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Twisted Central Configurations of the Eight-Body problem

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by

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Abstract

The N-body problem qualifies as the problem of the twenty-first century because of its fundamental importance and difficulty to solve [1]. A number of great mathematicians and physicists have tried but failed to come up with the general solution of the problem. Due to the complexity of the problem, even a partial result will help us in the understanding of the N-body problem. Central configurations play a 'central' role in the understanding of the N-body problem. The well known Euler and Lagrangian solutions are both generated from three-body central configurations. The existence and classifications of central configurations have attracted number of researchers in the past three hundred years. A number of results and papers have been published but the study of central configurations is still far from complete. We do not even know how to prove the finiteness of the number of central configurations for $N > 5$.

In this paper, we studied a special type of central configuration: twisted central configurations of the eight-body problem. Consider the eight-body problem where these bodies $m_1, m_2, \cdots, m_8$ are located on the vertices of two squares $\square 1234$ and $\square 5678$. We assume that both squares have the same centroid and they are symmetrical about the lines connecting m_5, m_7 and m_6, m_8 (see Figure 4.1). We show that the masses must satisfy $m_1 = m_2 = m_3 = m_4$ and $m_5 = m_6 = m_7 = m_8$ when the configuration forms a central configuration. When the two square configurations have a common centroid, the configuration can form a central configuration only if the ratio of the size of the two squares falls into one of three intervals. Moreover there are some numerical evidences that there are exactly three nested central configurations for each given mass ratio $\frac{m_1}{m_5}$.

Key Words: N-body problem, Central configuration, Twisted central configuration.

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Chapter 1

Introduction

1.1. Celestial Mechanics

Celestial mechanics is the branch of astronomy that deals with the motions of objects in outer space. The three aspects of celestial mechanics are: physics of motion, mathematics of motion and (numerical) calculation of motion. Ancient celestial mechanics starts with Appolonius and the idea of an epicyclic motion, Ptolemy and the idea of a geocentric system, Copernicus and the idea of a heliocentric system, Kepler and his three Kepler laws and Galileo with his study of satellites of Jupiter as a model for the Solar system. Modern celestial mechanics started with Newton with his mathematical formulation of mechanics and gravitational force. Later Einstein studied the problem of perihelion advance of Mercury and the general theory of relativity.

1.2. *N*-body Problem

The classical Newtonian *N*-body problem in celestial mechanics consists of studying the motion of *N* point-like masses under the interactions among themselves according to Newton's gravitational law (Newton 1687) [2].

$$m_i \ddot{r}_i = -\sum_{j=1, j\neq i}^{N} \frac{m_j m_i (r_i - r_j)}{r_{ij}^3} \tag{1.1}$$

for $i = 1, 2, .., N$. By choosing the units of mass, one can arrange that G = 1 and r_j is the position of the point mass m_j in the inertial barycentric system. $r_{ij} = |r_i - r_j|$ is the Euclidean distance between r_i and r_j.

1.3. The 2-Body Problem

In classical mechanics, the two-body problem is to predict the movement of two objects which are viewed as point particles. While working on the two body problem, all external forces are ignored. The only force affecting each mass arises from the other one. The

solution of the 2-Body problem is formulated as [3]:
Let x_1 and x_2 be the vector positions of the two bodies, and m_1 and m_2 be their masses. The goal is to determine the trajectories $x_1(t)$ and $x_2(t)$ for all times t, given the initial positions $x_1(t=0)$ and $x_2(t=0)$ and the initial velocities $v_1(t=0)$ and $v_2(t=0)$.
Newton's second law states that

$$F_{12}(x_1,x_2) = m_1\ddot{x}_1, \tag{1.2}$$

$$F_{21}(x_1,x_2) = m_2\ddot{x}_2, \tag{1.3}$$

where F_{12} is the force on mass 1 due to its interactions with mass 2, and F_{21} is the force on mass 2 due to its interactions with mass 1. $\ddot{x}_1$is the acceleration vector.

Adding and subtracting these two equations decouples them into two one-body problems, which can be solved independently. Adding equations (1.2) and (1.3) results in an equation describing the center of mass (barycenter) motion. By contrast, subtracting equation (1.3) from equation (1.2) results in an equation that describes how the vector $r = x_1 - x_2$ between the masses changes with time. The solutions of these independent one-body problems can be combined to obtain the solutions for the trajectories $x_1(t)$ and $x_2(t)$.

1.4. The 3-Body Problem

The famous three-body problem can be traced back to Newton [2] in the 1680s. It has attracted many famous mathematicians and physicists such as Euler [4] and Lagrange [5]. Poincare [6] found that not enough number of the first integrals exist to solve the three-body problem, and besides orbits of three-body system are rather sensitive to initial conditions. The discovery of the so-called sensitivity dependence on initial conditions (SDIC) laid the foundation of modern chaos theory.

The three body problem is a special case of N body problem. The closed-form solution does not exist for the two-body. The resulting dynamics are generally chaotic for most initial conditions and numerical methods are generally required.

The first nontrivial examples of central configurations were discovered by Euler in 1767, who studied the case $N = 3$, $d = 1$, that is, three bodies on a line (Euler (1767)). When two masses are equal, one can get a central configuration by putting an arbitrary mass at their midpoint (a centered 2-gon). For three unequal masses it is not obvious that any central configurations would exist. But Euler showed that, in fact, there will be exactly one equivalence class of collinear central configurations for each possible ordering of the masses along the line.

Similarly, Lagrange found a next example in the planar three-body problem when $N = 3$, $d = 2$. Remarkably, an equilateral triangle is a central configuration, not only for equal masses, but for any three masses m_1, m_2, m_3. Moreover, it is the only noncollinear central configuration for the three-body problem (Lagrange (1772)).

1.5. Central configuration

In celestial mechanics and the mathematics of the N-body problem, a central configuration (CC) is a system of point masses with the property that each mass is pulled by the combined gravitational force of the system directly towards the center of mass, with acceleration proportional to its distance from the center. Central configurations may be studied in Euclidean spaces of any dimension, although only dimensions one, two, and three are directly relevant for celestial mechanics.

Now, we have a relation for the center of the mass as given by

$$C = m_1 r_1 + \cdots + m_N r_N,$$

$$M = m_1 + m_2 + \cdots + m_N, c = C/M$$

If $\ddot{r}_i = -\lambda(r_i - c)$ with $\lambda \neq 0$ for all i, the configuration could generate a homographic solution, and the configuration $r = (r_1, ...r_n)$ is called a **central configuration** if the acceleration vectors of the bodies satisfy:

$$-\lambda(r_i - c) = -\sum_{j=1, j\neq i}^{N} \frac{m_j(r_i - r_j)}{r_{ij}^3} \tag{1.4}$$

Central configurations play a 'central' role in the study of the N-body problem and the problem on the finiteness of the number of central configurations for a given n positive masses was listed as a challenge question for 21st century mathematicians [7]. Due to the complexity of problems the study of central configuration is far from completed. There are only three collinear central configurations found by Euler (1767) and an equilateral central configuration found by Lagrange (1873) for the 3-body problem. The finiteness of the number of central configurations was proved by Hampton and Moeckel [10] in 2006 for the

4-body problem, and partially proved by Albouy and Kaloshin [15] in 2012 for the planar 5-body problem. A stacked central configuration is a central configuration where a proper subset of particles also forms a central configuration. This concept was first introduced by Hampton in 2005 [9]. He found a very interesting property of the configuration; the CC has a subset of three bodies forming a CC of the 3-body problem, the three bodies are in an equilateral triangle, and the remaining two bodies are in the interior of the triangle and are located symmetrically with respect to a perpendicular bisector.

Chapter 2

Literature review

In this section, we summarize a few recent results for central configurations. Since we study the twisted central configurations of an eight-body problem, we will focus on the recent papers of stacked central configurations, nested central configurations, or twisted central configurations.

Xiang Yu, Shiqing Zhang (2012) [11] studied the necessary conditions and sufficient conditions for the twisted angles of the central configurations formed by two twisted regular polygons, in particular, they proved that for the $2N$-body problem, the twisted angles must be $\theta = 0$ (nested central configurations) or $\theta = \pi/N$ (twisted central configurations).

Xia Su, Tianqing An (2012) [12] studied twisted stacked central configurations for the spatial seven-body problem. More precisely, the position vectors r_1 , r_2 , r_3 and r_4 are at the vertices of a regular tetrahedron and the position vectors r_5 , r_6 and r_7 are at the vertices of an equilateral triangle have twisted angle $\frac{\pi}{3}$.

Antonio Carlos Fernandes, Luis Fernando Mello (2012) [18]. They placed the four masses on the vertices of a square. They proved for non-collinear configurations, the only possible strictly planar central configuration for the five-body problem for which it can be removed one body such that the remaining four bodies are already in a central configuration is obtained with four bodies of equal masses at the vertices of a square and the fifth body of arbitrary mass at the center of the square.

Sen Zhang, and Furong Zhao (2013) [16] studied the configuration formed by two squares in two parallel layers separated by a distance. They picture the two layers horizontally with the Z-axis passing through the centers of the two squares. The masses on the vertices of each square are same but the masses of the top square are not equal to the bottom square. They proved that the above configuration of two squares forms a central configuration if and only if the twist angle is equal to $\frac{k\pi}{2}$ or $(\frac{\pi}{4} + \frac{k\pi}{2})$ $(k = 1,2,3,4)$.

Xiang Yu, Shiqing Zhang (2015) [13] studied central configurations formed by two twisted regular polygons and found out the necessary and sufficient conditions for the central configurations formed by two twisted regular polygons (one N-regular polygon and one L-regular polygon). They proved that the necessary condition on the central configurations is L = N.

Chunhua Deng, Xia Su (2015) [17] studied the existence of the twisted stacked central configurations for the nine-body problem. The position vectors r_1, r_2, r_3, r_4 and r_5 are at the vertices of a square pyramid; the position vectors r_6, r_7, r_8 and r_9 are at the vertices of a square. They found that the square (r_1, r_2, r_3, r_4) and the square (r_6, r_7, r_8, r_9) have twisted angle $\frac{\pi}{4}$.

Zhifu Xie, Gokul Bhusal and Hamas Tahir (2020) [14] studied six bodies which are located on two equilateral triangles Δ_{123} and Δ_{456} and found out that the six body configuration is not a central configuration if the triangle Δ_{123} is above or below the triangle Δ_{456}. When the two equilateral triangle configurations have a common centroid, masses on each equilateral triangle must be same respectively and the configuration can form a central configuration only if the ratio of the lengths of the sides between Δ_{123} and Δ_{456} falls into certain ranges. Moreover there are some numerical evidences that: first there are exactly two nested central configuration but there may be one, two, or three twisted nested CC for a given mass ratio; and second, there exist central configurations other than same centroid.

Chapter 3

Central Configurations in the Planar 6-body Problem Forming Two Equilateral Triangles

The results of this section are partially from the publication [14]. With the support of the Wright W. and Annie Rea Cross Scholarship, Hamas and I started to work on the central configurations under the supervision of Dr. Zhifu Xie. We started working on this project as a summer Research Experience for Undergraduate (REU 2017). We studied the classical N-body problem and proposed the idea of studying a stacked central configuration for the planar six-body problem. Our primary research question was to determine the possibility of a central configuration and to identify regions where it would be possible to choose positive masses, making the configuration central. By the end of the summer, we were able to find the family of central configurations, which was obtained when two triangles formed a regular hexagon. We also received MAA travel awards to present our work in the undergraduate poster session at the Joint Mathematics Meeting 2018 in San Diego, California.We continued to work on the project during the semesters and were able to partially answer our proposed research questions. There are lots of new questions coming out from the paper. One may ask whether the results in the twisted six-body central configurations in [14] can be extended to the twisted eight-body problem.

In this research, we are interested in a particular class of stacked central configurations formed by two equilateral triangles in a plane. More precisely, in the 6-body planar configu-

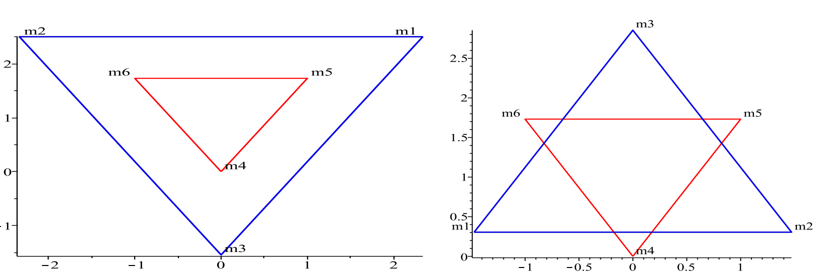

Figure 3.1: Two equilateral triangles share the axis of symmetry and triangle Δ_{456} is fixed. $r_4 = (0,0), r_5 = (1,\sqrt{3}), r_6 = (-1,\sqrt{3}); r_1 = (\frac{b-a}{\sqrt{3}}, b), r_2 = (\frac{a-b}{\sqrt{3}}, b), r_3 = (0,a)$. Left: $a < b$; Right: $a > b$.

ration (see Fig. 3.1), the position vectors r_4, r_5, r_6 are fixed at the vertices of an equilateral triangle and the position vectors r_1, r_2, r_3 depend on two parameters a and b as follows:

$$r_4 = (0,0), r_5 = (1,\sqrt{3}), r_6 = -1,\sqrt{3}, r_1 = (\frac{b-a}{\sqrt{3}}, b), r_2 = (\frac{a-b}{\sqrt{3}}, b), r_3 = (0,a). \quad (3.1)$$

3.1. Symmetries and Masses

For the planar central configurations we can use Dziobek-Laura-Andoyer equations (see Hagihara 1970, p.241).

$$f_{ij} = \sum_{k=1,k\neq i,j}^{N} m_k(R_{ik} - R_{jk})\Delta_{ijk} = 0 \quad (3.2)$$

for $1 \leq i < j \leq N$, where $R_{ij} = 1/r_{ij}^3$ and $\Delta_{ijk} = (r_i - r_j) \times (r_i - r_k)$, where $\times$ denotes the cross product of two vectors. Thus Δ_{ijk} gives twice the signed area of the triangle Δ_{ijk}.
For the 6-body problem (3.2) is a set of 15 equations.
$f_{12}, f_{13}, f_{14}, f_{15}, f_{16}, f_{23}, f_{24}, f_{25}, f_{26}, f_{34}, f_{35}, f_{36}, f_{45}, f_{46}, f_{56}$.
Due to the symmetries of the configurations studied here the following relations must be satisfied.

$$\begin{aligned} r_{12} = r_{23} = r_{13}, r_{45} = r_{46} = r_{56} \\ r_{14} = r_{24}, r_{16} = r_{25}, r_{35} = r_{36} \end{aligned} \quad (3.3)$$

$$\begin{aligned} \Delta_{125} = \Delta_{126}, \Delta_{136} = -\Delta_{235} \\ \Delta_{134} = -\Delta_{234}, \Delta_{453} = -\Delta_{463}, \end{aligned} \quad (3.4)$$

and much more. Using these symmetries we obtain the necessary conditions for central configurations.

Lemma 1. If the configuration r in (3.1) is a stacked central configuration for the 6-body problem, then $m_1 = m_2$ and $m_5 = m_6$.

Proof. It is sufficient to consider the equations $f_{12} = 0$ and $f_{56} = 0$. Using the symmetries, we have

$$\begin{aligned} f_{12} &= (m_5 - m_6)(R_{15} - R_{16})\Delta_{125}, \\ f_{56} &= (m_1 - m_2)(R_{15} - R_{16})\Delta_{156}. \end{aligned} \quad (3.5)$$

In the above equation $R_{15} - R_{16} \neq 0$, $\Delta_{156} \neq 0$ and $\Delta_{125} \neq 0$ which implies that $m_1 = m_2$ and $m_5 = m_6$. □

Due to Lemma 1, we restrict $m_1 = m_2$, $m_5 = m_6$ and substitute them into the other thirteen equations. It follows that $f_{34} = 0$ always holds and the other twelve equations are equivalent as follows.

$f_{14} = 0 \iff f_{24} = 0; f_{15} = 0 \iff f_{26} = 0; f_{13} = 0 \iff f_{23} = 0;$

$f_{45} = 0 \iff f_{46} = 0; f_{16} = 0 \iff f_{25} = 0; f_{36} = 0 \iff f_{35} = 0$

The six equations that the configuration r and mass vector m must satisfy to form a central configuration can be written respectively as:

$f_{14} = m_1(R_{12} - R_{42})\Delta_{142} + m_3(R_{13} - R_{43})\Delta_{143} + m_5[(R_{15} - R_{45})\Delta_{145} + (R_{16} - R_{46})\Delta_{146}] = 0.$

$f_{15} = m_1(R_{12} - R_{52})\Delta_{152} + m_3(R_{13} - R_{53})\Delta_{153} + m_4(R_{14} - R_{54})\Delta_{154} + m_5(R_{16} - R_{56})\Delta_{156} = 0.$

$f_{23} = m_4(R_{24} - R_{34})\Delta_{234} + m_5[(R_{25} - R_{35})\Delta_{235} + (R_{26} - R_{36})\Delta_{236}] = 0.$

$f_{45} = m_1[(R_{41} - R_{51})\Delta_{451} + (R_{42} - R_{52})\Delta_{452}] + m_3(R_{43} - R_{53})\Delta_{453} = 0.$

$f_{16} = m_1(R_{12} - R_{62})\Delta_{162} + m_3(R_{13} - R_{63})\Delta_{163} + m_4(R_{14} - R_{64})\Delta_{164} + m_5(R_{15} - R_{65})\Delta_{165} = 0.$

$f_{35} = m_1[(R_{31} - R_{15})\Delta_{135} + (R_{32} - R_{52})\Delta_{235}] + m_4(R_{34} - R_{54})\Delta_{354} + m_5(R_{36} - R_{65})\Delta_{356} = 0.$

3.2. Existence of stacked central configurations when both triangles have same centroids

Lemma 2. Assume that the centroids of both triangles Δ_{123} and Δ_{456} are same, i.e. $b = \sqrt{3} - a/2$. When a is in certain intervals in Table 3.1, there exist positive masses $m_1 = m_2 = m_3 > 0$ and $m_4 = m_5 = m_6 > 0$ such that the configuration is a stacked central configuration.

Table 3.1: Existence of Stacked C.C. when both triangles have same centroids

Range of a	Stacked C.C for positive masses	Ratio γ_0 of sides between Δ_{123} and Δ_{456}
$(-\infty, -1.20529627185787)$	Exist	$(2.043817191, +\infty)$
$(-1.20529627185787, 0.589727962083864)$	Doesn't exist	(0.4892806035, 2.043817191)
$(0.589727962083864, \frac{2\sqrt{3}}{3})$	Exist	(0, 0.4892806035)
$(\frac{2\sqrt{3}}{3}, 1.63261715743596)$	Exist	(0, 0.413887933)
(1.63261715743596, 1.86758318958456)	Doesn't exist	(0.413887933,0.617374486)
(1.86758318958456, 3.02507247551557)	Exist	(0.617374486,1.619789613)
(3.02507247551557, 3.94458889461316)	Doesn't exist	(1.619789613, 2.416114192)
$(3.94458889461316, +\infty)$	Exist	$(2.416114192, +\infty)$

Proof. When both triangles have same centroid, $(R_{24} - R_{25})$ is always zero because the line connecting m_2 and m_6 is always perpendicular and bisects to the line segment of m_4 and

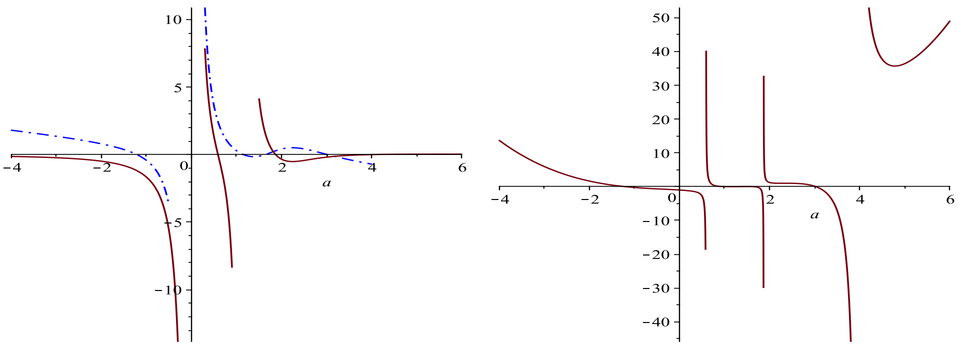

Figure 3.2: (Left) The graphs of s_{11} (solid red line) and s_{12} (dashdotted blue line); Central configurations are possible when the values of s_{11} and s_{12} have opposite signs. (Right) The mass ratio of $\frac{m_1}{m_5} = -\frac{s_{12}}{s_{11}}$ should be positive if it is a central configuration.

m_5. Because the line connecting m_1 and m_3 is parallel to the line connecting m_4 and m_5, $\Delta 451 = \Delta 453 \neq 0$ and $(R_{14} - R_{15}) = -(R_{34} - R_{35}) \neq 0$ for any non-collision configuration. So

$$\begin{aligned} f_{45} &= m_1[(R_{41} - R_{51})\Delta 451 + (R_{42} - R_{52})\Delta 452] + m_3(R_{43} - R_{53})\Delta 453 \\ &= (m_1 - m_3)(R_{41} - R_{51})\Delta 451 = 0 \end{aligned}$$

implies m_1=m_3.
Similarly, $(R_{25} - R_{35}) = 0$, $(R_{24} - R_{34}) = -(R_{26} - R_{36}) \neq 0$ and $\Delta 234 = \Delta 236 \neq 0$ for any non-collision configuration. The equation $f_{23} = m_4(R_{24} - R_{34})\Delta_{234} + m_5[(R_{25} - R_{35})\Delta_{235} + (R_{26} - R_{36})\Delta_{236}] = 0$ gives m_4=m_5.
When $m_1 = m_2 = m_3$ and $m_4 = m_5 = m_6$, $f_{15} = 0$ holds by symmetry and it is easy to check that all the other three equations $f_{14} = 0$, $f_{16} = 0$ and $f_{35} = 0$ are equivalent each other. So when both triangles have same centroid, the configuration is a stacked central configuration if and only if there exist positive masses m_1 and m_5 such that equation $f_{14} = s_{11}m_1 + s_{12}m_5 = 0$ holds, where

$$s_{11} = (R_{12} - R_{24})\Delta 142 + (R_{13} - R_{34})\Delta 143; \qquad s_{12} = (R_{15} - R_{45})\Delta 145 + (R_{16} - R_{46})\Delta 146. \tag{3.6}$$

By using Maple Solve, we obtain that s_{11} and s_{12} have opposite signs (see Figure 3.2) when a is in the following open intervals

$$(-\infty, \alpha_1), (\alpha_2, \alpha_3), (\alpha_3, \alpha_4), (\alpha_5, \alpha_6) \text{ or } (\alpha_7, \infty),$$

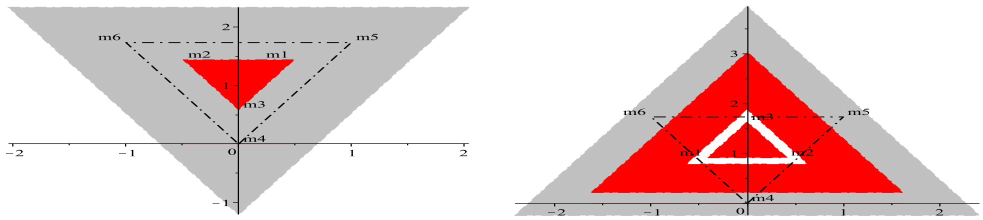

Figure 3.3: Common centroid central configurations. Left $(b > a)$: If the nested triangle Δ_{123} is in the gray area, it is not a central configuration. Right $(b < a)$: If the twisted nested triangle Δ_{123} falls into the gray region, it is not a central configuration.

where $\alpha_1 < \alpha_2 < \cdots < \alpha_7$ and

$$(\alpha_1, \alpha_2, \cdots, \alpha_7) \approx (-1.2052, 0.5898, \frac{2\sqrt{3}}{3}, 1.6326, 1.8676, 3.0250, 3.9446).$$

In fact, α_2, α_5, and α_7 are zeros of $s_{11} = 0$. $\alpha_1, \alpha_3, \alpha_4$ and α_6 are zeros of $s_{12} = 0$. s_{11} is also undefined at α_3 and $\lim_{a \to \alpha_3^-} s_{11} = -\infty$ and $\lim_{a \to \alpha_3^+} s_{11} = +\infty$. This means that the corresponding configurations are central configurations for some positive masses m_1 and m_5.

Geometrically, when the triangle Δ_{123} is in the red region or out of the color regions, the six bodies form a central configurations for some positive masses (see Figure 3.3). On the other hand, when a is out of the above intervals, there are no positive masses m_1 and m_5 to make the corresponding configurations central (see table 4.1). Geometrically, when the triangle Δ_{123} is in the gray region, no positive masses can make the corresponding configuration central.

$\square$

3.3. Existence of new stacked central configurations

In this section, we use a numerical method to show the existence of stacked central configurations other than those in Lemma 2 for some positive masses. The numerical method was carried out by Maple (2016). Recall that there are six equations $f_{14} = 0, f_{15} = 0$, $f_{23} = 0, f_{45} = 0$, $f_{16} = 0, f_{35} = 0$ with six parameters m_1, m_3, m_4, m_5, a, and b. In order to find values a, b and positive values m_1, m_3, m_4, m_5 to satisfy the six equations, we use the substitution method.

First from $f_{23} = 0$ one obtains

$$m_4 = -\frac{((R_{25} - R_{35})\,\Delta_{235} + (R_{26} - R_{36})\,\Delta_{236})\,m_5}{(R_{24} - R_{34})\,\Delta_{234}}.$$

From $f_{45} = 0$ one obtains

$$m_1 = -\frac{m_3\,(R_{34} - R_{35})\,\Delta_{453}}{(R_{14} - R_{15})\,\Delta_{451} + (R_{24} - R25)\,\Delta_{452}}.$$

Substituting the above expressions m_1 and m_4 into $f_{35} = 0$ and solving for m_3, one obtains

$$m_3 = \frac{m_5\,((R_{14} - R_{15})\,\Delta_{451} + (R_{24} - R_{25})\,\Delta_{452})}{(R_{34} - R_{35})\,\Delta_{453}\,((R_{13} - R_{15})\,\Delta_{135} + (R_{23} - R_{25})\,\Delta_{235})} \times$$
$$\left(-\frac{((R_{25} - R_{35})\,\Delta_{235} + (R_{26} - R_{36})\,\Delta_{236})\,(R_{34} - R_{45})\,\Delta_{354}}{(R_{24} - R34)\,\Delta_{234}} + (R_{36} - R_{56})\,\Delta_{356}\right)$$

Then substituting the above expressions of m_4, m_1, and m_3 into the other three equations. Then,

$f_{14} = m_1(R_{12} - R_{24})\Delta_{142} + m_3(R_{13} - R_{34})\Delta_{143} + m_5[(R_{15} - R_{45})\Delta_{145} - (R_{46} - R_{16})\Delta_{146}] = 0$;

$f_{15} = m_1(R_{12} - R_{25})\Delta_{125} - m_3(R_{35} - R_{13})\Delta_{143} - m_4(R_{45} - R_{14})\Delta_{154} - m_5(R_{56} - R_{16})\Delta_{156}] = 0$;

$f_{16} = m_1(R_{12} - R_{26})\Delta_{162} - m_3(R_{36} - R_{13})\Delta_{163} - m_4(R_{46} - R_{14})\Delta_{164} - m_5(R_{56} - R_{15})\Delta_{165}] = 0$.

We can assume $m_5 = 1$, since m_5 is a common factor for all three functions, to find zeros of the above three equations with variables a, b and m_5. Due to the complexity of the expression, it is not possible to simplify the expression by hand. First we use Maple 2016 to conduct symbolic computation and we find that f_{14} is the same as f_{15}, i.e. $f_{14} = f_{15}$. Unfortunately, we are not able to prove this analytically. Second, we use implicit plot to identify some possible regions for zeros of f_{14} and f_{16}. It is very interesting to note that the resulting graphs in the ab-plane of $f_{14} = 0$ and $f_{16} = 0$ are almost the same.

Now we use Maple to conduct a numerical search. Choosing an approximate value of a_0 based on the implicit plot, we find zeros of $f_{14} = 0$ to get the corresponding value b_0 (in general, there is more than one solution). Then we check whether the pair values (a_0, b_0) satisfy $f_{16} = 0$ and make $m_i > 0$ for all $i = 1, 3, 4$. If it is, the configuration with parameter (a_0, b_0) is a central configuration for some positive masses. Here are some examples from our numerical results (see Table 4.2).

Table 3.2: Examples of stacked central configurations with $m_5 = 1$

Common Centroid	Values of a	values of b	values of m_1	values of m_3	values of m_4
NO	1.5	0.4771831	0.5546158215	1.317318509	0.08200198396
YES	1.5	0.9820508080	0.02736916455	0.02736916441	1.00000000
NO	1.6	0.514212	0.4586126179	0.7910660856	0.1739112951
YES	1.6	0.9320508080	0.01635692886	0.01635692887	1.00000000
NO	1.7	0.379265	0.0154245	0.300804	0.191886
YES	1.7	0.8820508080	Negative	Negative	1
NO	2.1	1.1370092	0.5138598491	0.08245610014	4.310400458
Yes	2.1	0.682050808	1.038805286	1.038805286	1.00000000
NO	2.2	1.139994	0.1976244611	0.05433805186,	4.727708458
YES	2.2	0.63205080	1.002174732	1.002174729	1.00000000
NO	2.3	1.172536	0.02196732623	0.006323164009	5.718104210
YES	2.3	0.582050808	0.9997844739	0.9997844735	1.00000000

Chapter 4

Twisted Central Configurations for Planar Eight-body Problem

We present our main results in this chapter. We study the existence of twisted central configurations in an 8-body problem which has four bodies in the vertices of a square and the other four bodies also lie in vertices of a square (see Figure 4.1).

The position vector r_5, r_6, r_7, r_8 are fixed at the vertices of the square $r_5 = (0,-1)$, $r_6 = (1,0)$, $r_7 = (0,1)$ and $r_8 = (-1,0)$; the position vectors r_1, r_2, r_3, r_4 depend on one parameter d with $r_1 = (\frac{d}{\sqrt{2}}, -\frac{d}{\sqrt{2}})$, $r_2 = (\frac{d}{\sqrt{2}}, \frac{d}{\sqrt{2}})$, $r_3 = (-\frac{d}{\sqrt{2}}, \frac{d}{\sqrt{2}})$ and $r_4 = (-\frac{d}{\sqrt{2}}, -\frac{d}{\sqrt{2}})$. The size of square 1234 depends upon the value of d and $d = \frac{r_{12}}{r_{56}}$. The configuration is always symmetrical with both x-axis and y-axis.

The equations of motion for the planar Newtonian n-body problem are given by

$$\ddot{r}_i = -\sum_{j=1, j\neq i}^{N} m_j \frac{(r_i - r_j)}{(r_{ij}^3)} \tag{4.1}$$

for $i = 1, 2, ...n$. Hence the gravitation constant is taken equal to 1, $r_j \in R^2$ is the position vector of the punctual mass m_j in the inertial barycentric system, and as before

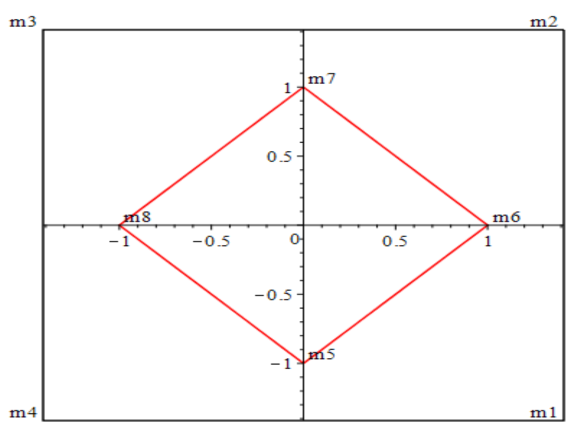

Figure 4.1: The configuration of eight bodies. (1) Square 5678 is fixed at $r_5 = (0,-1)$, $r_6 = (1,0)$, $r_7 = (0,1)$ and $r_8 = (-1,0)$; (2) Square 1234 is given by $r_1 = (\frac{d}{\sqrt{2}}, -\frac{d}{\sqrt{2}})$, $r_2 = (\frac{d}{\sqrt{2}}, \frac{d}{\sqrt{2}})$,$r_3 = (-\frac{d}{\sqrt{2}}, \frac{d}{\sqrt{2}})$ and $r_4 = (-\frac{d}{\sqrt{2}}, -\frac{d}{\sqrt{2}})$.

$r_{ij} = |r_i - r_j|$ is the Euclidean distance between r_i and r_j. For the central configurations we have $\ddot{r}_i = -\lambda(r_i - c)$ with $\lambda \neq 0$ for all $j = 1, ...n$. So from the Eq. 4.1 we have

$$-\lambda(r_i - c) = -\sum_{j=1, j\neq i}^{N} m_j \frac{(r_i - r_j)}{(r_{ij}^3)} \tag{4.2}$$

4.1. Symmetries and Masses

For the planar central configurations we can use Dziobek-Laura-Andoyer equations (see Hagihara 1970, p.241) which is equivalent to equation (1.4)

$$f_{ij} = \sum_{k=1, k\neq i,j}^{N} m_k (R_{ik} - R_{jk}) \Delta_{ijk} = 0 \tag{4.3}$$

for $1 \leq i < j \leq N$, where $R_{ij} = 1/r_{ij}^3$ and $\Delta_{ijk} = (r_i - r_j) \times (r_i - r_k)$, where $\times$ denotes the cross product of two vectors. Thus Δ_{ijk} gives the twice the signed area of the triangle Δijk. For the 8-body problem (4.3) is a set of $C_2^8 = 28$ equations.
f_{12},f_{23},f_{34},f_{14}, f_{56},f_{67},f_{78},f_{58}, f_{13},f_{24}, f_{57},f_{68}, f_{15},
f_{16},f_{26},f_{27},f_{37},f_{38},f_{48},f_{45}, f_{17},f_{25},f_{27},f_{36},f_{35},f_{46},f_{47},f_{18}

Due to the symmetries of the configurations studied here the following relations must be satisfied.

$$\begin{aligned} r_{12} = r_{23} = r_{34} = r_{14}, r_{56} = r_{67} = r_{78} = r_{58}, \\ r_{18} = r_{28} = r_{25} = r_{35} = r_{36} = r_{46} = r_{47} = r_{17}, \\ r_{15} = r_{16} = r_{26} = r_{27} = r_{37} = r_{38} = r_{48} = r_{45}. \end{aligned} \tag{4.4}$$

$$\begin{aligned} \Delta_{145} = \Delta_{354} = \Delta_{327} = \Delta_{162}, \\ \Delta_{156} = \Delta_{485} = \Delta_{378} = \Delta_{726}, \\ \Delta_{147} = \Delta_{374} = \Delta_{325} = \Delta_{182}. \end{aligned} \tag{4.5}$$

Using these symmetries we obtain the necessary conditions for central configurations.

Theorem 3. *If the configuration of eight bodies is given in Figure 4.1 and it forms a central configuration, the following statements are necessary conditions.*
1. The masses m_1, m_2, m_3 and m_4 must be equal.
2. The masses m_5, m_6, m_7 and m_8 must be equal.

Proof. It is sufficient to consider the equations $f_{13} = 0$, $f_{24} = 0$, $f_{23} = 0$, $f_{14} = 0$, $f_{34} = 0$, $f_{58} = 0$.

By symmetries we have:

$f_{13} = (m_5 - m_6 - m_7 + m_8)[(R_{15} - R_{35})\Delta_{135} + (R_{18} - R_{38})\Delta_{138}] = 0$
$f_{24} = (m_6 - m_7 + m_5 - m_8)[(R_{26} - R_{46})\Delta_2 46 + (R_{25} - R_{45})\Delta_{245}] = 0$
$f_{23} = (m_1 - m_4)(R_{21} - R_{31})\Delta_{231} + (m_6 - m_8)(R_{26} - R_{36})\Delta_{236} = 0$
$f_{14} = (m_2 - m_3)(R_{12} - R_{42})\Delta_{142} + (m_6 - m_8)(R_{16} - R_{46})\Delta_{146} = 0$
$f_{34} = (m_1 - m_2)(R_{31} - R_{41})\Delta_{341} + (m_5 - m_7)(R_{35} - R_{45})\Delta_{345} = 0$
$f_{58} = (m_1 - m_3)(R_{51} - R_{81})\Delta_{581} + (m_6 - m_7)(R_{56} - R_{86})\Delta_{586} = 0$

For our class of central configuration, we have,

$(R_{15} - R_{35}) \neq 0$, $(R_{18} - R_{38}) \neq 0$, $(R_{26} - R_{46}) \neq 0$, $(R_{25} - R_{45}) \neq 0$, $(R_{21} - R_{31}) \neq 0$, $(R_{26} - R_{36}) \neq 0$, $(R_{12} - R_{42}) \neq 0$, $(R_{16} - R_{46}) \neq 0$, $(R_{31} - R_{41}) \neq 0$, $(R_{35} - R_{45}) \neq 0$, $(R_{51} - R_{81}) \neq 0$, $(R_{56} - R_{86}) \neq 0$.
Similarly, $\Delta_{135} \neq 0$, $\Delta_{138} \neq 0$, $\Delta_{246} \neq 0$, $\Delta_{245} \neq 0$, $\Delta_{231} \neq 0$, $\Delta_{236} \neq 0$, $\Delta_{142} \neq 0$, $\Delta_{146} \neq 0$, $\Delta_{341} \neq 0$, $\Delta_{345} \neq 0$, $\Delta_{581} \neq 0$, $\Delta_{586} \neq 0$.

Solving $f_{13} = 0$ gives $(m_5 - m_6 - m_7 + m_8) = 0$ and solving $f_{24} = 0$ gives $(m_6 - m_7 + m_5 - m_8) = 0$.
Solving these two expression gives $m_7 = m_5$ and $m_8 = m_6$.
Solving f_{23} gives $m_1 = m_4$.
Solving $f_{14} = 0$ and $m_6 = m_8$ gives $m_2 = m_3$
Solving $f_{34} = 0$ and $m_5 = m_7$ gives $m_1 = m_2$
Solving $f_{58} = 0$ and $m_1 = m_3$ gives $m_6 = m_7$.
Hence $m_1 = m_2 = m_3 = m_4$ and $m_5 = m_6 = m_7 = m_8$. □

Due to Theorem 3, we restrict $m_1 = m_2 = m_3 = m_4$ and $m_5 = m_6 = m_7 = m_8$ and substitute into the other 22 equations, it follows that $f_{57} = 0$, $f_{68} = 0$ always holds. f_{17} and f_{25} are equivalent because their relative positions are equivalent. Since $m_1 = m_2 = m_3 = m_4$ and $m_5 = m_6 = m_7 = m_8$ we don't need to worry about the masses. By similar argument, the other twenty equations are equivalent as follows

$$f_{17} = 0 \iff f_{25} = 0 \iff f_{27} = 0 \iff f_{36} = 0$$

$$\iff f_{35}=0 \iff f_{46}=0 \iff f_{47}=0 \iff f_{18}=0;$$

$$f_{15}=0 \iff f_{16}=0 \iff f_{26}=0 \iff f_{27}=0$$

$$\iff f_{37}=0 \iff f_{38}=0 \iff f_{48}=0 \iff f_{45}=0.$$

The two equations that the configuration r and mass vector m must satisfy to form a central configuration can be written respectively as:
$f_{15} = M_1[(R_{12}-R_{52})\Delta_{152}+(R_{13}-R_{53})\Delta_{153}+(R_{14}-R_{54})\Delta_{154}]+M_5[(R_{16}-R_{56})\Delta_{156}+(R_{17}-R_{57})\Delta_{157}+(R_{18}-R_{58})\Delta_{158}]$ =0.
$f_{17} = M_1[(R_{12}-R_{72})\Delta_{172}+(R_{13}-R_{73})\Delta_{173}+(R_{14}-R_{74})\Delta_{174}]+M_5[(R_{15}-R_{75})\Delta_{175}+(R_{16}-R_{76})\Delta_{176}+(R_{18}-R_{78})\Delta_{178}]$=0.

Lemma 4. If the configuration of eight bodies in Figure 4.1 forms a central configuration, $f_{15}=0$ is equivalent to $f_{17}=0$.

Proof. We can write the equation f_{15} and f_{17} in terms of matrices as follows.
$f_{15} = m_1 * a_{11} + m_5 * a_{12}$, $f_{17} = m_1 * a_{21} + m_5 * a_{22}$,

$$\begin{bmatrix} f_{15} \\ f_{17} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} * \begin{bmatrix} m_1 \\ m_5 \end{bmatrix},$$

where
$a_{11} = [(R_{12}-R_{52})\Delta_{152}+(R_{13}-R_{53})\Delta_{153}+(R_{14}-R_{54})\Delta_{154}]$,
$a_{12} = [(R_{16}-R_{56})\Delta_{156}+(R_{17}-R_{57})\Delta_{157}+(R_{18}-R_{58})\Delta_{158}]$,
$a_{21} = [(R_{12}-R_{72})\Delta_{172}+(R_{13}-R_{73})\Delta_{173}+(R_{14}-R_{74})\Delta_{174}]$,
$a_{22} = [(R_{15}-R_{75})\Delta_{175}+(R_{16}-R_{76})\Delta_{176}+(R_{18}-R_{78})\Delta_{178}]$.

Using the geometric relations: Δ_{154}= $\Delta_{153}-\Delta_{152}$, Δ_{174}= $\Delta_{173}-\Delta_{172}$, $\Delta_{152}=\Delta_{172}$, $\Delta_{153}=-\Delta_{173}$, we have $a_{11}+a_{21} = [R_{12}-R_{52}-R_{14}+R_{45}+R_{12}-R_{72}-R_{14}+R_{74}]*\Delta_{152}+[R_{13}-R_{53}+R_{14}-R_{45}+R_{73}-R_{13}-R_{14}+R_{74}]*\Delta_{153}$.
Therefore $a_{11}+a_{21} = 0*\Delta_{152}+0*\Delta_{153} = 0$.
Similarly, using the geometric relations: Δ_{156}= $\Delta_{157}-\Delta_{158}$, Δ_{178}= $\Delta_{175}-\Delta_{176}$, $\Delta_{158}=\Delta_{176}$, $\Delta_{175}=-\Delta_{157}$, we have $a_{12}+a_{22} = [R_{16}-R_{56}+R_{17}R_{57}+R_{57}-R_{15}+R_{75}-R_{18}]*\Delta_{157}+[R_{56}-R_{16}+R_{18}-R_{58}+R_{16}-R_{67}-R_{18}+R_{78}]*\Delta_{158}$.

$a_{12} + a_{22} = 0 * \Delta_{157} + 0 * \Delta_{158} = 0.$
Here, the determinant of the matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is zero, so the equations are equivalent with each other.

$\square$

Remark: From theorem 3 and lemma 4, the configuration of eight bodies in Figure 4.1 with $m_1 = m_2 = m_3 = m_4$, $m_5 = m_6 = m_7 = m_8$ forms a central configuration if and only if m_1, m_5 and d satisfy the equation $f_{15} = 0$.

4.2. Existence of central configuration for given size d of the configuration

Theorem 5. *If the configuration of eight bodies is given in Figure 4.1, then it forms a central configuration for positive masses* $m_1 = m_2 = m_3 = m_4 > 0$ *and* $m_5 = m_6 = m_7 = m_8 > 0$ *with a unique mass ratio* $\alpha = \frac{m_1}{m_5}$ *for each d in one of three existence intervals in Table 4.1.*

Table 4.1: Existence of CC for given d

Range of d	C.C exist or doesn't exist
0 – 0.624060599185569	Exist
0.624060599185569 – 0.697380512488264	Doesn't exist
0.697380512488264 – 1.43393740484010	Exist
1.43393740484010 – 1.60240841085268	Doesn't exist
1.60240841085268 – ∞	Exist

Proof. From theorem 3 and lemma 4, the only equation to determine whether the configuration r is a central configuration for mass vector m is $f_{15} = 0$. It can be written as:

$$f_{15} = a_{11} * m_1 + a_{12} * m_5 = 0.$$

Let the mass ratio be $\alpha = \frac{m_1}{m_5}$. So this configuration is a central configuration for some positive masses if and only if the mass ratio $\alpha = -\frac{a_{12}}{a_{11}}$ is positive for a positive d.

Here,

$$a_{11} = -\frac{1}{2}\left(\frac{\sqrt{2}}{4*d^3} - \left(d^2 + d\sqrt{2} + 1\right)^{-3/2}\right)d^2 - \frac{1}{2}\left(\frac{1}{8}d^{-3} - \left(d^2 + d\sqrt{2} + 1\right)^{-3/2}\right)d\sqrt{2} + \frac{1}{2}\left(\frac{1}{4}\frac{\sqrt{2}}{d^3} - \left(d^2 - d\sqrt{2} + 1\right)^{-3/2}\right)d\left(d - \sqrt{2}\right)$$

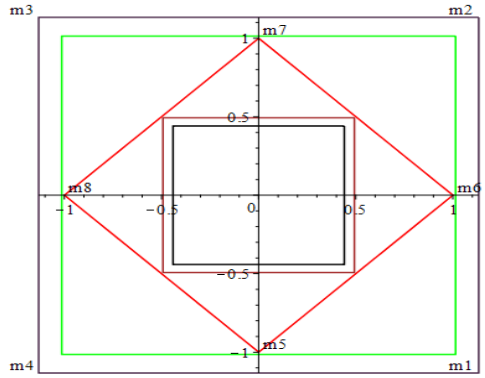

Figure 4.2: Different positions of $\square 1234$ when $d = \alpha_1$ (Black), $d = \alpha_2$ (Violet), $d = \alpha_3$ (Green) and $d = \alpha_4$ (Brown)

.

$$a_{12} = \left(\left(d^2 - d\sqrt{2}+1\right)^{-3/2} - \tfrac{1}{4}\sqrt{2}\right)\left(-\tfrac{1}{2}d\sqrt{2}+\tfrac{1}{2}\right) - \tfrac{1}{2}, \left(\left(d^2 + d\sqrt{2}+1\right)^{-3/2} - \tfrac{1}{8}\right) d\sqrt{2} - \tfrac{1}{2}\left(d^2 + d\sqrt{2}+1\right)^{-3/2} + \tfrac{1}{8}\sqrt{2}$$

By using Maple Solve, we obtain that a_{11} and a_{12} have opposite signs (see Figure 4.3) when d is in the following three open intervals

$$(0, \alpha_1), (\alpha_2, \alpha_3), (\alpha_4, \infty).$$

On the other hand a_{11} and a_{12} have same signs when d is in the following two open intervals (α_1, α_2), (α_3, α_4), where

$$0 < \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \infty$$

and

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \approx (0.624060588, 0.6973805, 1.43394, 1.602408).$$

In fact, α_1, and α_3 are zeros of $a_{11} = 0$. α_2 and α_4 are zeros of $a_{12} = 0$. This means that the corresponding configurations could be central configurations for some positive masses m_1 and m_5 when their size d falls into certain intervals. Geometrically, for each $d = \alpha_i$, it gives a square 1234 in the plane. So squares 1234 with $d = \alpha_i (i = 1, 2, 3, 4)$ can divide the plane into five regions (see Figure 4.2). The square 1234 with $d = \alpha_i$ is called the square α_i.

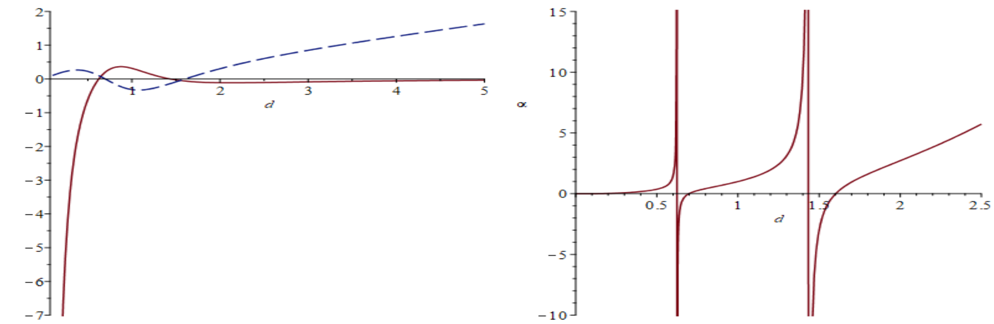

Figure 4.3: (Left) The graphs of a_{11} (solid red line) and a_{12} (dashdotted blue line); Central configurations are possible when the values of a_{11} and a_{12} have opposite signs. (Right) The mass ratio of $\frac{m_1}{m_5} = -\frac{a_{12}}{a_{11}}$ should be positive if it is a central configuration.

Region 1: region inside black square, or inside the smallest square α_1.
Region 2: region between black square and violet square, or between the square α_1 and the square α_2.
Region 3: region between violet square and green square, or between the square α_2 and the square α_3.
Region 4: region between green square and brown square, or between the square α_3 and the square α_4.
Region 5: region outside brown square, or outside the square α_4.
When the square 1234 is in the regions 1, 3 and 5 (see Figure 4.2), the eight bodies can form a central configurations for some positive masses. On the other hand, when the square 1234 is in the regions 2 and 4, there are no positive masses m_1 and m_5 to make the corresponding configurations central (also see table (4.1)). This completes the proof of the theorem.

□

4.3. Existence of central configurations for given mass ratio

Theorem 6. *[Numerical observation] If the configuration of eight bodies is given in Figure 4.1, then for positive masses $m_1 = m_2 = m_3 = m_4 > 0$ and $m_5 = m_6 = m_7 = m_8 > 0$ with a given mass ratio $\alpha = \frac{m_1}{m_5}$, there exist three distinct values of $d(d_1, d_2, d_3)$ to make the configuration central.*

Proof. The equation that we are left with is:

$$f_{15} = a_{11} * m_1 + a_{12} * m_5 = 0.$$

$$\frac{m_1}{m_5} = -\frac{a_{12}}{a_{11}}.$$

$$\alpha = -\frac{a_{12}}{a_{11}}.$$

The expressions of the equations are complicated and it is not feasible to perform analytical study by hand. We use Maple 2016 to conduct numerical search. First, we choose some values for mass ratio then use that to get the corresponding values of d. The result is shown in the table (4.2). Moreover, we use implicit plot to identify the graph of the expression on $d\alpha$-axis. From the graph 4.4 we can conclude that for any given mass ratio there are always three different values of d to make the configuration central.

Table 4.2: Values of d for given mass ratio α

Mass ratio	values of d_1	values of d_2	values of d_3
0.1	0.3310656186	0.7138990769	1.610111945
0.4	0.5102811859	0.7948610330	1.636490416
0.8	0.5741148859	0.9363384964	1.680226573
0.9	0.5809058720	0.9691657272	1.692791850,
1.0	0.586162697428852	1	1. 70601098361671
1.2	0.5937036998	1.055418880	1.734349219
1.5	0.6007819199	1.123675708	1.781168803
2	0.607361742836975	1.2050333883483	1.86775101074181
4	0.616271664387084	1.33589180236931	2.22931621134475,
8	0.620301733791366	1.39119110583675	2.79770691531376

□

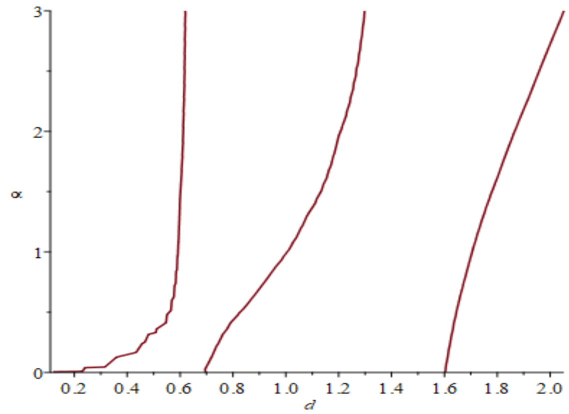

Figure 4.4: Graph of $\alpha = -\frac{a_{12}}{a_{a11}}$ over $d\alpha$-axis. It is observed that there exist three distinct d for each given mass ratio α, which means there are three different central configurations for each given mass ratio.

Chapter 5

Conclusion

We studied the existence of the central configurations for the planar six-body and eight-body problems. In the six body configurations, we showed that there exist some central configurations when both the triangles share a common centroid. We found new CC when the triangles do not share a common centroid. We proved that the six body configuration is not a central configuration if the triangle Δ_{123} is above or below the Δ_{456}.

We used the same approach to study the eight body problem. In eight-body configurations, we showed that the masses on each square should be equal i.e $m_1 = m_2 = m_3 = m_4$ and $m_5 = m_6 = m_7 = m_8$ to form a central configuration. We found out the existence and non-existence of CC when we varied the ratio of size of two squares. We also showed the numerical observation that there are three different twisted central configurations for the eight-body problem for any mass ratio.

There is still a lot that can be done for this project. In the future, we hope to make our proof, especially theorem 6 more rigorous by using analytical techniques to prove it. We also want to write a paper for a possible journal publication.

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Bibliography

[1] S. Smale, Mathematical problems for the next century, in Mathematics: Frontiers and Perspectives, ed. V. Arnold, M. Atiyah, P. Lax, and B. Mazur, American Math. Soc. 2000, 271-294.

[2] I. Newton, Philosophiae naturalis principia mathematica (London: Royal Society Press, 1687).

[3] Two-Body Problem. 17 Mar. 2020, en.wikipedia.org/wiki/Two-body problem.

[4] L. Euler, Novo Comm. Acad. Sci. Imp. Petrop. 11, 144 (1767).

[5] J.L. Lagrange, Prix de lacademie royale des Sciences de paris 9, 292 (1772).

[6] J. H. Poincare, Acta Math. 13, 1 (1890).

[7] S.Smale,Mathematical problems for the next century, the Mathematical Intelligencer,Volume20, Num- ber 2, 1998.

[8] Y. Hagihara, Celestial Mechanics, Volume I, Dynamical Principles and Transformation Theory, The MIT Press, 1970. ISBN 0-262-08037-0.

[9] M. Hampton, Stacked central configurations: new examples in the planar five-body problem, Nonlinearity 18 (2005), no. 5, 2299–2304.

[10] M. Hampton and R. Moeckel, Finiteness of relative equilibria of the four body problem, Invent. Math. 163 (2006), 289–312.

[11] X. Yu and S. Zhang, Twisted angles for central configurations formed by two twisted regular polygons, Journal of Differential Equations.

[12] X. Su, T. An, Twisted stacked central configurations for the spatial seven-body problem. J. Geom. Phys. 70 (2013), 164–171

[13] Yu, Xiang, and Shiqing Zhang. "Central configurations formed by two twisted regular polygons." Journal of Mathematical Analysis and Applications 425.1 (2015): 372-380.

[14] Z.Xie, G.Bhusal and H.Tahir, Central configuration in the planar 6-body problem forming two equilateral triangles, Journal of Geometry and Physics (2020).

[15] A. Albouy, and V. Kaloshin, Finiteness of central configurations of five bodies in the plane, Annals of mathematics (2012): 535-588.

[16] S. Zhang, and F. Zhao, Spatial central configurations with two twisted regular 4-gons, Journal of Applied Mathematics 2013 (2013).

[17] C. Deng and S. Xia, Twisted stacked central configurations for the spatial nine-body problem, Zeitschrift angewandte Mathematik und Physik 66.4 (2015): 1329-1339.

[18] A. Fernandes and L. Mello, On stacked planar central configurations with five bodies when one body is removed, Qualitative theory of dynamical systems 12.2 (2013): 293-303.