text{and let} \quad \widehat{B} := \begin{bmatrix} b_1 & b_2 & \cdots & b_m \\ 0 & b_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_2 \\ 0 & \cdots & 0 & b_1 \end{bmatrix}$$

*be m*m upper triangle banded matrices with bandwidth $k = m-1$. Then, there are numbers $c_1, c_2, \ldots, c_m$ such that the product AB equals*

$$\begin{bmatrix} c_1 & c_2 & \cdots & c_m \\ 0 & c_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_2 \\ 0 & \cdots & 0 & c_1 \end{bmatrix}$$

which is also an upper triangle banded matrix.

Proof:

The product of A and B is given by:

$$AB = \begin{bmatrix} a_1 & a_2 & \cdots & a_m \\ 0 & a_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_2 \\ 0 & \cdots & 0 & a_1 \end{bmatrix} \times \begin{bmatrix} b_1 & b_2 & \cdots & b_m \\ 0 & b_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_2 \\ 0 & \cdots & 0 & b_1 \end{bmatrix}$$

by the rules of matrix multiplication,

$$= \begin{bmatrix} a_1b_1 & a_1b_2 + a_2b_1 & \cdots & a_1b_{s+1} + a_2b_s + \cdots + a_{s+1}b_1 \\ 0 & a_1b_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_1b_2 + a_2b_1 \\ 0 & \cdots & 0 & a_1b_1 \end{bmatrix}$$

because all of the nonzero entries on each diagonal are the same,

$$= \begin{bmatrix} c_1 & c_2 & \cdots & c_m \\ 0 & c_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_2 \\ 0 & \cdots & 0 & c_1 \end{bmatrix}$$

Thus, AB is also an upper triangle banded matrix. ■

Lemma 3.4 ***Generic Entry in first row of n^{th} power of Transition Matrix***

$$w \sum_{i=1}^{k-((c-1)-2)} \binom{i+c-4}{(c-1)-2} p^{i-1} w^{(c-1)-1} + p \sum_{i=1}^{k-(c-2)} \binom{i+c-3}{c-2} p^{i-1} w^{c-1}$$
$$= \sum_{j=1}^{(k+1)-(c-2)} \binom{(j-1)+(c-2)}{c-2} p^{j-1} w^{c-1}$$

Proof:
The statement is trivial for $k < c-2$ (all sums are zero) and it can be verified by inspection for $k = c-2$ (LHS and RHS are equal to w^{c-1}. Hence, for the remainder, we can assume that $k > c_2$.)

$$w \sum_{i=1}^{k-((c-1)-2)} \binom{i+c-4}{(c-1)-2} p^{i-1} w^{(c-1)-1} + p \sum_{i=1}^{k-(c-2)} \binom{i+c-3}{c-2} p^{i-1} w^{c-1}$$

let j=i and substitute in first sum,

let j = i + 1 then i = j - 1 and substitute in second sum,

$$= \sum_{j=1}^{k-(c-3)} \binom{j+c-4}{(c-2)-1} p^{j-1} w^{c-1} + \sum_{j=2}^{k-(c-3)} \binom{j+c-4}{c-2} p^{j-1} w^{c-1}$$

$$= \binom{1+c-4}{c-3} p^{0} w^{c-1} + \sum_{j=2}^{k-c+3} \binom{j+c-4}{c-3} p^{j-1} w^{c-1} + \sum_{j=2}^{k-c+3} \binom{j+c-4}{c-2} p^{j-1} w^{c-1}$$

$$= 1 \times p^{0} w^{c-1} + \sum_{j=2}^{k-c+3} \left(\binom{j+c-4}{(c-2)-1} + \binom{j+c-4}{c-2} \right) p^{j-1} w^{c-1}$$

We know: $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$

thus on further simplification, we get:

$$= 1 \times p^{0} w^{c-1} + \sum_{j=2}^{k-c+3} \binom{(j+c-4)+1}{c-2} p^{j-1} w^{c-1}$$

$$= \binom{c-2}{c-2} p^{0} w^{c-1} + \sum_{j=2}^{k-c+3} \binom{(j-1)+(c-2)}{c-2} p^{j-1} w^{c-1}$$

$$= \binom{(1-1)+(c-2)}{c-2} p^{1-1} w^{c-1} + \sum_{j=2}^{k-(c-3)} \binom{(j-1)+(c-2)}{c-2} p^{j-1} w^{c-1}$$

$$= \sum_{j=1}^{(k+1)-(c-2)} \binom{(j-1)+(c-2)}{c-2} p^{j-1} w^{c-1}$$

■

3.2 Theorems on Powers of the Transition Matrix

Theorem 3.5 ***First row of higher powers of transition matrix***
Let $[T]$ be the $m \times m$ transition matrix and let n be the power under consideration. The

elements in the first row of this matrix $[T]^n$ *are*

$$\begin{bmatrix}1 & \sum_{i=1}^{n-(2-2)} \binom{i-1+(2-2)}{2-2} p^{i-1} w^{2-1} & \cdots & \sum_{i=1}^{n-(m-2)} \binom{i-1+(m-2)}{m-2} p^{i-1} w^{m-1}\end{bmatrix}$$

That is, the element in arbitrarily chosen column (c) is,

$$[T]^n_{(1,c)} = \begin{cases} \sum_{i=1}^{n-(c-2)} \binom{i-1+(c-2)}{c-2} p^{i-1} w^{c-1}; & \text{for } n > (c-2), \\ 1; & \text{for } n \leq (c-2). \end{cases}$$

Proof:

Let $p(n)$ be the statement to be proved. *Base Case* $(n = 2)$

L.H.S:

$$[T]^2_{(1,c)} = [T]_{(1,c)} \times [T]$$

$$= \begin{bmatrix}1 & w & 0 & \cdots & \cdots & 0\end{bmatrix} \times \begin{bmatrix}1 & w & 0 & \cdots & 0 \\ 0 & p & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & w \\ 0 & \cdots & \cdots & 0 & p\end{bmatrix}$$

applying the rules of matrix multiplication,

$$= \begin{bmatrix}1 \times 1 & 1 \times w + w \times p & w \times w & 0 & \cdots & 0\end{bmatrix}$$

$$= \begin{bmatrix}1 & w(1+p) & w^2 & 0 & \cdots & 0\end{bmatrix}$$

R.H.S:

$$\begin{bmatrix}1 & \sum_{i=1}^{2-(2-2)} \binom{i-1+(2-2)}{2-2} p^{i-1} w^{2-1} & \cdots & \sum_{i=1}^{2-(m-2)} \binom{i-1+(m-2)}{m-2} p^{i-1} w^{m-1}\end{bmatrix}$$

simplifying,

$$= \begin{bmatrix}1 & \sum_{i=1}^{2} \binom{i-1}{0} p^{i-1} w^1 & \sum_{i=1}^{1} \binom{i}{1} p^{i-1} w^2 \cdots & \sum_{i=1}^{4-m} \binom{i+m-3}{m-2} p^{i-1} w^{m-1}\end{bmatrix}$$

expanding the terms,

$$= \begin{bmatrix}1 & w(1+p) & w^2 & 0 & \cdots & 0\end{bmatrix}$$

Since L.H.S = R.H.S, $P(2)$ is true.

Inductive Hypothesis $(n = k)$

Assume, $P(k)$ is true, i.e:

$$[T]^k_{(1,c)} = \begin{bmatrix}1 & \sum_{i=1}^{k-(2-2)} \binom{i-1+(2-2)}{2-2} p^{i-1} w^{2-1} & \cdots & \sum_{i=1}^{k-(m-2)} \binom{i-1+(m-2)}{m-2} p^{i-1} w^{m-1}\end{bmatrix}$$

Now, for $P(k+1)$, L.H.S:

$$[T]^{k+1}_{(1,c)}$$
$$= [T]^{k}_{(1,c)} \times [T]$$

from inductive hypothesis,

$$= \begin{bmatrix} 1 & \sum_{i=1}^{k-(2-2)} \binom{i-1+(2-2)}{2-2} p^{i-1} w^{2-1} & \cdots & \sum_{i=1}^{k-(m-2)} \binom{i-1+(m-2)}{m-2} p^{i-1} w^{m-1} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & w & 0 & \cdots & 0 \\ 0 & p & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & w \\ 0 & \cdots & \cdots & 0 & p \end{bmatrix}$$

applying the rules of matrix multiplication and simplifying

$$= \Bigg[1 \times 1 \qquad 1 \times w + \sum_{i=1}^{k} \binom{i-1}{0} p^i w \qquad \cdots$$

$$\cdots \sum_{i=1}^{k-(m-3)} \binom{i+m-4}{m-3} p^{i-1} w^{m-2} \times w + \sum_{i=1}^{k-(m-2)} \binom{i+m-3}{m-2} p^{i-1} w^{m-1} \times p \Bigg]$$

here, c^{th} *entry on LHS is the LHS of the lemma 3.4*

$$w \sum_{i=1}^{k-((c-1)-2)} \binom{i+c-4}{(c-1)-2} p^{i-1} w^{(c-1)-1} + p \sum_{i=1}^{k-(c-2)} \binom{i+c-3}{c-2} p^{i-1} w^{c-1}$$

thus, using lemma 3.4 to simplify, we get,

$$\sum_{i=1}^{k-c+3} \binom{(i-1)+(c-2)}{c-2} p^{i-1} w^{c-1}$$

R.H.S:

$$= \begin{bmatrix} 1 & \sum_{i=1}^{k+1-(2-2)} \binom{i-1+(2-2)}{2-2} p^{i-1} w^{2-1} & \cdots & \sum_{i=1}^{k+1-(m-2)} \binom{i-1+(m-2)}{m-2} p^{i-1} w^{m-1} \end{bmatrix}$$

simplifying,

$$= \begin{bmatrix} 1 & \sum_{i=1}^{k+1} \binom{i-1}{0} p^{i-1} w^1 & \sum_{i=1}^{k} \binom{i}{1} p^{i-1} w^2 & \cdots & \sum_{i=1}^{k+1-(m-2)} \binom{i-1+(m-2)}{m-2} p^{i-1} w^{m-1} \end{bmatrix}$$

so the generic term is,

$$\sum_{i=1}^{k-c+3} \binom{(i-1)+(c-2)}{c-2} p^{i-1} w^{c-1}$$

Since L.H.S = R.H.S, $P(k+1)$ is true.
Thus, $P(n)$ is true. ∎

Lemma 3.6

$$w \binom{k}{c-2} p^{k-(c-2)} w^{c-2} + p \binom{k}{c-1} p^{k-(c-1)} w^{c-1} = \binom{k+1}{c-1} p^{k-(c-2)} w^{c-1}$$

Proof:

$$
\begin{aligned}
& w\binom{k}{c-2}p^{k-(c-2)}w^{c-2} + p\binom{k}{c-1}p^{k-(c-1)}w^{c-1} \\
&= \binom{k}{c-2}p^{k-(c-2)}w^{c-1} + \binom{k}{c-1}p^{k-(c-2)}w^{c-1} \\
&= p^{k-(c-2)}w^{c-1}\left(\binom{k}{c-2} + \binom{k}{c-1}\right) \\
& \qquad \textit{We know: } \binom{a}{b-1} + \binom{a}{b} = \binom{a+1}{b}, \\
& \textit{thus on further simplification, we get:} \\
&= \binom{k+1}{c-1}p^{k-(c-2)}w^{c-1}
\end{aligned}
$$

■

Theorem 3.7 ***Lower right block of higher powers of transition matrix***
The n^{th} power of the $m * m$ matrix of the form

$$
\begin{bmatrix}
p & w & 0 & \cdots & 0 \\
0 & p & w & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
\vdots & & \ddots & \ddots & w \\
0 & \cdots & \cdots & 0 & p
\end{bmatrix}
$$

is given by

$$
\begin{bmatrix}
p^n & \binom{n}{1}p^{n-1}w^1 & \binom{n}{2}p^{n-2}w^2 & \cdots & \binom{n}{m-1}p^{n-(m-1)}w^{m-1} \\
0 & p^n & \binom{n}{1}p^{n-1}w^1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \binom{n}{2}p^{n-2}w^2 \\
\vdots & & \ddots & \ddots & \binom{n}{1}p^{n-1}w^1 \\
0 & \cdots & \cdots & 0 & p^n
\end{bmatrix}
$$

where $n > m$ and we define $\binom{n}{k} := 0$ for $n < k$.

Proof:

Let $P(n)$ be the statement to be proved.

Now, for $P(1)$,

$$L.H.S = \begin{bmatrix} p & w & 0 & \cdots & 0 \\ 0 & p & w & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & w \\ 0 & \cdots & \cdots & 0 & p \end{bmatrix}$$

Similarly,

$$R.H.S = \begin{bmatrix} p^1 & \binom{1}{1}p^{1-1}w^1 & \binom{n}{2}p^{1-2}w^2 & \cdots & \binom{1}{m-1}p^{1-(m-1)}w^{m-1} \\ 0 & p^1 & \binom{1}{1}p^{1-1}w^1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \binom{1}{2}p^{1-2}w^2 \\ \vdots & & \ddots & \ddots & \binom{1}{1}p^{1-1}w^1 \\ 0 & \cdots & \cdots & 0 & p^1 \end{bmatrix}$$

since $\binom{n}{k} := 0$ *for* $n < k$, *on simplification we get,*

$$= \begin{bmatrix} p & w & 0 & \cdots & 0 \\ 0 & p & w & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & w \\ 0 & \cdots & \cdots & 0 & p \end{bmatrix}$$

Here, L.H.S = R.H.S, thus $P(1)$ is true.

Inductive Hypothesis:
Let us assume that $P(k)$ is true i.e.

$$\begin{bmatrix} p & w & 0 & \cdots & 0 \\ 0 & p & w & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & w \\ 0 & \cdots & \cdots & 0 & p \end{bmatrix}^k$$

$$= \begin{bmatrix} p^k & \binom{k}{1}p^{k-1}w^1 & \binom{k}{2}p^{k-2}w^2 & \cdots & \binom{k}{m-1}p^{k-(m-1)}w^{m-1} \\ 0 & p^k & \binom{k}{1}p^{k-1}w^1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \binom{k}{2}p^{k-2}w^2 \\ \vdots & & \ddots & \ddots & \binom{k}{1}p^{k-1}w^1 \\ 0 & \cdots & \cdots & 0 & p^k \end{bmatrix}$$

Thus, for $P(k+1)$,

L.H.S:

$$\Rightarrow \begin{bmatrix} p & w & 0 & \cdots & 0 \\ 0 & p & w & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & w \\ 0 & \cdots & \cdots & 0 & p \end{bmatrix}^{k+1}$$

$$= \begin{bmatrix} p & w & 0 & \cdots & 0 \\ 0 & p & w & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & w \\ 0 & \cdots & \cdots & 0 & p \end{bmatrix}^{k} \cdot \begin{bmatrix} p & w & 0 & \cdots & 0 \\ 0 & p & w & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & w \\ 0 & \cdots & \cdots & 0 & p \end{bmatrix}$$

from inductive hypothesis,

$$= \begin{bmatrix} p^k & \binom{k}{1}p^{k-1}w^1 & \binom{k}{2}p^{k-2}w^2 & \cdots & \binom{k}{m-1}p^{k-(m-1)}w^{m-1} \\ 0 & p^k & \binom{k}{1}p^{k-1}w^1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \binom{k}{2}p^{k-2}w^2 \\ \vdots & & \ddots & \ddots & \binom{k}{1}p^{k-1}w^1 \\ 0 & \cdots & \cdots & 0 & p^k \end{bmatrix}$$

$$\cdot \begin{bmatrix} p & w & 0 & \cdots & 0 \\ 0 & p & w & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & w \\ 0 & \cdots & \cdots & 0 & p \end{bmatrix}$$

from the rules of matrix multiplication, entries in main diagonal are the product of p^k and p, entries below the main diagonal are 0s, and entries above the main diagonal are, with c being the number of the "off-diagonal" and $c = 1$ being the diagonal, obtained from:

$$w\binom{k}{c-2}p^{k-(c-2)}w^{c-2} + p\binom{k}{c-1}p^{k-(c-1)}w^{c-1}$$

which on simplification using lemma 3.6 gives:

$$\binom{k+1}{c-1}p^{k-(c-2)}w^{c-1}$$

banded structures are preserved so, filling in all the entries, we get:

$$= \begin{bmatrix} p^{k+1} & \binom{k+1}{1}p^{k+1-1}w^1 & \binom{k+1}{2}p^{k+1-2}w^2 & \cdots & \binom{k+1}{m-1}p^{k+1-(m-1)}w^{m-1} \\ 0 & p^{k+1} & \binom{k+1}{1}p^{k+1-1}w^1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \binom{k+1}{2}p^{k+1-2}w^2 \\ \vdots & & \ddots & \ddots & \binom{k+1}{1}p^{k+1-1}w^1 \\ 0 & \cdots & \cdots & 0 & p^{k+1} \end{bmatrix}$$

Similarly,

$$R.H.S = \begin{bmatrix} p^{k+1} & \binom{k+1}{1}p^{k+1-1}w^1 & \binom{k+1}{2}p^{k+1-2}w^2 & \cdots & \binom{k+1}{m-1}p^{k+1-(m-1)}w^{m-1} \\ 0 & p^{k+1} & \binom{k+1}{1}p^{k+1-1}w^1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \binom{k+1}{2}p^{k+1-2}w^2 \\ \vdots & & \ddots & \ddots & \binom{k+1}{1}p^{k+1-1}w^1 \\ 0 & \cdots & \cdots & 0 & p^{k+1} \end{bmatrix}$$

Here, L.H.S = R.H.S, thus $P(k+1)$ is also true.
Hence, $P(n)$ is true. ■

3.2.1 General Formula For Powers Of Transition Matrix

We can now combine the above two theorems, to obtain the n^{th} power of a transition matrix.

$$T^n : \begin{bmatrix} 1 & w & 0 & \cdots & \cdots & 0 \\ 0 & p & w & \ddots & & \vdots \\ \vdots & \ddots & p & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & p & w \\ 0 & \cdots & \cdots & \cdots & 0 & p \end{bmatrix}^n$$

is given by:

$$\begin{bmatrix} 1 & \sum\limits_{i=1}^{n-(2-2)} \binom{i-1+(2-2)}{2-2}p^{i-1}w^{2-1} & \cdots & \cdots & \sum\limits_{i=1}^{n-(m-2)} \binom{i-1+(m-2)}{m-2}p^{i-1}w^{m-1} \\ 0 & p^n & \binom{n}{1}p^{n-1}w^1 & \cdots & \binom{n}{m-1}p^{n-(m-1)}w^{m-1} \\ \vdots & \ddots & p^n & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \binom{n}{1}p^{n-1}w^1 \\ 0 & \cdots & \cdots & \cdots & p^n \end{bmatrix}$$

Chapter 4

Stationary Form

As $n \to \infty$, T^n attains a constant form, we call this the stationary form [1]. The first row attains constant values $c_i, i \in \{1, 2, ...r\}$ and all other entries ultimately approach 0. This matrix is the stationary transition matrix.

$$\lim_{n\to\infty} [T]^n = \begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_r \\ 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & 0 \\ \vdots & & & & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

4.1 Limits of the first row elements in the n^{th} power Transition Matrix

Lemma 4.1

$$(-1)^k \binom{k+\alpha-1}{\alpha-1} = \binom{-\alpha}{k}$$

where $\alpha \in \mathbb{Z}$ *and* $k \in \mathbb{N}$

Proof:

Case 1: when $\alpha \leq k$

$$\begin{aligned} &(-1)^k \binom{k+\alpha-1}{\alpha-1} \\ &= (-1)^k \frac{(k+\alpha-1)\times(k+\alpha-2)\times ...\times(k+1)}{(\alpha-1)!} \\ &= (-1)^k \frac{(k+\alpha-1)\times(k+\alpha-2)\times ...\times(k+1)}{(\alpha-1)!} \times \frac{k\times(k-1)\times ...\times\alpha}{k\times(k-1)\times ...\times\alpha} \\ &= \frac{(-(k+\alpha-1))\times(-(k+\alpha-2))\times ...\times(-(\alpha+1))\times(-\alpha)}{k!} \\ &= \frac{(-\alpha)\times(-\alpha-1))\times ...\times(-\alpha-(k-2))\times(-\alpha-(k-1))}{k!} \\ &= \binom{-\alpha}{k} \end{aligned}$$

Case 2: when $\alpha > k$

$$(-1)^k \binom{k+\alpha-1}{\alpha-1}$$
$$= (-1)^k \frac{(k+\alpha-1)\times(k+\alpha-2)\times ... \times(\alpha+1)\times\alpha\times(\alpha-1)\times ...\times(k+1)}{(\alpha-1)\times ...\times(k+1)\times k!}$$
$$= (-1)^k \frac{(k+\alpha-1)\times(k+\alpha-2)\times ... \times(\alpha+1)\times\alpha}{k!}$$
$$= \frac{(-\alpha)\times(-\alpha-1))\times ... \times(-\alpha-(k-2))\times(-\alpha-(k-1))}{k!}$$
$$= \binom{-\alpha}{k}$$

■

Theorem 4.2 *The limit as $n \to \infty$ of the terms in the first row at position $(1,c)$ of the n^{th} power of the transition matrix is:*

$$\lim_{n\to\infty} \sum_{i=1}^{n-(c-2)} \binom{i-1+(c-2)}{c-2} p^{i-1} w^{c-1} = \left(\frac{w}{1-p}\right)^{c-1}$$

where c is the column index of the elements in the first row of the n^{th} power of the transition matrix.

Proof:

From theorem 3.4.2, the c^{th} element in the first row of the n^{th} power of the transition matrix is given by:

$$[T]^n_{(1,c)} = \begin{cases} \sum_{i=1}^{n-(c-2)} \binom{i-1+(c-2)}{c-2} p^{i-1} w^{c-1}; & \text{for } n > (c-2), \\ 1; & \text{for } n \leq (c-2). \end{cases}$$

Considering the case that $n > (c-2)$, and applying the limit,

$$\lim_{n\to\infty} \sum_{i=1}^{n-(c-2)} \binom{i-1+(c-2)}{c-2} p^{i-1} w^{c-1}$$
$$= w^{c-1} \lim_{n\to\infty} \sum_{i-1=0}^{n-(c-2)-1} \binom{i-1+(c-1)-1}{(c-1)-1} p^{i-1}$$

let, $(i-1) \to k, (c-1) \to \alpha$

$$= w^{\alpha} \lim_{n\to\infty} \sum_{k=0}^{n-\alpha} \binom{k+\alpha-1}{\alpha-1} p^k$$

multiplying the terms by $(-1)^k/(-1)^k$ and simplifying,

$$= w^{\alpha} \lim_{n\to\infty} \sum_{k=0}^{n-\alpha} (-1)^k \binom{k+(\alpha-1)}{\alpha-1} p^k (-1)^{-k}$$

from lemma 4.1,

$$= w^{\alpha} \lim_{n\to\infty} \sum_{k=0}^{n-\alpha} \binom{-\alpha}{k} (-p)^k$$

Using the negative binomial series,

$$
\begin{aligned}
w^{\alpha} \lim_{n\to\infty} \sum_{k=0}^{n-(\alpha-1)} \binom{-\alpha}{k}(-p)^k &= w^{\alpha} \sum_{k=0}^{\infty} \binom{-\alpha}{k}(-p)^k \\
&= w^{\alpha}(1+(-p))^{-\alpha} \\
&= \left(\frac{w}{1-p}\right)^{\alpha} \\
&= \left(\frac{w}{1-p}\right)^{c-1}
\end{aligned} \tag{4.1}
$$

■

As the game progresses, the number of players decreases in every stage other than the "winner stage" $r+1$, because since players from the tables are always eliminated if they do not finish first or second in a particular game. For larger time value, we can expect that players in each stage are either eliminated from the game or have moved to the next stage so much so that the only ones left are the winners collected in the last possible stage. Hence, the limit of powers of the transition matrix is sufficient to help us compute the revenues by giving us the proportion and hence the number of players to be rewarded.

Chapter 5

Player's Perspective

It is also possible to analyze the system from the player's perspective. Keeping our notations consistent, let w be the probability of a player to win the game in a stage and p be the probability to repeat the stage. Assuming p_{ne}^{k} is the probability of winning the stage after not being eliminated by game k, the probability of winning the first stage is given as:

$$\begin{aligned}
&w + pp_{ne} \\
&= w + p(w + pp_{ne}^{2}) \\
&= w + wp + p^{2}p_{ne}^{2} \\
&= w + wp + p^{2}(w + pp_{ne}^{3}) \\
&= w + wp + wp^{2} + p^{3}p_{ne}^{3} \\
&\textit{Continuing this further, we get:} \\
&= w + wp + wp^{2} + p^{3}w + wp^{4} + ... + p^{i+1}p_{ne}^{i+1}
\end{aligned}$$

We observe that,

$$p_{ne}^{k} = p_{ne}^{k+1} = p_{ne}^{k+2} = ... = p_{ne}^{k+i} = p_{ne}$$

Thus,

$$\begin{aligned}
w + pp_{ne} &= \sum_{k=0}^{\infty} p^{k}w \\
&= w\sum_{k=0}^{\infty} p^{k} \\
&= \frac{w}{1-p}
\end{aligned}$$

Because the tournament has r stages and the probability of winning the next stage is independent of that of winning the previous, the probability of consecutively winning r stages is:

$$\left(\frac{w}{1-p}\right)^{r} \tag{5.1}$$

■

As a matter of fact, the (1,c) entry from the first row of the stationary transition matrix (result 4.1) is also the probability of a player winning $c-1$ stages consecutively. Hence, the probability of player winning r stages (result 5.1) is the $(1, r+1)$ entry from the first row of the stationary transition matrix (theorem 4.2)

Chapter 6

Sample Simulation

Let us demonstrate how we can use our work so far in estimating the revenues of a game provider with the similar tournament format as our model considers.

6.1 Generalization

For w=1/4 and p=1/4 i.e $(1/4)^{th}$ of the players in the table advances the next stage and $(1/4)^{th}$ of the players repeat the stage, let us compute the revenues. Consider an m-staged-tournament format for an online game. The transition matrix $[T]$ for this format will be an $(m+1) \times (m+1)$ matrix of the form:

$$[T] = \begin{bmatrix} 1 & 1/4 & 0 & \cdots & 0 \\ 0 & 1/4 & 1/4 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1/4 \\ 0 & \cdots & \cdots & 0 & 1/4 \end{bmatrix}$$

Computing $\lim_{n\to\infty}[T]^n$ using the general formula for powers of transition matrix and the stationary form, we get.

$$\lim_{n\to\infty}[T]^n = \begin{bmatrix} 1 & 1/3 & 1/9 & \cdots & 1/3^m \\ 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & 0 & & \vdots \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

The entries in the top row of this matrix (left to right) give the proportion of players in the current stage advancing to the final stage. The last element of the top row is the proportion of winners overall. If we have N players at the start of the tournament, the numbers of players in each of the stages are given as:

- # Players reaching stage 2 = $\frac{1}{3} \times N$
- # Players reaching stage 3 = $\frac{1}{9} \times N$
- # Players reaching stage 4 = $\frac{1}{27} \times N$

  Continuing this further,

- # Players reaching stage $m = \frac{1}{3^{m-1}} \times N$
- # Players completing stage m = # Players reaching stage $m+1 = \frac{1}{3^m} \times N$

The players reaching a particular stage are the winners of the earlier stage. The number of games played is the number of total winners throughout the stages of the tournament. Thus, the total number of games ($\#G$) is given by the sum of elements in the first row excluding the first element.

$$\#G = \sum_{i=1}^{m} \left(\frac{1}{3^i}\right) \times N$$

However, only the winners from the m^{th} stage are to be rewarded. Thus, the number of winners to be rewarded (W_r) is given by:

$$W_r = \frac{1}{3^m} \times N$$

If an entrey fee of E was paid by N players, the total collected amount (Amount) is given as:

$$Amount = N \times E$$

If f is the factor by which the reward amount is greater than the entry fee, the total payout amount (Payout) is given as:

$$Payout = f \times E \times W_r$$

Hence, the revenue generated (Revenue) from N players for the provider is:

$$\begin{aligned} Revenue &= Amount - Payout \\ &= N \times E - f \times E \times W_r \\ &= N \times E - f \times E \times \left(\frac{1}{3^m} \times N\right) \\ &= N \times E \left(1 - f \times \frac{1}{3^m}\right) \end{aligned}$$

Now, finding the revenue per game (R/G):

$$\begin{aligned} R/G &= \frac{Revenue}{\#G} \\ &= \frac{\cancel{N} \times E \left(1 - f \times \frac{1}{3^m}\right)}{\sum_{i=1}^{m} \left(\frac{1}{3^i}\right) \times \cancel{N}} \\ &= \frac{E \left(1 - f \times \frac{1}{3^m}\right)}{\sum_{i=1}^{m} \left(\frac{1}{3^i}\right)} \end{aligned} \tag{6.1}$$

Simplifying further,

$$
\begin{aligned}
R/G &= \frac{E\left(3^m - f\right)}{3^m \times \sum\limits_{i=1}^{m} \left(\frac{1}{3^i}\right)} \\
&= \frac{E\left(3^m - f\right)}{\sum\limits_{i=1}^{m} \left(\frac{3^m}{3^i}\right)} \\
&= \frac{E\left(3^m - f\right)}{3^m \sum\limits_{i=1}^{m} \left(\frac{1}{3^i}\right)}
\end{aligned}
$$

Computing the general progression formula for the summation,

$$
= \frac{E\left(3^m - f\right)}{3^m \left(\frac{1-\left(\frac{1}{3}\right)^m}{2}\right)}
$$

Simplifying,

$$
= \frac{2E\left(3^m - f\right)}{3^m - 1}
$$

■

To show the revenue per game increases in m, we consider the following:

$$
\frac{R}{G}(m) = \frac{2E\left(3^m - f\right)}{3^m - 1}
$$

$$
\frac{R}{G}(m+1) = \frac{2E\left(3^{m+1} - f\right)}{3^{m+1} - 1}
$$

dividing $\frac{R}{G}(m+1)$ *by* $\frac{R}{G}(m)$,

$$
\begin{aligned}
\frac{\frac{R}{G}(m+1)}{\frac{R}{G}(m)} &= \frac{\frac{2E\left(3^{m+1}-f\right)}{3^{m+1}-1}}{\frac{2E(3^m-f)}{3^m-1}} \\
&= \frac{\cancel{2E}(3^{m+1} - f)}{3^{m+1} - 1} \times \frac{3^m - 1}{\cancel{2E}(3^m - f)} \\
&= \frac{3^{2m+1} - 3^{m+1} - 3^m f + f}{3^{2m+1} - 3^{m+1} f - 3^m + f} \\
&= \frac{(3^{2m+1} + f) - 3^m(3 + f)}{(3^{2m+1} + f) - 3^m(3f + 1)}
\end{aligned}
$$

Here, m and f are positive numbers. So, $(3f + 1) > (3 + f)$ for $f > 1$.
Thus, $(3^{2m+1} + f) - 3^m(3 + f) > (3^{2m+1} + f) - 3^m(3f + 1)$.
Hence, $\frac{\frac{R}{G}(m+1)}{\frac{R}{G}(m)} > 1$ for $f > 1$ and any $m \geq 0$.
This shows, R/G is increasing in m. ■

Finding limit of R/G, using (6.1)

$$\lim_{m\to\infty} R/G = \lim_{m\to\infty} \frac{E\left(1 - f \times \frac{1}{3^m}\right)}{\sum\limits_{i=1}^{m} \left(\frac{1}{3^i}\right)}$$

Applying limit in the numerator and using formula for sum of infinite geometric series in the denominator,

$$= \frac{E\left(1 - f \times \frac{1}{3^\infty}\right)}{\frac{1/3}{1-1/3}}$$

$$= \frac{E(1-0)}{1/2}$$

$$= 2 \times E$$

■

This shows that the per-game revenue will never be more than twice the per player entry fee.

6.2 Numerical Results

For $N = 10000000$ players entering the game assume the following conventions of the game: proportion of players moving to the next stage $(w) = 1/4$, proportion of players repeating the stage $(p) = 1/4$, entry fee $E = 50,000$ unit currency, reward factor $(f) = 20$. If m is the number of stages to be completed to collect the reward, then table (6.1) shows the results for the revenue, number of games (#G) and per-game revenue (R/G).

m	Revenue	$\#G$	R/G
3	129629629630	4814815	26923
4	376543209877	4938272	76250
5	458847736626	4979424	92148
6	486282578875	4993141	97390
7	495427526292	4997714	99130
8	498475842097	4999238	99710
9	499491947366	4999746	99903

Table 6.1: Numerical results for revenue for different m

As seen from the table, when m increases, R/G approaches 1,00,000 which is twice as much as E. This validates our results for limiting value of the per game revenue for the game provider.

Chapter 7

Conclusion

With the help of the transition matrix, we can calculate the revenues for the game providers and compare per game revenues for different tournament formats to ultimately decide on the optimal format that yields greater revenue with minimal cost for facilitating games. The matrix can be adjusted by changing the proportion of players making it to the next stage according to the tournament format. Thus, this method is scalable and can be used across different game formats.

Future works could address more analysis from the player's perspective. The analysis could be done to compute the expected time to win a stage or a sequence of stages. Players can then decide on the basis of these computations on whether or not to play a game. Because these games involve paying entry fees and the prospect of winning rewards that have some monetary values, having even a handful of information that relates to the odds of winning the games, and hence the tournament, will be helpful to the players.

Bibliography

[1] D.L Isaacson, R.W Madsen. *Markov Chains Theory and Applications.* Wiley Series In Probability and Mathematical Statistics, John Wiley and Sons, New York, The United States of America. (1992)

[2] J.R. Norris. *Markov Chains Theory and Applications.* Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, Cambridge, United Kingdom.(1998)