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# Using Modern Portfolio Theory to Analyze Virgil's Aeneid (or Any Other Poem)

David Patterson

# USING MODERN PORTFOLIO THEORY TO ANALYZE

# VIRGIL'S AENEID (OR ANY OTHER POEM)

by

David Eugene Patterson

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Approved by:

Dr. Haiyan Tian, Committee Chair
Dr. Jiu Ding
Dr. John Perry

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| Dr. Haiyan Tian | Dr. Bernd Schroeder | Dr. Karen S. Coats |
| Committee Chair | Director of School | Dean of the Graduate School |

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ABSTRACT

This paper demonstrates that it is possible to use mathematics to study literature as it has been used to study the social sciences. By focusing on mathematically defining economic and literary terms, it can be shown that the underlying mathematical structure behind key concepts in economics and literature are analogous.

This opens the possibility of applying economic models in literature. Specifically, it is demonstrated that the economic mathematical model of modern portfolio theory can answer long standing questions around the Roman epic Aeneid by Virgil. The poet died before completing his poem. The relative completeness of the books of the Aeneid are quantitatively determined by applying Markowitz's modern portfolio theory.

Modern portfolio theory optimizes the weights of the securities in a portfolio to maximize the expected return and minimize standard deviation in financial models. The literary tradition around the Aeneid records that Virgil's style sought to create a uniform structure while heightening the drama. Thus that optimization for the poet would correspond to the conventional optimization in modern portfolio theory. The extant books can then be compared with the minimum variance portfolio using Capital Asset Price Model (CAPM) and its Beta factor. The books with the Betas further away from 1 are more incomplete.

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# Notation

| | |
|---|---|
| $\rho_{ij}$ | Correlation coefficient of the $i$-th and $j$-th securities |
| $\mathbb{E}(X)$ | Expected value of $X$ |
| $\mu_i$ | Expected return of the $i$-th security |
| $\prod_{i=1}^{n}$ | Indexed Cartesian product from 1 to $n$ |
| $\bigotimes_{i=1}^{n}$ | Indexed tensor product from 1 to $n$ |
| $(\Omega, \mathcal{F})$ | Measurable space |
| $(\Omega, \mathcal{F}, m)$ | Measure space with measure $m$ |
| $(\Omega, \mathcal{F}, \mathcal{P})$ | Probabilistic model |
| $(\Omega, \mathcal{F}, P)$ | Probability space with probability $P$ |
| $K_i$ | Return function of the $i$-th security |
| $A \triangle B$ | Symmetric difference of sets $A$ and $B$ |
| $\sigma_i^2$ | Variance of return of the $i$-th security |

# Chapter 1

# INTRODUCTION

The opening sentence of M. L. West *Greek Meter* makes the observation that, "Meter is Measure." He continues,

> What is measured, and how strictly, is different in different poetries. It may be syllables, or accents, or assonances of one sort or another; it may be units of meaning. But fundamental to all poetic systems is the arrangement of language into segments.... [66]

This truism makes the very study of numerical patterns in poetry universally possible. Anyone can take any poem composed in any culture at any point in history and discover some numerical pattern. Even the poetry of illiterate and innumerate societies lies ready for some sort of numerical extraction because of the very nature of the art, and therein lies the problem [67]. There are too many things that can be counted in any poem, and every one of those things is counted using real numbers.

This is the existential crisis of the digital humanities[1] [25]. One must not underestimate the power that modern computing has brought to the study of the humanities. The mere proliferation of documents on digital archives has dramatically expanded the abilities of scholars to do research.

For example, today, any professor (or any graduate student) can look at high quality digital photographs of all of the ancient manuscripts in the Vatican Archives. These images have the identical chronographic information as the originals [6]. This provides a universal opportunity to do research that only the most privileged would have had a few years ago. However, what is amazing about this is not that Vatican does this, but that hundreds of institutions do the same. This is transformative.

Computers have done much much more for the humanities than facilitate older methods of research. Modern searching and sorting as well as data collection have revolutionized humanities research. Yet, with every positive new route that computing has opened to the

[1] Using statistical and probabilistic methods in the humanities predated the widespread use of computers, and early researchers using these techniques, of course, did not refer to their work as the digital humanities. However, this early work did build a professional foundation for digital humanities [61].

humanities a plethora of dead ends—most of which involve using statistics and probability or extracting simple patterns [25].

The simplest way to use computers is to search for numerical patterns, but this is not always productive. Counting the number of full stops in a poem [13] or performing a linear regression on the pattern created by location of spondees[2] in Latin Hexameter [18] can provide only so much enlightenment. The purpose of scholarship is to expand human knowledge, but a good portion of what is done in digital humanities cannot. Furthermore, some numerical work can become a barrier to knowledge by not adequately taking into account the non-digital work that came before [5].

To search for a numerical pattern in a poem is equivalent to searching for a numerical pattern on the real number line given a finite set of real numbers. This is certainly doable, but just because a numerical pattern (or worse a single example of a numerical pattern) can be found within a poem, it does not necessarily follow that that pattern elucidates anything about that piece of poetry.

Even the smallest poem presents a number of possible numerical patterns that is astronomical. To "find" meaning in one of these patterns muddies the waters of literary criticism. Take William Carlos Williams' poem "A Red Wheelbarrow," for example:

> so much depends
> upon
>
> a red wheel
> barrow
>
> glazed with rain
> water
>
> beside the white
> chickens [69]

In this poem there are 4 stanzas, 8 lines, and 16 words. These numbers can be taken as integers for a Diophantine equation

$$4x + 8y = 16.$$

[2] A spondee is a metric foot with two long (or stressed) syllables.

A solution for $(x,y)$ is clearly $(2,1)$. One could stop here and say that the William Carlos Williams had intended this Diophantine equation because solutions can refer to the plural chickens and the singular red wheelbarrow, but the pattern gets more interesting if one continues.

From Bézout's Identity all of the solutions to this equation are $(2+2k, 1-1k \mid k \in \mathbb{Z})$; thus the tuple $(42,19)$ is also a solution [58]. It takes 42 days for a broiler chicken to mature and there are 19 Billion chickens in the world [43, 42]. This is the point where, if this were intended as an example of Williams' use of mathematics in his poetry, that the hapless scholar would claim that William Carlos Williams definitely used this Diophantine equation to structure his poem. Could this be just a coincidence?

It probably is. The difficulty is that one cannot prove that William Carlos Williams did not shape "A Red Wheelbarrow" with a Diophantine equation in mind. The given example is far from convincing or inspiring. Other ventures attempting to manipulate literary texts to "find" mathematical patterns have had more traction, and through similar deficiency in the "proof" that Williams uses Diophantine equations to structure "A Red Wheelbarrow" have discredited the use of mathematics in studying literature. The most notable example of this is George Duckworth's work finding examples of the golden ratio in the section[3] lengths of Virgil's *Aenied* [11].

Duckworth was neither the first nor the last Classicist to explore the golden ratio in Virgil's epic [46, 41]. His work faced scathing criticism from fellow Classicist immediately after being presented. His work diverged considerably from the norm of how research in Classical Literature was done in the era that he was writing [17]. In the 1980's two mathematicians Fischler and Markowsky published articles pointing out how un-mathematical Duckworth's exploration is [22, 46].

Without going far into Duckworth's convoluted theory, suffice it to observe that Duckworth finds ratios of number of lines of two sections where the longer divides the shorter, i.e. there exists a ratio $r \in [0,1]$. Duckworth claims that

$$\frac{1}{1+r} = \frac{1}{\phi}$$

where $\phi$ is the golden ratio [17, 22, 46]. Fischler and Markowsky observe the expected value of the previous fraction taking all possible values for a variable $x \in [0,1]$ is

$$\int_0^1 \frac{1}{1+x} dx = \ln(2) \approx \frac{1}{\phi}.$$

[3]Duckworth uses the term section, and how he divides them varies considerably. Sections, as defined by Duckworth, have one quality in common—they are self contained. He uses verse paragraphs, poetic cola, and even Books, as well as other divisions. His usage is somewhat ambiguous, but he is searching for a numerological pattern that he believes is an underlying pattern for the whole epic.

Thus, for any random set of numbers satisfying the condition of $r$ it should be possible to approximate the inverse of the golden ratio [22, 46]. This would have been noteworthy observation if it hadn't been made in the first book reviews of Duckworth's *Structural Patterns and Proportion in Vergil's*[4] *Aeneid* [24, 11]. Duckworth wasn't engaging in mathematics; he was trying to reconstruct Roman numerology [17].

In his conclusion Duckworth acknowledges that his theory about Virgil's use of the golden ratio is extremely controversial. He writes,

> Many lovers of Vergil will find all this very disturbing and will be reluctant to admit that Vergil counted lines and arranged his material on the basis of mathematical proportion. On the other hand, such a procedure explains the statement in the Donatus-Suetonius Life that Vergil devoted especial attention to the study of mathematics and that he composed the study of mathematics and that he composed the *Aeneid* in short and separate units working from a prose outline [17].

However, far from wishing to sanitize his theory for the credulous, he drives home the point that mathematical proportions are vitally important for the aesthetics of all the other arts [17, 16]. These norms, hardly being recent innovations like epic poetry itself, were inherited from the Greeks and Romans [27, 68]. Forcing his reader to ask the rhetorical question: why would poetry be any different? To which he implies–it isn't. For Duckworth the important thing is not the golden ratio, but the existence of quantitative values that could be extracted from the text to better elucidate its meaning and the author's desires [17, 16]. The golden ratio was simply the quantitative tool that he found that produced some results.

Duckworth's argument has some modest traction because its greatest weakness is also its greatest strength [2]. This makes it as hard to disprove as it is to prove. As shown in the example of William Carlos Williams and Diophantine equations, this kind of reasoning is the difficulty proving a negative. Which side requires the burden of proof? In a field like Classics proof means something very different than it does in mathematics or physics, through necessity. Using conventional means it is simply impossible to know Virgil's intentions that weren't recorded by ancient grammarians [63, 21]. This ambiguity can be presented to both support and undermine Duckworth's theory [63, 21, 2]. In this line of reasoning or similar lines of reasoning it becomes a question of whether it is worth the effort to follow the logic. When I was an undergraduate taking my first course in Ethics the

[4] The English form of the name of the Roman poet Publius Vergilius Maro can be spelled either Virgil or Vergil. The later was more common in the first half of the 20th Century the later spelling is more common today.

professor gave a wonderful thought project highlighting that the inability to disprove does not prove. He said, “One cannot prove that Oliver Cromwell did not have nuclear weapons, but it is highly unlikely that he did.”

Outside of the study of poetic meter, extracting mathematical patterns from poetry is a colossal waste of time. It is unknown if William Carlos Williams even knew what a Diophantine equation was, much less found any interest in them, and a similar objection can be made for a host of other spurious claims about literature that have come out of digital humanities. Far from helping literary criticism this type of work actively hinders it. A good rule of thumb when analyzing the patterns in literature is if there is no linguistic or known poetic constraint creating a pattern it is best to consider that pattern accidental [38]. However, the whole point of literary criticism is to explore the broader meaning of a piece of literature [37]. A technique in literary criticism is only interesting if it elucidates some dark corner of the mind of the reader or the mind of the author [37]. The previously given constraint seems to prohibit mathematics from being a tool for literary analysis, but this is not necessarily true. Here enters economics.

Economics, per se, is not mathematics, but it is a field that heavily relies upon mathematics to quantify human preferences and human behavior. This can be seen in the most fundamental concept in economics—price. What is a price? A price is a quantification of the amorphous quality value. Value has no intrinsic number; it is only the subconscious appreciation of utility. A price, especially in a fluid market, quantifies unconscious human preference. Not all, but most mathematical methods in economics take price and derive meaning from it [8].

The lived practice of economics and finance does on a daily basis what purportedly cannot be done in literature. It quantitatively measures human preferences. What if literature had something analogous to a price? It would not essential for this object to be exactly the same as a price. It only needs to be a numerical quantity that has a value influenced by the preferences of an author. Mathematics could be used to elucidate aspects of this field that are impossible to explore by conventional means if such a quantity could be found. Its apparent absence has made literature a field best studied with dialectical methods.

What is a price? Among other things the price of an object can be viewed as a real valued function mapping the underlying value of that object to a number. Is there a real valued function in a piece of literature? Yes. One example is the word count of a narrative. A word count maps the partitions of words to a natural number. Another example is the number of lines of a poem. However, if one wants to measure human preferences in literature, human preferences must act upon a real valued function that can be found in literature. The former depends upon whether the author feels that it is complete, among other constraints. The number of lines of a poem may or may not reflect the preferences of a poet.

The number of lines of a smaller poem is determined by convention. A sonnet must have 14 lines to be a sonnet. Conversely, a longer poem tend to have a number of lines determined exclusively by the preferences of the poet. T.S. Elliot's "Wasteland" has 433 lines because that is the number of lines it had when the poet felt that the poem was complete. This does not diminish the importance of poetic features. The syllable count of a line directly effects the number of words that can be contained in any given line. Likewise, stanza structure will determine the number of lines not just in the stanza itself but also in the Book or Canto using the stanza. Poetry is a game where the poet tries to express emotions and ideas through the sounds and meaning of words while submitting to artificial constraints [37]. Poetic preferences are not expressed as arbitrarily as in prose. Even the freedom found in modern and contemporary poetry is not antinomianism [37]. The rules may be isolated to an individual poem but in being so they are all the more important. It is the very structure of poetry that forces the poet to actively express his or her will by playing within the rules.

Poetry is not the only human activity that demands that will be expressed within the framework of a set of rules. A sports game and a opera are each an embodiment of human will through actions bounded by consciously followed rules. These are not the only two examples. Much of human activity does this. From traffic to markets, human will is expressed through conventions that limit action. Though some activities are more inclined to conscious will than others. The desire of any given driver can easily be assumed to get from point A to point B with the least hassle possible because the overwhelming desire of most drivers at any given point in time is just that. Trading in a market has a more limited avenue of expression and a wider range of desires to express. The will of a trader is given through buying or selling. In its most simple form the an act of buying or selling can mean that the buyer/seller has a need or a surplus, but it can imply a great deal more. Buying or selling a stock at a price is an expectation of how that security will perform in the future as well as an evaluation of the issuer of that stock. The act can also be contingent upon the personal needs of the buyer or seller or his or her view of the behavior of the market as a whole. The root causes of an individual act in a market can be quite complex.

Taking the previous observations to their logical conclusion would imply that, theoretically, it could be possible to use mathematics in literary criticism. The natural response to this assertion is pure skepticism. Literary criticism has been immune to mathematicization for twenty-five hundred years; it hardly seems plausible that could change. Nevertheless, literature is not only field to have evolved in this manner. The inspiration for applying mathematics to literature comes from the history of economics. Before the Twentieth Century economics was regarded as a field of philosophy, and use of mathematics in the field was disparaged [1]. In the first decades of the Twenty-First Century, most notable economists are also mathematicians. This evolution in economics will be discussed in more detail in the next chapter.

To show that there exists a mathematical method from economics that can be used in studying literature productively a single example shall be selected from each. Given that the number of words in a logically self-contained piece of text is highly dependent upon the idiosyncrasies of the language in which it is being written, using the number of lines in a verse paragraph of a poem as the price equivalent would give a random variable that is more independent of factors external to the preferences of the author than a word count. Also since it has been suggested that there is an underlying similarity to the length of a narrative and a market price, looking at a mathematical model of pricing would be a good start.

This, however, would be an abortive start. Although narrative length has the freedom that price does, it is too ambiguous. Narrative is used here to mean a complete story. The length of text can vary considerably depending upon the story under consideration. For example, the entire *Iliad* is a story. It is long one. Likewise, Homer's description of Hector's infant son crying when he sees his father in the armor of Achilles is also a story. It is quite a bit shorter, and significantly less intricate. The latter is contained in the former. The two lengths are dependent upon too many external factors. Any conclusions drawn between them would be trivial at best or, more likely, meaningless. Another quantity independent of meter must be chosen. A good alternative to blocks of narrative are verse paragraphs. A Verse paragraphs is the smallest self-contained rhetorical unit in a poem. A verse paragraph is composed of cola and clauses which cannot stand independently.

A poem is composed of a finite number of verse paragraphs. They are relatively easy to define and they are disjoint. For a longer poem that is divided into cantos or books, its books or cantos each have a finite number of verse paragraphs. Most importantly, a verse paragraph has a set of lines that can be counted. The line counts of each subsequent verse paragraph in a book in a longer poem will vary. The number of lines in verse paragraphs can stand in for a price if a poem is taken as a security.

A natural choice of a mathematical model is modern portfolio theory. Modern portfolio theory has two primary parts, viz. portfolio optimization and CAPM (Capital Asset Pricing Model) [9]. If a longer poem is chosen, like Virgil's *Aeneid*, which has verse paragraphs divided among multiple books, portfolio optimization can be used to determine what the poet felt the expected verse paragraph length and variance of verse paragraph lengths of an ideal book should be. The second part, CAPM, has a Beta factor that can be used to compare the behavior of each extant book with the behavior of the ideal book found in portfolio optimization.

This type of analysis can be of interest in studying the *Aeneid* because Virgil died before completing the poem [52, 63]. It has been impossible to know how close Virgil was to finishing his epic. The stories surround Virgil's final years and his intent for the *Aeneid* are incomplete at best and shrouded in legend at worst [21]. The incompleteness of the *Aeneid* is important to the study of the text [63]. There are few certainties known about the composition of any piece of ancient literature; this is one. Every study of the *Aeneid* that has been written on it in the last 2000 years has addressed that the poem is incomplete, but what that means for the text has been unknowable. This is because no one recorded how much Virgil wanted to do to each book of the *Aeneid* when he died. However, his general preferences for poetry were written down. Modern portfolio theory can create determine the expected difference and variance of verse paragraph lengths of a book that would satisfy Virgil's general preferences for poetry. These values can then be compared with the expected differences and variances of verse paragraph lengths of each of the extant books. Specifically, having expected differences and variances of verse paragraph lengths of an optimal book will allow for the calculation of Beta factors[5] for each book. The Beta factor of a book will show how closely it mirrors optimal behavior, i.e. how much this book meets Virgil's preferences.

[5] The Beta factor will be described in detail in Chapter 4.

# Chapter 2

## HISTORY BEHIND MODERN PORTFOLIO THEORY

Modern portfolio theory is a system of quantifiable portfolio optimization. Introduced in 1952 by Harry Markowitz in his seminal article, "Portfolio Selection," modern portfolio theory was the first successful attempt to apply a rigorous mathematical approach to investing [44, 55]. This method stood in stark contrast to traditional portfolio theory. The older method allocated the proportions of various investments in a portfolio on preset professional models and qualitative analysis of the institutions issuing the securities.

The development of modern portfolio theory was a watershed moment in finance, and the impetus behind using modern portfolio theory in literature can be found in its history. To capture its importance, Mark Rubinstein, in "Markowitz's 'Portfolio Selection': A Fifty-Year Retrospective," describes this innovation in flowing language,

> Near the end of his reign in 14 AD, the Roman emperor Augustus could boast that he had found Rome a city of brick and left it a city of marble. Markowitz can boast that he found the field of finance awash in the imprecision of English and left it with the scientific precision and insight made possible only by mathematics [55].

Comparing any living academic, even a Nobel Prize winner in his nineties, with the first emperor of Rome, may be a bit hyperbolic; nevertheless, Markowitz's importance in finance is truly monumental. Before him financiers were equipped only with the tools of professional gamblers; his work made finance scientific [55]. Finance used to be dominated by men who had degrees in humanities (if they had a college education at all). Markowitz's work created a field that would be intellectually stimulating to scientists and engineers like James Simons and Mike Bloomberg. Rubenstein's praise of Markowitz might seem excessive to someone outside of the field, but this quote shows how highly people in finance regard Markowitz.

Finance is, broadly speaking, the study of diversification. Markowitz didn't just show how to select the most advantageous weights in a diversified portfolio; he mathematically proved the utility of diversification of a financial portfolio [45]. Markowitz wasn't nearly the first to study diversification mathematically.[1] Empirically, diversification reduces financial

[1] For example, Daniel Bernoulli proved in 1738 that given a number of investments with uncertain expected

risk and improves return, but this does not necessarily follow [45]. Assume that a producer wants to maximize the return on all money invested. Return will be redefined later to fit the assumptions of modern portfolio theory. For simplicity, return is defined here as

$$return = \frac{price - cost}{cost}.$$

Given a selection of commodities, each priced at the equilibrium price,[2] it would be irrational for the producer to invest in making all of the commodities. The producer must select the single commodity with the highest individual return to maximize his or her return on the entire investment. A certainty of an optimal pricing guarantees a return. Markowitz's first stroke of genius was to recognize that conditions effecting equilibrium prices can change quickly and randomly; thus all future prices are inherently uncertain [44, 45]. All returns have some risk.

In a world where economists were pursuing, with newfound zeal, the certainty that can only come from comprehensive and accurate mathematical modeling, Markowitz embraced uncertainty. His train of thought was so unique that his first paper on modern portfolio theory seemed to come out of nowhere [55]. Yet, modern portfolio theory was a natural extension of the penchant for mathematics that swept through economics in that era. Markowitz's paper was published only six years after Paul Samuelson's *Foundations of Economic Analysis*. It was novel when first presented, and its simplicity has made it robust. Modern portfolio theory has evolved considerably in the intervening decades, but the theory is still built upon Markowitz's foundational work. This itself is amazing. Furthermore, Modern portfolio theory was invented very early in the development of mathematical economics. One would expect only the barest kernel of the original work to remain. Nevertheless, Markowitz's work has stood the test of time. In new fields, inspiration gives birth to concrete ideas which are gradually replaced by more comprehensive abstractions. Reading contemporary economics books it is easy to forget how early it really was. Before the work of John Hicks, Franco Modigliani, and Paul Samuelson economists did not view their profession through the lens of mathematical models.

It cannot be denied that economists used mathematics or that mathematicians created models of economic phenomena before the twentieth century. Between Cournot's Duoh-

returns diversification can reduce variance and increase expected return [4]. But Bernoulli's demonstration that the geometric mean of a set is inversely proportional to its variance is a far from proving that diversification is an essential strategy for maximizing return and minimizing risk for all financial portfolios.

[2] The market equilibrium of a commodity is an ordered pair of price $p$ and quantity $Q$ such that

$$(p_S, Q_S) = (p_D, Q_D),$$

where $S$ is supply and $D$ is demand.

poly,[3] and Walras' Law[4] the Marshallian cross,[5] not to mention the Malthusian Trap, there is enough evidence that nineteenth century mathematicians enjoyed modeling aspects of economics. However, what they did not do, that Samuelson did, was construct a set of models that could be used like scientific laws in economics.

Almost all of the Classical economists (and their dissenting contemporaries) were educated in philosophy, and most of them wrote extensively on other branches of philosophy. Adam Smith, J. S. Mill, and Jeremy Bentham were predominately ethical philosophers. Thomas Malthus was a polymath who briefly taught philosophy as a Fellow at Jesus College, Cambridge. The only major exception was David Ricardo, who never earned a degree.

The acceptance of mathematics in economics changed over the subsequent two centuries after Malthus but not very quickly. The most notable proponent of using mathematical modeling before Paul Samuelson was Alfred Marshall. He was one of the leading economists of turn of the Twentieth Century and an early proponent of Neoclassical economics. He is, now, best remembered for association with John Maynard Keynes, but his work has had a indelible influence in economics. For instance, he invented the supply and demand curves. Marshall's work divorced economics from philosophy making it a more a scientific subject.

Like Maltus, Marshall sought to build his theory of economics upon a mathematical foundation, but unlike Ebenezer Scrooge's spiritual father,[6] Marshall's theories found universal approbation in the field. It might have been reasonable to place the Marshall as the beginning of the mathematical revolution in economics rather than Samuelson. That is what Samuelson did [1]. This paper follows the contemporary delineation and relegates Marshall to being the Giotto of modern economics. There are a number of reasons why historians have made this decision, not the least of which is the predisposition to portray the First World War a universal turning point. Great movements in human history rarely have definitive beginnings.

Yet, this historical division is not completely arbitrary. Marshall's approach does warrant

[3]The equilibrium in an economy of two firms selling the same commodity.

[4]Walras' Law concerns the equilibrium of supply and demand. Given $k$ number of commodities

$$\sum_{j=1}^{k} p_j(D_j - S_j) = 0$$

where $p$ is the price, $D$ is demand, and $S$ is supply.

[5]This is a graph of supply and demand curves. It was first introduced by Alfred Marshal in *Principles of Economics* in 1890 [47].

[6]Ebenezer Scrooge has become a stock character in literature typifying miserliness. Wealthy villains from Mr. Potter in *It's a Wonderful Life* to Gordon Gekko in *Wall Street* are cast in the mold of Scrooge, but Charles Dickens didn't intend Scrooge to be an archetype of greed. Scrooge was to be the living embodiment of the *homo economicus.* To emphasize how disconnected economics had become from life, Dickens selected whom he believed was the most unrelatable economist to provide Scrooge's ideology—Thomas Malthus [33].

the contemporary classification. Like his predecessors, he shied away from assuming that mathematics could accurately describe economic behavior. Marshall assumed that his models were a necessarily vague theoretical abstraction, while Samuelson saw good mathematical models as an approximation of fact, and this makes all the difference [1].

In many ways the mathematical revolution in economics is still on going. It is still a minority opinion held by some heterodox economists. Although far from the most influential contemporary opponent of the mathematization of economics, Richard Wolff expresses the minority position eloquently in his book *Contending Economic Theories*:

> There is certainly no necessity to use mathematics. Everything in economics can be explained just as clearly and logically without it. However, since mathematics became the preferred language of modern Neo-Classical and [Neo-]Keynesian economics, we use some mathematical language to convey their structures [70].

It is hard to reject Samuelson's main idea more thoroughly than this. Gradations of this view don't simply influence the theories of economists outside the mainstream. Key historical figures that dominate economic theory today, e.g. John Maynard Keynes and Milton Friedman, were overtly non-mathematical.

It is an interesting paradox of history that Keynes has been remembered as being apathetic to use of mathematics in economics [49]. He was a mathematician. Keynes read mathematics at King's College Cambridge, placing high in that University's prestigious Mathematical Tripos. Economics was simply an undergraduate hobby that he maintained into his career as a civil servant in the India Office.

Keynes' aptitude for applying mathematical models to solve practical problems in economics intrigued the economist Alfred Marshall, and in 1909 Marshall personally funded a lectureship at Cambridge for Keynes [59]. Keynes had fist met Marshall at Cambridge as a undergraduate where Marshall was a professor. Marshall's mathematical mind sparked Keynes original interest in economics. The economic tumult of the Interwar period shook Keynes' faith in Neoclassical economics to its foundation[59]. His vehement hostility to the certitude of Neoclassical economics in the late 1920's spilled over into his personal relationships, even souring his friendship with Marshall's widow [30]. His belief in the fundamental irrationality of markets should be seen as a reaction to what he saw as the misapplication of Marshallian principles to macroeconomics.

It has been argued that Keynes never fully abandoned a mathematical approach to economics [49]. Neo-Keynesianism's ability to transform Keynes' core ideas into a strictly analytical formulation suggest that there may be some truth to that. It would be very surprising to see someone take Milton Friedman's ideas and convert them into mathematical theorems. Yet, the consensus opinion differs. It must be sufficient to observe that Keynes' work is less mathematical than either his immediate predecessors or successors. Samuelson himself asserted that his rigorous mathematical approach was a response to Keynes' skepticism about the ability of mathematics to accurately explain economic behavior [1].

Although Keynes worked at a very different time in the field, his reticence towards a strictly analytical approach to economics set a path that many contemporary opponents have followed. Keynes saw irregularities in the market that could not be explained with extant mathematical models [59]. He was correct to abandon the tendency towards mathematization of early Neoclassical economics. The first Neo-Keynesians depended upon mathematics discovered in the 1940's and 1950's to explain the theories that Keynes developed in the 1920's and 1930's [1]. Most non-mathematically inclined economists lack a comparably good excuse. Austrian School or Chicago School economists are simply wedded to an ideological interpretation of the market that is simply not quantifiable [70]. However, other schools of thought, like Behavioral Economics, focus on economic actors that cannot satisfactorily be modeled mathematically.

Samuelson's textbook *Economics* had swept the country in the late 1940's. His assertion that it was possible to account for all or nearly all of the variables of economic behavior redefined what economic certainty means. Markowitz's theory was both a product of the early days of this revolution and, likewise, a response to it. It is important to note that Markowitz does not directly address Samuelson. In his 1991 article expounding the origins of modern portfolio theory he doesn't even mention Samuelson's name [45]. However, without the context of Samuelson's work, the bulk of Markowitz's thought projects would have been inapplicable.

Take the example given above. There are $n$ possible commodities for a producer to produce. Obviously, the return of commodity with the maximum return has a return greater than any other possible weighted average of the $n$ returns. Thus diversification can be sub-optimal. The producer should invest in the one commodity. Nothing could be more simple. However, the unspoken assumption is that market equilibrium can be computed from the supply and demand curves for each commodity. It is an anomaly of post-Samuelson economics that this supposition is tacitly accepted.

Partially this is due to the current state of mathematics. More importantly this a confidence in economic models inherited from Samuelson. Earlier generations of economists would not have shared this approach. To be honest, it would be anachronistic to ask how David Ricardo or Karl Marx would have read the previous equilibrium problem. This is a paradox following from the assumptions of a Marshallian cross. This and its corresponding parts of supply and demand curves and market equilibrium were first defined by Alfred Marshall in the late nineteenth century. However, in similar examples, the Classical economists refused to take the first step and move beyond the broad brush strokes of economic behavior.

Over half a century later, Marshall accepted that specific cases could be determinable, but he intentionally avoided generalization. That had broad ramifications in economics. He was as concerned with uncertainty as his predecessors, but for different reasons. Marshall was less concerned about the fog of economic activity. He navigated circumspectly because of the gaping holes in his profession that he saw were present [47].

By the 1950's such caution had faded into distant memory. Economic uncertainty had become a synonym for inaccuracy and miscalculation. Randomness was regarded as virtually an anomaly plaguing the fringes the economics. Markowitz was brilliant enough to recognize that uncertainty was central to finance and that financial uncertainty could be modeled accurately using probability theory [55]. The first resurrected an antiquated concept. The later connected the latest developments in both mathematics and economics. The breadth and depth of the material that Markowitz drew upon in formulating modern portfolio theory makes his work all the more impressive.

In many ways the section might seem to be an apology of Keynesianism and Neo-Keynesianism, but this apparent bias is through necessity. Markowitz invented MPT in the 1950's when the last vestiges of Neoclassical economics were dying out in the West. His idea was made consciously in opposition to Neoclassical economics. More importantly Markowitz specifically says that his ideas were framed in opposition to the implications of economic thought on finance [55]. The world was Keyesian. It cannot be forgotten that the impetus for Keynes' theories was the failure of Neoclassical economics to explain the First World War and the interwar period. The Second World War and Keynes' dominance afterwards set the stage for an era in which Richard Nixon could nonchalantly say that "We're all Keynesians." The fact that in the nearly seven decades since Markowitz invented MPT Neoclassical economics has made a resurgence, becoming the dominant economic theory. It would be more descriptive to distinguish between contemporary Neoclassical economics and the earlier Neoclassical economics of Alfred Marshall and GE Moore. This division is not done, and this paper is not the place to coin the neologism of Neo-Neoclassical economics.

Contemporary Neoclassical economics has answered most of the questions posed by Kenesianism. It has superseded Keynesian economics because using its assumptions simplify many mathematical models. Ironically this is a legacy of the Soviet Union. Neoclassical economics never died in the USSR [39]. After the Revolution Russian economists overlaid a Marxist veneer over their preexisting theories [39]. When Neoclassical economics was in full retreat in the West, Stalin had come to power. Challenging the status quo became somewhat dangerous. Even Kondratiev was murdered in one of Stalin's purges. Soviet economists made progress by developing mathematical models for Neoclassical economics, rather than challenging key assumptions which is what we were doing in the West [39]. The current favor of Neoclassical economics is built upon a desire to take advantage of the predictive ability of advanced mathematical models that were built upon Soviet work and a sea-change in what the core ideas of economics are [3].

# Chapter 3

# HIGHER MATHEMATICS AND MODERN PORTFOLIO THEORY

## 3.1 The story begins

The story of modern portfolio theory begins with a security. The term security is one that is not common outside of finance; yet, almost everyone is familiar with the object. A security is a financial instrument that has a market price [14]. Stocks, bonds, derivatives, ETFs, are Mutual Funds are all securities. Anyone who has a 401(k) or follows the markets has an intimate knowledge of what a security is and how it behaves.

To continue with the mathematical narrative, let there be a security $\mathbb{S}$. Furthermore, let its price at time $t$ be $S(t)$. For simplicity assume that there are only two values of $t$, viz. $t = 0$ and $t = 1$. There are two prices: the current price $S(0)$ and the future price $S(1)$. Stepping slightly back from this nascent model and returning to the financial reality shaping the mathematical representation, attributes of current price and future price become clear.

In practice, prices of securities are positive real numbers. It is impossible to find any security that has ever sold for a negative amount of money. Many sellers have had to face the annoyance of the value of owning a security that was worth nothing, but no seller has ever been forced by the market into a place where he or she would have to pay a buyer to take a security. Correspondingly, prices of securities lie on the real number line. There are no multi-dimensional prices. One price may be higher (or lower) than another price, but in the world of securities one will never find two prices that are orthogonal to each other. Thus it can be inferred that $S(t) \in \mathbb{R}_{\geq 0}$.

The current price is the price at which security $\mathbb{S}$ is selling on the market. It is known and fixed by the market. The price may remain constant only for a moment, but for that moment it is a constant. Thus for a constant $k \in \mathbb{R}_{\geq 0}$ the current price $S(0) = k$. In a similar fashion the future price is unknown. Just as the next number on the roulette wheel is unknown and unknowable, it is impossible to know the exact value that the future price will take. The future price of any security is random, and for the security $\mathbb{S}$ the future price $S(1)$ is a random variable. This leads to the question: What is a random variable?

### 3.2 The big question in probability

This is a query far easier to ask than to properly answer because the answer rest deep within probability theory. The deceptive phantasm of simplicity created by the dull hyper-macho world of finance drops. Simple this is not. Standing bare, a complex reality is reveled. A reality built upon Kolmogorov's axioms, spinning with probability fields, Lebesgue integration, and Markov chains.

It is necessary to furbish a description the foundations of axiomatic probability as well as the rudiments of measure theory before defining what a random variable is. The world of finance is traded for the world of probability after only after examining the nature of the current and future prices. The looming rabbit hole, nevertheless, is bounded. The full breadth and depth of financial mathematics is beyond the scope of this paper. In due course this paper shall return back to the reality of finance after a brief sojourn.

Probability models the possible outcomes of an experiment or game [56]. Let $\omega_1, \omega_2, \ldots, \omega_N$ be a collection of $N$ possible outcomes of an experiment $\mathcal{E}$. It doesn't matter what the experiment is or what the outcomes are. The could be anything. The experiment could be the single spin of roulette where the outcomes are the numbers 0 to 36, or, with the same experiment, the possible outcomes could be the colors red and black. The experiment could be the health of a corporation over a period of time and the outcomes could be actions and external influences upon the corporation. The experiment could also be the growth rings on a unicorn's horn and the possible outcomes could be environmental conditions that effect the density of the rings. The physical (or imaginary) nature of the experiment and the outcomes has no bearing on the probability mode [32].

The set of all possible outcomes is a sample set or sample space. For the given hypothetical experiment $\mathcal{E}$ let the finite set

$$\Omega_0 = \{\omega_1, \omega_2, \ldots, \omega_N\}$$

be the sample space for the probability model. As an aside, a sample space does not need to be a finite set. A sample space can be infinite, but it does need to be countable. For example, the set of all real numbers cannot be a sample space.

A simple example of a sample space is the sample space of the experiment of a single flip of a single coin. Let $\Omega_{coin}$ be such a sample space. Hence,

$$\Omega_{coin} = \{H, T\}$$

where $H$ is heads and $T$ is tails. In this example $\Omega_{coin}$ only has two possible outcomes.

Using only the tools of classical probability this would be the full extent of theoretical examination both of this example and the original sample space $\Omega_0$ [32]. More concrete information would need to be given, in particular the probability for each outcome [65]. Taking an example like the coin toss, these probabilities can be determined by experiment. Logical conclusions could be used to flesh out other facets of the model. Some analytical methods did exist in probability before the 20th Century, but for the most part these methods depended upon relationships that would be discovered empirically.

A full history of the development of probability is beyond the scope of this paper; however, must be observed that historically the field of probability was quite different that it is today. Before axiomatization the study of probability resembled the study of subjects like physics and chemistry than pure mathematics [57]. Looking back on mathematical history, this little detail is easy to forget since most major advancements in probability were achieved by mathematicians, e.g. Cardano, Bernoulli, Laplace, and Chebyshev [57, 34]. But the fact that many mathematicians enjoyed the study of probability as well no more established the field within mathematics than the coincidence that many mathematicians were also astronomers made astronomy a branch of mathematics [57, 7].

To be fair classical probability did pose questions that needed a mathematical framework [65]. The lack of such a theoretical foundation in classical probability did propel some mathematicians in the late 19th Century, most notably Chebyshev, to begin the search for an acceptable set of axioms for probability [57]. By the turn of the century this became one of the most interesting problems in mathematics earning a place as Hilbert's Sixth Problem [57]. Yet, in his lecture to the Second Mathematical Congress David Hilbert posed the problem of the axiomatization of all physical disciplines that depended heavily upon mathematics, including probability in the list as a partner to mechanics and physics [71]. This underlined the fact that before Kolmogorov's axioms mathematicians did not view probability as a full-fledged discipline of mathematics.

Axiomatic probability still requires a probability function to be a given for an experiment, but it need not be said that defining a function is a far different proposition that defining a set of real numbers [57]. Classical probability simply does not allow for the same level of abstraction that axiomatic probability does [57]. Returning to the example of the single coin toss, centuries of empirical observations have marked this subject as the quintessential example of even odds. A comparable conclusion cannot be drawn about the sample set $\Omega_0$.

Assuming that the coin is a fair coin the likelihood of either $H$ or $T$ occurring will be the same. Thus, if $p_H$ is the probability of $H$ and $p_T$ is the probability of tales,

$$p_H = p_T.$$

For the case of a single coin toss it can be further inferred that

$$p_H = \frac{1}{2} \qquad p_T = \frac{1}{2}.$$

Hence,

$$p_H + p_T = 1.$$

But this begs the question: What is a probability?

In the classical definition, as given by Laplace, "The probability of an event is the ratio of the number of cases favorable to it, to the number of all cases possible when nothing leads us to expect that any one of these cases should occur more than any other, which renders them, for us, equally possible" [36]. In comparison to the descriptive nature of modern pure mathematics the ambiguity and limited scope of this definition is highly unsatisfactory. This is why Pafnuty Chebyshev and, later, his colleagues in St. Petersburg abandoned the classical definitions of probability, spurring the drive for axiomatization. This project was eventually completed by Kolmogorov in 1933 [57, 34].

Andrey Kolmogorov's brilliant insight was to recognize that a probability was a measure on the interval $[0,1]$. Building upon the work of Henri Lebesgue and Maurice RenÃľ FrÃľchet in generalizing integration, Kolmogorov was able to extend the concepts developed for measure theory to complete the axiomatization of probability [57, 34]. This, of course, requires defining the concept of a measure and the rudiments of measure theory.

### 3.3 Measure theory

The development of measure theory, or rather the need to develop measure theory, was a byproduct of two driving forces behind early modern mathematics, viz. the abstraction of integration and the exploration of analytic geometry [60]. For Greek mathematics it was an unstated presupposition that an $d$-dimensional geometric object was measurable in the $d$-th dimension [62, 29]. The existence of a non-measurable object was simply inconceivable [62]. The limited abstraction of their mathematics prevented the construction of geometric objects that might be non-measurable [29].

Even though the assumption that all geometric objects could be measured in pre-modern Euclidean geometry ultimately proved to be false, it played a profound role in the development of Western mathematics. It is doubtful that Archimedes would have pursued his tentative venture into integration without the certainty that a method could be found that could give measure of the area under a curve. Likewise the same can be said of Napier, Briggs, Newton, and Leibniz in their studies of infinitesimals [7].

With the re-conceptualization of Euclidean geometry as the exploration of Cartesian products of the real number line, the concept of a measure ceased to be self-evident [60]. The general subset $E \subseteq \mathbb{R}^d$ and $d \in \mathbb{N}$ can have too many strange meanings for just anyone to visualize $m(E)$ (or even to guarantee that it exists). Here, after establishing the essential difference between ancient and modern axiomatic mathematics, the preliminary discussions of measure theory must end.

No mathematical object addressed in this paper will be more fundamental than the concept of a field. ZFC[1] shall be assumed throughout. A truly satisfactory definition of a field requires giving a complete definition of abelian groups and algebraic structures. Taking these demands to their logical conclusion would be a process that would end with the beginning of Bourbaki's *Elements*, but given the limited subject matter it is hardly necessary to follow in the footsteps of the Bourbaki Group by starting with first-order logic. Suffice it say that an algebraic structure is an ordered tuple with a set and one or more binary operators [53]. An abelian group is a commutative group [53]. A group satisfies four properties, it is an algebraic structure that is closed, associative, it has an identity element, and every element has a corresponding inverse element [53].

Having put check marks in the most necessary boxes in Algebra, it is possible to proceed with the definition of a field.

**Definition 3.3.1.** (Field). Let $F$ be a set, the operator $+$ represent addition, and the operator $\cdot$ represent multiplication. The algebraic structure $(F, +, \cdot)$ is a *field* if and only if:

i. the algebraic structure $(F, +)$ is an abelian group,

ii. the algebraic structure $(F \backslash \{0_F\}, \cdot)$ is an abelian group, where $0_F$ is the additive identity, and

iii. Multiplication distributes over addition [53, 54].

Measure theory like probability is built upon a set. Let $E$ be that set. Unlike our previous sample space $\Omega_0$, $E$ can be finite or infinite. Taking the limitations of ZFC on sets for granted, there are no other restrictions on $E$ than fact that it divides the universe into two parts, objects in $E$ and objects in its complement $E^c$. The concept of measure requires a few more concepts than a simple set. First and foremost, it is necessary to define what is measurable. Before defining a measurable space it is necessary to define a $\sigma$-algebra. First, the definition of an algebra of sets is as follows.

[1]ZermeloâĂŞFraenkel set theory with the Axiom of Choice

**Definition 3.3.2.** (Algebra). Let $E$ be set. A collection $\mathcal{A}$ of subsets of $E$ is an *algebra* over $E$ if the following properties are satisfied.

i. The unit $E$ is in $\mathcal{A}$. ($E \in \mathcal{A}$)

ii. $\mathcal{R}$ is closed under complement relative to $E$. ($\forall A \in \mathcal{A} : C_E(A) \in \mathcal{A}$)

iii. $\mathcal{R}$ is closed under union. ($\forall A, B \in \mathcal{A} : A \cup B \in \mathcal{A}$) [56]

For an algebra $\mathcal{A}$ over a set $E$, this definition implies that $\mathcal{A} \subseteq \mathcal{P}(E)$, where $\mathcal{P}(E)$ is the power set of $E$. Likewise, $\emptyset \in \mathcal{A}$ because the complement relative to $E$ of $E$ is the null set. Here enters $\sigma$-algebra.

**Definition 3.3.3.** (Sigma Algebra). Let $E$ be set. A collection $\mathcal{R}$ of subsets of $E$ is a *$\sigma$-algebra* over $E$ if the following properties are satisfied.

i. The unit $E$ is in $\mathcal{R}$. ($E \in \mathcal{R}$)

ii. $\mathcal{R}$ is closed under complement relative to $E$. ($\forall A \in \mathcal{R} : C_E(A) \in \mathcal{R}$)

iii. $\mathcal{R}$ is closed under countable unions. ($\forall A_1, A_2, \dots \in \mathcal{R} : \bigcup_{k \in \mathbb{N}} A_k \in \mathcal{R}$) [32, 56]

From this definition it clearly follows what the largest and smallest $\sigma$-algebras are for any given set. For a set $E$ the smallest $\sigma$-algebra, denoted by $\sigma(E)$, is $\{\emptyset, E\}$ [56]. Similarly the largest $\sigma$-algebra over $E$ is the power set $\mathcal{P}(E)$ [56].

Mixing and matching the elements of a set $E$ might have seemed like it a step away from measuring $E$ to the pre-modern mind, but it is not. Before Descartes a measure was in essence only a real number [15]. This 17th Century mathematician showed that a measure could be a mapping [15]. Following the distant past, the range of a measure function is given, viz. $\mathbb{R}_{\geq 0}$. But what is the domain?

Before addressing the domain of a measure function it is necessary to take a step back. Obviously it is necessary define a measure space, but the complications hardly end there. It cannot be assumed that all sets can have a measure, or even that there is only one type of measure [60]. The very concrete example Descartes gives by proving that the area of a 2-dimensional object can be mapped onto a 1-dimensional line falls far short a universal description of taking a measure. Taking Descartes idea beyond the realm of the mathematics of his day is only the similitude of progress. A mathematical superstructure for measure (and measurability) is essential.

**Definition 3.3.4.** (Measurable Space). If $E$ is a set and $\mathcal{R}$ is a $\sigma$-algebra generated over $E$ then the algebraic structure $(E, \mathcal{R})$ is a *measurable space* [19, 60].

To make a long story short, there is more than just one type of measurability and, by extension, more than a single kind of measure. Measures are functions, and they fall into two theories: the absolute integrable theory where measures map onto $(-\infty,\infty)$ and the non-negative integrable theory where measures that map onto $[0,\infty]$ [60]. These theories require different fundamental convergence. The former requires the dominated convergence theorem and the latter needs the monotone convergence theorem. Of these two theories only non-negative integrable theory is of interest to probability theory.

**Definition 3.3.5.** (Measure). A *measure* $m$ on a measurable space $(E,\mathcal{R})$ is a function where $m:\mathcal{R}\to R\subseteq\mathbb{R}$ such that

i. $m(\emptyset)=0$, and

ii. For any sequence $\{A_n\}_{n=1}^{\infty}$ of mutually disjoint measurable sets in $\{\mathcal{R}\}$, the equality $m(\bigcup_{n=1}^{\infty}A_n)=\sum_{n=1}^{\infty}m(A_n)$ is always satisfied.

**Definition 3.3.6.** (Measure Space). If $(E,\mathcal{R})$ a measurable space and $m$ is a measure on $(E,\mathcal{R})$ then the algebraic structure $(E,\mathcal{R},m)$ is a *measure space* [60, 32].

Rather than falling deeper into Alice's world to have tea with the Mad Hatter and the White Rabbit, it is time to turn back. The necessary questions to make progress in probability theory have been answered. For more concepts from measure theory used in this paper see Appendix A.1.

### 3.4 Axioms of probability

Kolmogorov's original axioms of probability, in his words, are:

> Let $E$ be a collection of elements $\xi,\eta,\zeta\ldots$, which we shall call *elementary events*, and $\mathcal{F}$ a set of subsets of $E$; the elements of the set $\mathcal{F}$ will be called *random events*.
>
> I. $\mathcal{F}$ is a field of sets.
>
> II. $\mathcal{F}$ contains the set $E$.
>
> III. To each set $A$ in $\mathcal{F}$ is assigned a real number $P(A)$. This number is called the probability of the event $A$.
>
> IV. $P(E)$ equals 1.

V. If $A$ and $B$ have no element in common, then

$$P(A+B) = P(A) + P(B)$$

[34].

These first words of *Foundations of the Theory of Probability* changed the world. In less than forty years after Hilbert posed his 6th Problem, probability jettisoned its mooring to physical reality becoming bounded only by theory. As trivial as it may seem now, this accomplishment is one of the greatest testaments of the power of modern mathematics [57]. By jumping this hurdle modern mathematics surmounted a barrier that had confounded human intelligence for centuries. Kolmogorov didn't just finish a question posed by Hilbert. With his axioms Kolmogorov finished the mathematization of probability begun by Gerolamo Cardano five centuries earlier and laying this field bare to human exploration twenty centuries after the Emperor Claudius penned the first known inquiry into chance.

The contemporary formulation of the axioms of probability are a little different [56]. These changes are mostly due to theoretical constraints and developments that are beyond the scope of this paper. Before giving the axioms of probability it is necessary to define a probability measure.

**Definition 3.4.1.** (Probability Measure). Let $(\Omega, \mathcal{F}, P)$ be a measure space. The measure $P : \Omega \to [0,1]$ is a *probability measure* if it has the following properties:

i. $P(\Omega) = 1$,

ii. $P(\emptyset) = 0$, and

iii. If $A \subseteq B \in \mathcal{F}$ then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

**Axiom 1.** *(Probability). An algebraic structure $(\Omega, \mathcal{F}, P)$ is a probability space if*

*i. $\Omega$ is a set of points $\omega$,*

*ii. $\mathcal{F}$ is a $\sigma$-algebra over $\Omega$, and*

*iii. $P : \mathcal{F} \to [0,1]$ is a probability measure.*

*The set $\Omega$ is called a sample set, and $\mathcal{F}$ is called a set of events.*

**Definition 3.4.2.** (Probabilistic Model). Let $(\Omega, \mathcal{F})$ be a measurable space. If the set $\mathcal{P}$ is a set of one or more probability measures $P$ where each $P : \Omega \to [0,1]$ [56, 32].

Now it is possible to answer the original question—what is a random variable?

**Definition 3.4.3.** (Random Variable). Let $(\Omega, \mathcal{F})$ and $(\Psi, \mathcal{G})$ be two measurable spaces. The function $X : \Omega \to \Psi$ is a *random variable* if for every $B \in \mathcal{G}$ the inverse image

$$\xi^{-1}(B) \equiv \{\omega \in \Omega : \xi(\omega) \in \Psi\}$$

or equivalently the set

$$\{\omega \in \Omega : \xi(\omega) \in \Psi\} \in \mathcal{F}$$

[32, 56, 35].

Before departing the world of probability several important functions should be defined.

**Definition 3.4.4.** (Expected Value). Let $\xi$ be a random variable on the probability space $(\Omega, \mathcal{F}, P)$. The *expected value* of $\xi$ is

$$\mathbb{E}(\xi) := \int_{\Omega} \xi(x)\, dP(x)$$

[32, 56].

**Definition 3.4.5.** (Variance). Let $\xi$ be a random variable on the probability space $(\Omega, \mathcal{F}, P)$. The *variance* of $\xi$ is

$$\text{Var}(\xi) := \mathbb{E}(\xi - \mathbb{E}(\xi)).$$

Also the square of the standard deviation of $\xi$ is

$$\sigma_{\xi}^2 := \text{Var}(\xi)$$

[32, 56].

**Definition 3.4.6.** (Variance). Let $\xi$ and $\eta$ be two random variables on the probability space $(\Omega, \mathcal{F}, P)$. The *covariance* of $\xi$ and $\eta$ is

$$\text{Cov}(\xi, \eta) := \mathbb{E}(\xi - \mathbb{E}(\xi))(\eta - \mathbb{E}(\eta))$$

[56].

# Chapter 4

# A THEORY OF PORTFOLIOS

## 4.1 Returning to modern portfolio theory

The first venture into modern portfolio theory was somewhat abortive. Armed with new mathematical tools it should be possible to make more progress. Modern portfolio theory needs the mathematics of probability and measure theory [9, 20]. Once again, let $\mathbb{S}$ be a security, and let $S(t)$ be the price of $\mathbb{S}$ at time $t$. Also assume that $t = \{0, 1\}$ where $S(0) = k$ for some constant $k$ and $S(1) \in \mathbb{R}_{\leq 0}$.

What has just been said? In mathematical terms, it is not very much. A security is a financial object, not a mathematical object. Likewise, "price" is one of those ubiquitous words with a meaning that is so innate to the English language that it is hard to find a context in which the word has no meaning. Yet, that is exactly the problem this word has in mathematics. There is no mathematical object called a price. The first paragraph essentially says that there is an undefined mathematical object $\mathbb{S}$ vaguely associated with another undefined object $S(t)$ that somehow takes a time variable, etc. It is not surprising that nothing useful can be concluded from this formulation. This formulation is not mathematics. Using the language of probability a far better formulation can be constructed.

Here is where this paper shall venture beyond the extant literature on modern portfolio theory. The overwhelming majority of books on the subject simply present a series of formulas that can be used by plugging in numbers [9, 20]. It is hard to find the mathematics in such a presentation. Even the more mathematically adept texts focus on computation and deriving the formulas for computation. Occasionally the underlying geometry of a sample problem will be presented to give a visualization of what is going on [9]. No one has given mathematical definitions of the fundamental objects of modern portfolio theory. This is exactly what this paper shall do. From an understanding of the objects being examined an understanding of their relationship can be ascertained and the applicability of the model expanded.

It is not an accident that other authors on modern portfolio theory have avoided defining fundamental objects, like security and portfolio, mathematically. The overwhelming majority of authors on modern portfolio theory are not mathematicians writing books for practical courses on finance. Beyond a lack of interest in generalizing modern portfolio theory beyond finance, doing this adds a level of unnecessary complexity that gets in the way of its application to finance. Even the mathematically intense *Portfolio Theory and Risk Management* by Capinski intentionally avoids defining financial objects in mathematical language [9]. His focus remains on the linear algebra (which at times is quite complex) that has developed in and around modern portfolio [9]. Generalizing modern portfolio theory would require him to engage in lengthy discussions of probability that would obstruct his goal.

This paper requires generalization because it needs to be demonstrated that a long poem and a portfolio are mathematically identical. It is not necessary to have all of the mathematical developments around portfolio theory to analyze a poem. If it can be established that a poem is a portfolio it is only necessary to have the most basic tools of modern portfolio theory. The added complexity by generalizing the method hardly makes the mathematical proofs and subsequent computations used here unwieldy (or even difficult).

First, price should be defined mathematically. By convention a price is a non-negative[9]. A price maps a time and an outcome onto a non-negative real number. The price function only maps a domain onto the positive real numbers because of the economic nature of a price, not a mathematical constraint. For simplicity, proofs for theorems included below can be found in Appendix A.2.

**Definition 4.1.1.** (General Price Function). Let $T$ be a set of times and $\Psi$ be a set of events. The Cartesian product $T \times \Psi$ is a Linear Space and its elements are vectors $(t, \psi)$ for $t \in T$ and $\psi \in \Psi$. The price $C$ is a linear operator such that $C : T \times \Psi \to \mathbb{R}_{\leq 0}$.

The initial conditions given for the hypothetical price restrict the values of time $t = \{0, 1\}$. It was also assumed that for time $t = 0$ the price function produced a constant. Under these conditions there is essentially one element in $T$. If these conditions are assumed for price $C$, there are $n$ elements in $\Psi$, and $C(0, \psi) = k$ for all $\psi \in \Psi$, price $C$ is equivalent to the concatenation of an $n$-vector of $k$'s and the function $C(1)$.

$$C : T \times \Psi \to \mathbb{R}_{\leq 0} \equiv k\mathbf{1}_n \,\|\, C(1) : \Psi \to \mathbb{R}_{\leq 0}$$

The assumed restrictions on price negate the need to define it as a linear operator (and the complexity that such a definition would bring). Thus there are two price functions that map from a set of outcomes to $\mathbb{R}_{\leq 0}$, viz. current price $C(0) : \Psi \to k$ and future price $C(1) : \Psi \to \mathbb{R}_{\leq 0}$. From Definition 3.4.3 a future price is a random variable. The $t$ value shall be retained in the notation of the price functions as a legacy of the original linear operator, e.g. $C(1, \omega)$ is a future price function taking the single variable $\omega$.

Now it a security can be defined mathematically.

**Definition 4.1.2.** (Security). A security $\mathbb{S}$ is an experiment that generates a probability space $(\Omega, \mathcal{F}, P)$.

Let $\mathbb{S}$ be a security, generating the probability space $(\Omega, \mathcal{F}, P)$. Likewise let the $\mathbb{S}$ current price $S(0) : \Omega \to k$ and the $\mathbb{S}$ future price $S(1) : \Omega \to \mathbb{R}_{\leq 0}$ be a measurable function over $\Omega$.

These two sentences have made more progress than the entire first section. The power of mathematics is in abstraction of relationships. Continuing with the example, assume for simplicity that $\Omega$ is a discrete and finite set[1] with only $N$ elements, i.e.

$$\Omega = \{\omega_1, \omega_2, \omega_3, \ldots, \omega_N\}.$$

By extension there are $N$ possible values of $S(1)$. They are

$$S(1, \omega_1), S(1, \omega_2), \ldots, S(1, \omega_N).$$

Since each of these values is a real number and $\Omega$ is discrete, from Definition A.1.1 $S(1)$ is a simple function.

Also from above and Definition 3.4.3 the sets $\{\omega : \forall \omega \in \Omega\}$ are in the $\sigma$-algebra $\mathcal{F}$. Thus corresponding to each $\omega \in \Omega$ there is a probability $P(\{\omega\})$. For a more convenient notation let

$$p_i = P(\{\omega_i\})$$

for $i = 1, 2, 3, \ldots, N$.

From Definition 3.4.4 expected value of $S(1)$ is

$$\mathbb{E}(S(1)) = \int_{\Omega} S(1)\, dP$$

which from Definition A.1.2 is equivalent to

$$\mathbb{E}(S(1)) = \sum_{i=1}^{N} S(1, \omega_i) p_i,$$

[1] If the future price $S(1)$ were continuous with a density function $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ the expected value and variance of this random variable would be

$$\mathbb{E}(S(1)) = \int_{-\infty}^{\infty} x f(x)\, dx$$

and

$$\mathrm{Var}(S(1)) = \int_{-\infty}^{\infty} (x - \mathbb{E}(S(1)))^2 f(x)\, dx$$

and following a similar logic the variance of $S(1)$ is

$$\mathrm{Var}(S(1)) = \sum_{i=1}^{N} \Big( S(1,\omega_i) - \mathbb{E}(S(1)) \Big)^2 p_i.$$

With the expected value and variance of the future price $S(1)$ it is possible to engage in primitive betting on the future of security $\mathbb{S}$. This analysis of the security would be a far cry from the nuanced and opaque world of modern betting odds, but it would also be more satisfactory than using the techniques that were available to the Emperor Augustus in his many games of dice. More is needed to engage in mathematical investment analysis of $\mathbb{S}$, much less modern portfolio theory.

It is necessary to define a return function. A return function $K$ of a security $\mathbb{S}$, with a corresponding probabilistic model $(\Omega, \mathcal{F}, P)$, a current price $S(0)$ and a future price $S(1)$, is a measurable function over that probabilistic model that maps the values of $S(0)$ and $S(1)$ onto a real number. Different securities need different return functions [9].

To keep the working example as uncomplicated as possible, let the return of security $\mathbb{S}$ be defined

$$K := \frac{S(1) - S(0)}{S(0)}.$$

Like $S(1)$, $K$ is a real valued random variable on the probability space $(\Omega, \mathcal{F}, P)$. The expected return $\mathbb{E}(K)$, denoted by $\mu$, is

$$\mu = \mathbb{E}\left( \frac{S(1) - S(0)}{S(0)} \right).$$

Since the expected value of a constant is that constant, expected return simplifies to

$$\mu = \frac{\mathbb{E}(S(1)) - S(0)}{S(0)},$$

and the variance of return $\mathrm{Var}(K)$, denoted by $\sigma^2$, is

$$\sigma^2 = \mathbb{E}(K - \mathbb{E}(K))^2$$

This simplifies

$$\begin{aligned}
\sigma^2 &= \mathbb{E}(K - \mathbb{E}(K))^2 \\
&= \mathbb{E}(K^2 - 2K\mathbb{E}(K) + \mathbb{E}(K)^2) \\
&= \mathbb{E}(K^2) - 2\mathbb{E}(K)^2 + \mathbb{E}(K)^2 \\
&= \mathbb{E}(K^2) - \mathbb{E}(K)^2.
\end{aligned}$$

Stepping back, what are the values of $\Omega$ in a security? The values of $\Omega$ are the possible forces that can act on the market. This might seem somewhat vague, and it is. Predicting market movements are somewhat difficult because it is hard to know what the possible outcomes are much less what their probabilities are. This has a knock on effect. It is very hard to determine the expected return of a security. Markowitz was a genius to recognize that even though the tools did not exist in the 1950's to accurately determine an expected return of a security such a value did exists [45]. This allowed him to take the first steps in analyzing portfolios.

### 4.2 Modern portfolio theory with two securities

Now let there be two securities $\mathbb{S}_1$ and $\mathbb{S}_2$ with the corresponding price functions[2] $S_1(t)$, and $S_2(t)$. These securities have the same simplifications as the initial security $\mathbb{S}$, viz. the sample sets are finite and they both use the previously given definition of return. Also it should be assumed that the returns of these two securities are not equal, i.e. $\mu_1 \neq \mu_2$.

A portfolio comprised of these two securities will have a value at time $t$ of

$$V_{(x_1,x_2)}(t) = x_1 S_1(t) + x_2 S_2(t)$$

where $x_1$ and $x_2$ are the number of shares of securities $\mathbb{S}_1$ and $\mathbb{S}_2$ respectively. The future value of the portfolio can also be expressed in terms of the return of the two securities. Nevertheless, it is easier to do so if another object is defined first—weights.

The weight of a security in a portfolio is the percentage of the value of the portfolio that that security comprises. The weight for the $i$-th security in a portfolio of $m$ securities is defined as

$$w_i := \frac{x_i S_i(0)}{V_{(x_1,x_2,\ldots,x_m)}(0)}.$$

Recalling that the return of the $i$-th security is

$$\begin{aligned} K_i &= \frac{S_i(1) - S_i(0)}{S_i(0)} \\ K_i S_i(0) &= S_i(1) - S_i(0) \\ S_i(1) &= S_i(0)(1 + K_i). \end{aligned}$$

Thus,

$$\begin{aligned} V_{(x_1,x_2)}(1) &= x_1 S_1(1) + x_2 S_2(1) \\ &= x_1 S_1(0)(1 + K_1) + x_2 S_2(0)(1 + K_2) \end{aligned}$$

[2]The two functions $S_1(t)$, and $S_2(t)$ shall stand in for four functions $S_1(0)$, $S_2(0)$, $S_1(1)$, and $S_2(1)$ to simplify the notation in general formula that are the same for the current value and future value. The theoretical distinction of the current price and future price functions still apply here.

$$
\begin{aligned}
&= \frac{w_1 V_{(x_1,x_2)}(0)}{S_1(0)} S_1(0)(1+K_1) + \frac{w_2 V_{(x_1,x_2)}(0)}{S_2(0)} S_2(0)(1+K_2) \\
&= w_1 V_{(x_1,x_2)}(0)(1+K_1) + w_2 V_{(x_1,x_2)}(0)(1+K_2) \\
&= V_{(x_1,x_2)}(0)(w_1(1+K_1) + w_2(1+K_2)) \\
&= V_{(x_1,x_2)}(0)(w_1 + w_2 + w_1 K_1 + w_2 K_2)
\end{aligned}
$$

Since, by definition, the sum of the weights equals 1,

$$
V_{(x_1,x_2)}(1) = V_{(x_1,x_2)}(0)(1 + w_1 K_1 + w_2 K_2).
$$

Following the return function given above, the return of the portfolio $K_{port}$ is

$$
K_{port} = \frac{V_{(x_1,x_2)}(1) - V_{(x_1,x_2)}(0)}{V_{(x_1,x_2)}(0)}.
$$

The return function can also be expressed as a function of the returns of the individual securities. Combining both preceding equations,

$$
\begin{aligned}
K_{port} &= \frac{V_{(x_1,x_2)}(0)(1 + w_1 K_1 + w_2 K_2) - V_{(x_1,x_2)}(0)}{V_{(x_1,x_2)}(0)} \\
&= \frac{V_{(x_1,x_2)}(0)(w_1 K_1 + w_2 K_2)}{V_{(x_1,x_2)}(0)} \\
&= w_1 K_1 + w_2 K_2.
\end{aligned}
$$

The expected return of the portfolio $\mu_{port}$ is

$$
\begin{aligned}
\mu_{port} &= \mathbb{E}(w_1 K_1 + w_2 K_2) \\
&= \mathbb{E}(w_1 K_1) + \mathbb{E}(w_2 K_2) \\
&= w_1 \mu_1 + w_2 \mu_2. \qquad (4.1)
\end{aligned}
$$

The variance of the portfolio is

$$
\begin{aligned}
\sigma^2_{port} &= \mathbb{E}(K^2_{port}) - \mu^2_{port} \\
&= \mathbb{E}(w_1^2 K_1^2 + w_2^2 K_2^2 + 2 w_1 w_2 K_1 K_2) - w_1^2 \mu_1^2 - w_2^2 \mu_2^2 - 2 w_1 w_2 \mu_1 \mu_2 \\
&= w_1(\mathbb{E}(K_1^2) - \mu_1^2) + w_2(\mathbb{E}(K_2^2) - \mu_2^2) + 2 w_1 w_2(\mathbb{E}(K_1 K_2) - \mu_1 \mu_2)
\end{aligned}
$$

If the covariance[3] of the $i$-th and the $j$-th securities is denoted by $\sigma_{ij}$, then

$$\sigma_{port}^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}.$$

Similarly, if correlation coefficient $\rho_{ij}$ of the $i$-th and the $j$-th securities is defined as

$$\rho_{ij} := \frac{\sigma_{ij}}{\sigma_i\sigma_j}$$

then

$$\sigma_{port}^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2. \tag{4.2}$$

From the Cauchy-Schwarz inequality[4] it follows that $-1 \le \rho \le 1$.

The portfolio return and variance can be easily converted to matrix notation. If

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad C = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix},$$

then

$$\begin{aligned} \mu_{port} &= \mathbf{w}^T\boldsymbol{\mu} \\ \sigma_{port}^2 &= \mathbf{w}^T C\mathbf{w}. \end{aligned}$$

Now it is possible to plot $(\sigma_1, \mu_1)$, $(\sigma_2, \mu_2)$, and $(\sigma_{port}, \mu_{port})$, and compare their performance by plotting them on a $(\sigma, \mu)$-plane. Modern portfolio theory assumes that all investors are risk averse, i.e. they want the highest expected return for the lowest risk. The weights chosen to construct this portfolio.

[3]For two random variables $X, Y$

$$\begin{aligned} \mathrm{Cov}(X,Y) &= \mathbb{E}((X-\mathbb{E}(X))(Y-\mathbb{E}(Y))) \\ &= \mathbb{E}(XY - \mathbb{E}(X)Y - X\mathbb{E}(Y) + \mathbb{E}(X)\mathbb{E}(Y)) \\ &= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) - \mathbb{E}(X)\mathbb{E}(Y) + \mathbb{E}(X)\mathbb{E}(Y) \\ &= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) \end{aligned}$$

[4]If $X, Y$ are two random variables then the Cauchy-Schwarz inequality is

$$|\mathbb{E}(XY)|^2 \le \mathbb{E}(X^2)\mathbb{E}(Y^2).$$

Let $X = A - \mathbb{E}(A)$ and $Y = B - \mathbb{E}(B)$, where $A, B$ are also random variable. From the definition of covariance and variance

$$|\mathrm{Cov}(A,B)|^2 \le \mathrm{Var}(A)\mathrm{Var}(B).$$

Given any two securities it is possible to construct any number of portfolios by selecting different weights. There are countless possibilities, and this set, the set of all possible portfolios, is called the attainable field. But the introduction of the attainable field has demonstrated that one concept has been left poorly defined. What is a portfolio? Without defining portfolio mathematically, it is impossible to define attainable field. Remember that a security is in the language of probability an experiment.

Combining two securities is combining the outcomes of two experiments into one. Let $\mathbb{S}_1$ and $\mathbb{S}_2$ be two securities generating the measurable spaces $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ respectively. The portfolio $\mathbb{S}_1 \cup \mathbb{S}_2$ must generate the product measurable space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$. However, for there to be a portfolio value or portfolio expected value there must be a probability. Let $P_{port}$ be such a probability[5] for $\mathbb{S}_1 \cup \mathbb{S}_2$.

A portfolio is composed of the two given securities is simply the measure space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, P_{port})$. Mathematically, a portfolio and a security are the same type of object. The attainable field for these two securities is a probabilistic model $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mathcal{P})$, where the collection $\mathcal{P}$ contains all possible probabilities for this portfolio. Generalizing these concepts give a mathematical definition of a portfolio and an attainable field.

**Definition 4.2.1.** (Portfolio). A portfolio $\mathbb{W}$ comprised of $n$ securities $\mathbb{S}_1, \mathbb{S}_2, \mathbb{S}_3, \ldots, \mathbb{S}_n$ is an experiment that generates from the probability spaces of the component securities a probability space $(\prod_{i=1}^{n} \Omega_i, \bigotimes_{i=1}^{n} \mathcal{F}_i, P)$.

In other words a portfolio is a security composed of securities. Not every security is a portfolio, but all portfolios are securities. This is not as strange as it may seem *prima facie*. Mutual funds are portfolios sold as securities.

**Definition 4.2.2.** (Attainable Field). An attainable field $\mathfrak{A}$ comprised of $n$ securities $\mathbb{S}_1, \mathbb{S}_2, \mathbb{S}_3, \ldots, \mathbb{S}_n$ is an experiment that generates the probabilistic model $(\prod_{i=1}^{n} \Omega_i, \bigotimes_{i=1}^{n} \mathcal{F}_i, \mathcal{P})$, where $\mathcal{P}$ is the collection of all possible probabilities for the measurable space $(\prod_{i=1}^{n} \Omega_i, \bigotimes_{i=1}^{n} \mathcal{F}_i)$.

Setting $w_1 = w$ it follows that $w_2 = (1 - w)$ because there are only two weights. Thus,

[5] In practice the probability of any security influence by forces external to it sample set. Maintaining an independence of the probabilities all measurable spaces preserves this idiosyncrasy. In practice the actual probability of a portfolio is a composition of a weight function and the Cartesian product of the probabilities of the individual securities. For the given example

$$P_{port} \equiv w \circ (P_1 \times P_2)$$

where $w$ is a weight function. That is also why weights of a portfolio are called weights and not probabilities.

the expected portfolio return and variance are

$$\mu_{port} = w\mu_1 + (1-w)\mu_2$$
$$\sigma^2_{port} = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_{12}.$$

The return function is a linear mapping and the standard deviation, being the square root of a quadratic function, is the graph of a hyperbola. The shape of the graph of the standard deviations and expected returns of the attainable field will be on a hyperbola almost surely. There are a few cases where it will not be. The shape of the curve depends upon the correlation coefficient. If $\rho_{12} = \pm 1$ it is possible to minimize variance of the portfolio to 0, i.e. it is no longer on a hyperbola with a center on the vertical axis.

**Theorem 4.2.1.** *(Tobin two security theorem[6]). Let $\mathbb{S}_1$ and $\mathbb{S}_2$ be two securities. If $\mu_1 \neq \mu_2$ and $-1 < \rho_{12} < 1$ then the $(\sigma, \mu)$ plot of the attainable field $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mathcal{P})$ will lie on a hyperbola that has a center on the $\mu$-axis [9].*

### 4.3 Two security theorem in practice

There are several important implications of Theorem 4.2.1. Risk, as variance is known in finance, and return of a portfolio of two assets are inextricably tied together [45, 9]. Since securities and portfolios are the same type of mathematical object, it is possible to recursively optimize any countable number of securities using this theorem. Risk and return are two interconnected attributes for any portfolio or security. In the words of an old proverb—nothing ventured, nothing gained. Harry Markowitz's research in portfolio theory was driven to find this mathematical relationship [45, 55].

Alfred Marshall's supply and demand curves create a paradox [45]. Without going into the finer details of micro-economics, the existence of a fixed equilibrium point where the supply and demand functions for an economic venture meet seems to guarantee the possibility of a risk free return for this venture. Taking this to its logical conclusion, if enough information is known, any venture could be risk free. Risk would be byproduct of poor information, not the nature of economic ventures [45]. This would make financial diversification illogical. An investor should simply do enough research to find the single investment with best guaranteed return and invest.

Not only would this course of action be contrary to good financial practice, it contradicts key assumptions in both Neo-Classical and Keynesian macro-economic theories [44].

[6]Most sources call this theorem the Two Mutual Fund Theorem or the Tobin Theorem. The decision to not use the words "mutual fund" is intended to refrain from introducing another non-mathematical term and tohighlight the abstraction of modern portfolio theory pursued in this paper [9, 20].

Markowitz noticed that the Marshallian Cross stood as a barrier to a scientific understanding of economics, even though it had been the first tentative step towards mathematizing economics [45]. Theorem 4.2.1 was discovered by a colleague of Markowitz, James Tobin in the 1958 [31]. This one discovery was the key needed to unlock modern portfolio theory.

Theorem 4.2.1 also shows that the relationship between the risk and return of a portfolio of two assets is predictable. This means if an optimal level of risk for an investor of two securities portfolio weights for these two securities can be found to maximize return. Not only is diversification mathematically an advantageous decision, the proper proportions of diversification can be ascertained [20]. This is the beating heart of modern portfolio theory. A practical example will help elucidate what is going on.

**Example 4.3.1.** (Apple and Facebook Stock). Take two securities, the stock of Apple AAPL and the stock of Facebook FB.[7] To make a portfolio of these two stocks it is necessary to have the ordered pairs $(\sigma_{AAPL}, \mu_{AAPL})_{AAPL}$ and $(\sigma_{FB}, \mu_{FB})_{FB}$. Rather than starting at this point, the current price and three possible future prices with their probabilities for each stock shall be given.

For AAPL , let the given prices be

$$S(0)_{AAPL} = 313.05, \qquad S(1)_{AAPL} = \begin{cases} 400 & \mathrm{p}_1 = 1/4 \\ 350 & \mathrm{p}_2 = 1/2 \\ 190 & \mathrm{p}_3 = 1/4. \end{cases}$$

Let the given prices of FB be

$$S(0)_{FB} = 210.18, \qquad S(1)_{FB} = \begin{cases} 300 & \mathrm{p}_1 = 1/4 \\ 250 & \mathrm{p}_2 = 1/2 \\ 178 & \mathrm{p}_3 = 1/4. \end{cases}$$

The expected values of each future price are

$$\begin{aligned} \mathbb{E}(S(1)_{AAPL}) &= 400 * 0.25 + 350 * 0.5 + 190 * 0.25 = 322.50 \\ \mathbb{E}(S(1)_{FB}) &= 300 * 0.25 + 250 * 0.5 + 178 * 0.25 = 244.50, \end{aligned}$$

and their corresponding variances are

$$\begin{aligned} \mathrm{Var}(S(1)_{AAPL}) &= 0.25(400 - 322.50)^2 + 0.5(350 - 322.50)^2 + 0.25(190 - 322.50)^2 \\ &= 6268.75 \\ \mathrm{Var}(S(1)_{FB}) &= 0.25(300 - 244.50)^2 + 0.5(250 - 244.50)^2 + 0.25(178 - 244.50)^2 \\ &= 1890.75. \end{aligned}$$

[7] The numbers given for the stocks of Apple and Facebook are purely hypothetical.

It is now possible to calculate the expected returns and variance of returns. They are:

$$\mu_{AAPL} = \frac{322.50 - 313.05}{313.05} = 3.01\%$$
$$\mu_{FB} = \frac{244.50 - 210.18}{210.18} = 16.33\%.$$

Before computing the variances of the returns, the computation can be simplified in another manner.

$$\begin{aligned}\sigma^2 &= \text{Var}\left(\frac{S(1) - S(0)}{S(0)}\right) \\ &= \frac{1}{S(0)^2}\text{Var}\left(S(1) - S(0)\right) \\ &= \frac{1}{S(0)^2}\text{Var}\left(S(1)\right)\end{aligned}$$

Using this formula, the variances of return are

$$\sigma^2_{AAPL} = \frac{6268.75}{313.05^2} = 0.0639$$
$$\sigma^2_{FB} = \frac{1890.75}{210.18^2} = 0.0428.$$

and the standard deviation is

$$\sigma_{AAPL} = \sqrt{0.0639} = 0.2527$$
$$\sigma_{FB} = \sqrt{0.0428} = 0.2068.$$

The desired $(\sigma, \mu)$-tuples for the stocks AAPL and FB are $(0.2527, 0.0301)_{AAPL}$ and $(0.2068, 0.1633)_{FB}$. It is possible to find every possible $(\sigma, \mu)$-tuples for every portfolio in the corresponding attainable field by using (4.1) and (4.2). The plot of these $(\sigma, \mu)$-tuples is in Figure 4.1. At first glace this plot does not look like an hyperbola. The segment created by the AAPL FB portfolio $(\sigma, \mu)$-tuples is so small that it almost looks like a straight line. However, Figure 4.2 visually shows the complete hyperbola traced by these tuples.

After looking at these figures it is easy to understand why Theorem 4.3.1 was not immediately self-evident. The mathematics is clear, but taking those first few steps to be able to see the broader structure. Sir Andrew Wiles once likened mathematics to stumbling around in a pitch black room [40]. The basic geography of the room is identified by seemingly blind forays into the unknown with unknowable results, until a light switch is found. Nevertheless, the utility of Theorem 4.3.1 depends upon being able to select one of many portfolios in an attainable set. The first and most important task in portfolio optimization is defining what optimization means [9, 44].

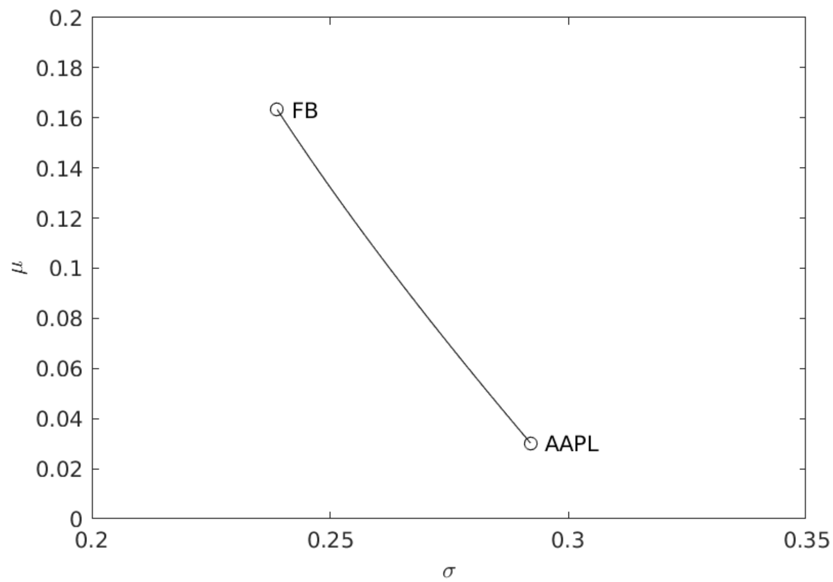


*Figure 4.1*: Plot of the $(\sigma, \mu)$-tuples of every portfolio in the attainable field generated by the Stocks APPL and FB.

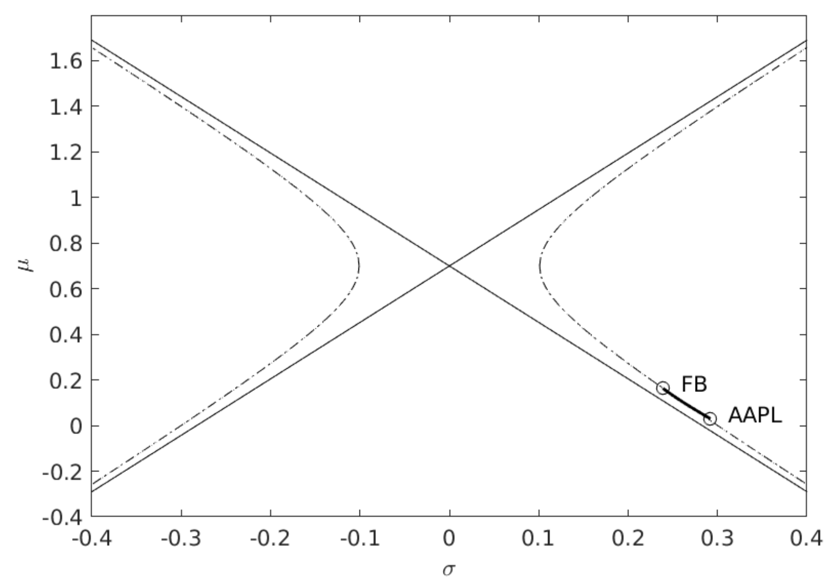


*Figure 4.2*: Plot of the $(\sigma, \mu)$-tuples of every portfolio in the attainable field generated by the Stocks APPL and FB mapped onto the corresponding hyperbola.

**Definition 4.3.1.** (Optimization). For an attainable field $(\prod_{i=1}^{n} \Omega_i, \bigotimes_{i=1}^{n} \mathcal{F}_i, \mathcal{P})$ generated by

$n$ securities the *optimal portfolio* $(\Omega_{opt}, \mathcal{F}_{opt}, P_{opt})$ is the portfolio where

$$\boldsymbol{\mu}_{opt} = \max\left(\left\{\bigcap_{\sigma_i \leq \sigma_0} \{\boldsymbol{\mu}_i\}_{i=1}^{n}\right\}\right)$$

for a given $\sigma_0$.

In Example 4.3.1 selecting optimal portfolio is trivial. The optimal portfolio is composed exclusively of Facebook stock. It has a higher expected return with a smaller variance. To include any Apple stock would reduce the portfolio expected return and increase risk. Not every pair of stocks behaves this way. Glowering in the corner, like Banquo's Ghost, lies the concept of the indifference curve. The indifference curve is a function that models investor preferences towards risk [9].

To address this object properly would require diving headlong into modeling Behavioral economics with differential equations. In practice, most theoretical examples of indifference curves desperately oversimplify the concept to make progress elsewhere and most practical applications of financial advisers pick a maximum variance that seems right for their client [20]. Indifference curves are a subject ripe for making progress, but that task must be left to other adventurers.

Beyond the optimal portfolio there are several other key portfolios. The most important one other than the optimal portfolio is the minimum variance portfolio. The following theorem provides a means of determining the weights of the minimum variance portfolio for an attainable set.

**Theorem 4.3.1.** *(Minimum variance portfolio, two securities). Let* $\mathbb{S}_1$ *and* $\mathbb{S}_2$ *be two securities generating the attainable field* $\mathfrak{A}$*. Allowing for short-selling,*[8] *the minimum variance portfolio* $\mathfrak{A}_{\min}$ *has the weights* $\mathbf{w}_{min} = [w_1, w_2]$ *where*

$$w_1 = \frac{a}{a+b} \qquad\qquad w_2 = \frac{b}{a+b}$$

*for*

$$a = \sigma_2^2 - \rho_{12}\sigma_1\sigma_2$$
$$b = \sigma_1^2 - \rho_{12}\sigma_1\sigma_2.$$

*except when both* $\rho_{12} = 1$ *and* $\sigma_1 = \sigma_2$ *[9].*

[8] Short-selling is the legal obligation of a seller to provide securities to a buyer that are not owned by the seller with the intent of converting the obligation to money. A simple way to think of short-selling is as the inverse of buying-long.

**Corollary 4.3.2.** *(Weights minimum variance portfolio). Using the same givens as the above theorem, for portfolio* $\mathfrak{A}_{\min}$ *the vector of weights* $\mathbf{w}_{min}$ *is*

$$\mathbf{w}_{min} = \frac{C^{-1}\mathbf{1}_2}{\mathbf{1}_2^T C^{-1}\mathbf{1}_2}$$

*where C is the covariance matrix for the security and* $\mathbf{1}_2$ *is a vector of 1's with length 2 [9].*

Before leaving this exploration of a portfolio with two securities two last definitions should be given, viz. market portfolio and capital market line. The concepts of the market portfolio and the capital market line are essential to understanding CAPM, which will be introduced later.

**Definition 4.3.2.** (Capital market line). Given an attainable field $\mathfrak{A}$ of $n$ securities the capital market line $m : \sigma \to \mu$ is a linear function that is tangent to the hyperbola drawn by $\mathfrak{A}$ on $(\sigma, mu)$-graph and passes through a point $(0,R)$, where $R < \mu_{min}$. $R$ is called the risk free return.[9]

**Definition 4.3.3.** (Market portfolio). Given an attainable field $\mathfrak{A}$ of $n$ securities and a capital market line $m : \sigma \to \mu$, the market portfolio $\mathbb{W}_m$ is the portfolio with a $(\sigma, \mu)$ that is the tangent point of hyperbola touched by $m$.

The equation of the capital market line $m$ is

$$m(\sigma) = R + \frac{\mu_m - R}{\sigma_m}\sigma.$$

A formula to find the weight vector of the market portfolio will be given in the next section. Observe that if $R = 0$ then the market portfolio is the minimum variance portfolio.

### 4.4 Modern portfolio theory with more than two securities

The real theoretical progress was made with the Two Security Theorem (Theorem 4.2.1). Since a portfolio and a security are the same mathematical object, viz. an experiment generating a probability space, any finite number of portfolios can be divided into recursive pairs and optimized using this theorem. Although this is theoretically possible, there is a difference between an obvious conclusion and doing so. A brief overview of modern portfolio theory with $n$ securities is given here.

Consider a portfolio $\mathbb{W}$ of $n < \infty$ securities $\mathbb{S}_1, \mathbb{S}_2, \mathbb{S}_3, \ldots, \mathbb{S}_n$. The weights of these securities in the portfolio are a vector

$$\mathbf{w} = [w_1, w_2, w_3, \ldots, w_n]$$

[9]If $R \geq \mu_{min}$ the linear function $m$ will intersect the hyperbola.

where $\sum_i^n w_i = 1$ and $w \geq 0$ for all $w \in \mathbf{w}$. The vector of the number of shares of each security is

$$\mathbf{x} = [x_1, x_2, x_3, \ldots, x_n]$$

where $x \geq 0$ for all $x \in \mathbf{x}$.

Similarly, the expected returns of these securities are the vector

$$\boldsymbol{\mu} = [\mu_1, \mu_2, \mu_3, \ldots, \mu_n].$$

The covariance between the returns $K_i, K_j$ for any $i, j \leq n$ is denoted by $\sigma_{ij}$. The $n \times n$ covariance matrix of this portfolio is

$$C = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \ldots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \ldots & \sigma_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \ldots & \sigma_n^2 \end{bmatrix}.$$

The return of the portfolio $\mathbb{W}$ is

$$K_w = \sum_{i=1}^n w_i K_i.$$

The expected return is

$$\begin{aligned} \mu_w &= \mathbb{E}\left(\sum_{i=1}^n w_i K_i\right) \\ &= \sum_{i=1}^n w_i \mathbb{E}(K_i) \\ &= \sum_{i=1}^n w_i \mu_i \\ &= \mathbf{w}^T \boldsymbol{\mu}. \end{aligned} \tag{4.3}$$

The variance of return is

$$\begin{aligned} \sigma_w^2 &= \text{Var}(K_w) \\ &= \text{Cov}(K_w, K_w) \\ &= \text{Cov}\left(\sum_{i=1}^n w_i K_i, \sum_{j=1}^n w_j K_j\right) \\ &= \sum_{i,j=1}^n w_i w_j \text{Cov}(K_i, K_j) \\ &= \sum_{i,j=1}^n w_i w_j \sigma_{ij} \\ &= \mathbf{w}^T C \mathbf{w}. \end{aligned} \tag{4.4}$$

Before moving onto the minimum variance portfolio for $n$ securities, it is necessary to prove a few lemmas.

**Lemma 4.4.1.** *(Covariance of two portfolios). Given any two portfolios $\mathbb{W}_A$ and $\mathbb{W}_B$ of $n$ securities, the covariance*

$$\mathrm{Cov}(K_A, K_B) = \mathbf{w}_A^T C \mathbf{w}_B^T$$

*[9].*

**Lemma 4.4.2.** *(Hessian $2C$). Given portfolio $\mathbb{W}$ of $n$ securities, the hessian matrix of $\mathbf{w}^T C\mathbf{w}$ is $2C$. Furthermore, the following gradients with respect to $\mathbf{w}$ are*

$$\nabla(\mathbf{w}^T \boldsymbol{\mu}) = \boldsymbol{\mu} \tag{4.5}$$

$$\nabla(\mathbf{w}^T \mathbf{1}_n) = \mathbf{1}_n \tag{4.6}$$

$$\nabla(\mathbf{w}^T C\mathbf{w}) = 2C\mathbf{w} \tag{4.7}$$

[9].

**Theorem 4.4.3.** *(General minimum variance portfolio). Given an attainable field $\mathfrak{A} = (\prod_{i=1}^n \Omega_i, \bigotimes_{i=1}^n \mathcal{F}_i, \mathcal{P})$ with $n$ securities, the minimum variance portfolio $\mathfrak{A}_{\min}$ has the vector of weights*

$$\mathbf{w}_{min} = \frac{C^{-1}\mathbf{1}_n}{\mathbf{1}_n^T C^{-1}\mathbf{1}_n}.$$

*where $C$ is the covariance matrix for the security and $\mathbf{1}_2$ is a vector of 1's with length $n$ [9].*

It is essential to define the minimum variance line and introduce several theorems about this line before generalizing the Two Security Theorem.

**Definition 4.4.1.** (Minimum variance line). The minimum variance line of an attainable field $(\prod_{i=1}^n \Omega_i, \bigotimes_{i=1}^n \mathcal{F}_i, \mathcal{P})$ is the probabilistic model composed of the portfolios in the attainable field that have an expected return equal to or greater than the expected return of the minimum variance portfolio $\mu_{min}$ and they have the smallest variance of all portfolios with the same expected return i.e.

$$\left( \prod_{i=1}^n \Omega_i, \bigotimes_{i=1}^n \mathcal{F}_i, \bigcup_{\substack{\mu_{min} \le \mu_j \\ \sigma_j^2 = \min(\sigma_k^2 | \mu_k = \mu_j)}} P_j \right).$$

**Theorem 4.4.4.** *Given an attainable field of n securities $\mathfrak{A}$, let M be a matrix*

$$M = \begin{bmatrix} \boldsymbol{\mu}^T C^{-1} \boldsymbol{\mu} & \boldsymbol{\mu}^T C^{-1} \mathbf{1}_n \\ \boldsymbol{\mu}^T C^{-1} \mathbf{1}_n & \mathbf{1}_n^T C^{-1} \mathbf{1}_n \end{bmatrix}.$$

*If both C and M are invertible, the weight vector of a portfolio on the minimum variance line is*

$$\mathbf{w} = \frac{1}{|M|} C^{-1} \left( |M_1| \boldsymbol{\mu} + |M_2| \mathbf{1}_n \right)$$

*where*

$$M_1 = \begin{bmatrix} \mu & \boldsymbol{\mu}^T C^{-1} \mathbf{1}_n \\ 1 & \mathbf{1}_n^T C^{-1} \mathbf{1}_n \end{bmatrix}, \qquad M_2 = \begin{bmatrix} \boldsymbol{\mu}^T C^{-1} \mathbf{1}_n & \mu \\ \mathbf{1}_n^T C^{-1} \mathbf{1}_n & 1 \end{bmatrix}$$

*[9].*

**Corollary 4.4.5.** *Assuming the givens of Theorem 4.4.4 there exists two vectors* $\mathbf{a}, \mathbf{b}$ *such that*

$$\mathbf{w} = \mu \mathbf{a} + \mathbf{b}$$

*[9].*

**Corollary 4.4.6.** *Let $\mathfrak{A}$ be an attainable field of n securities with a minimum variance line $\mathfrak{A}_{MVL}$. If there are two distinct portfolios $\mathbb{W}_1, \mathbb{W}_2 \in \mathfrak{A}_{MVL}$, then the vector of weights* $\mathbf{w}$ *of all portfolios $\mathbb{W} \in \mathfrak{A}_{MVL}$ are*

$$\mathbf{w} = \alpha \mathbf{w}_1 + (1 - \alpha) \mathbf{w}_2$$

*for some number $\alpha \in \mathbb{R}$.*

**Theorem 4.4.7.** *(Generalization of Tobin's two security theorem). Given an attainable field $\mathfrak{A}$, its minimum variance line $\mathfrak{A}_{MVL}$ is a segment of a hyperbola with its center on the vertical axis.*

[9, 31]

The last step before moving on to CAPM is to describe the weights of a market portfolio.

**Theorem 4.4.8.** *(Market portfolio weights). Given an attainable field $\mathfrak{A}$ of n securities and a capital market line $m : \sigma \to \mu$ and a risk free rate R, the market portfolio $\mathbb{W}_m$ has a weight vector*

$$\mathbf{w}_m = \frac{C^{-1}(\boldsymbol{\mu} - \mathbf{1}_n R)}{\mathbf{1}_n^T C^{-1}(\boldsymbol{\mu} - \mathbf{1}_n R)}$$

*[9].*

## 4.5 CAPM

Modern portfolio theory may have begun as a search for portfolio optimization, but it has hardly rested upon its early laurels. Understanding the nature of a security or a portfolio is as important as constructing a diversified portfolio [9, 20]. The motivation behind this obvious. There are hundreds of thousands of securities for sale on the open market. It is impracticable to try to analyze all securities in the world to find the most optimal portfolio possible. A short hand to filter out undesirable securities is important. Capital Asset Pricing Model (abbreviated CAPM) provides this short hand.

Another important attribute that CAPM brings to Modern Portfolio Optimization is price analysis [20, 9]. CAPM is not by any means the only set of methods used to determine the future prices of securities and portfolios, but it is one of the most common [20, 9, 8]. Trying to determine the future price a security is fraught with countless troubles. Market prices are the most relatable example of Brownian Motion. CAPM's very name focuses on its methods for determining future prices. This is, sadly, beyond the scope of this paper.

This section is brief. Most of formulas in CAPM are not necessary for analyzing a piece of poetry. It certainly wouldn't be impossible, but trying to extract financial information from something that is not financial can be impractical. Being too convoluted in computation can obscure any meaning. Although the mathematics is from modern portfolio theory, in the end the goal of this paper is to engage in meaningful literary criticism. Financial mathematics is a tool and only a tool. Simplicity and utility are vitally important, but this shall be addressed in more detail later on. However, one thing from CAPM will be essential in analyzing a poem, like the *Aeneid*. This is the Beta ($\beta$) factor.

**Definition 4.5.1.** (Beta factor). Given a market portfolio[10] $\mathbb{W}_m$ and another $\mathbb{W}_i$, the Beta ($\beta$) factor of $\mathbb{W}_i$ is defined as

$$\beta_i := \frac{\text{Cov}(K_i, K_m)}{\sigma_m^2}$$

[9].

The Beta factor measures the sensitivity to the underlying forces that effect prices, the correlation to the market, and how related a stock is to the market [9, 20]. Thus, the Beta

[10] A market portfolio is a baseline to compare performance. It can be portfolio in the attainable field of the portfolio being compared, but in many cases this is not necessary. The most common used market portfolio (outside of theoretical analysis) is a major market like the DJIA, NASDAQ, S & P 500, etc.

factor of the market portfolio is

$$\beta_m = \frac{\text{Cov}(K_m, K_m)}{\sigma_m^2} = \frac{\sigma_m^2}{\sigma_m^2} = 1$$

Even though most computations from CAPM are unnecessary for the topic in question. Introducing the fundamental theorem of CAPM is still important. It is

**Theorem 4.5.1.** *(Fundamental theorem of CAPM). Given an attainable field $\mathcal{A}$ of n securities and a risk-free return R the expected return of the i-th security is*

$$\mu_i = R + \beta_i(\mu_m - R)$$

*[9].*

Since mathematically portfolios and securities are identical. The Theorem 4.5.1 applies to both. This venture into CAPM ends here with one last corollary. There is a simpler method of computing a Beta Factor.

**Corollary 4.5.2.** *(Alternative formula for Beta). Given the assumptions of Theorem 4.5.1, the Beta factor for the i-th security (or portfolio) is*

$$\beta = \frac{\mu_i - R}{\mu_m - R}$$

*[9].*

Corollary A.2.13 allows the Beta factor to move beyond the cumbersome calculation of covariance and variance of two random variables. These computations and the data that doing so would require are costly and, in practice, damaging to any mathematical model that uses a Beta factor.

The expected return of a security or portfolio, as mentioned earlier, depends heavily upon the definition of the return function and the expected value integral. These functions can change rapidly as market behavior changes [9, 20]. Past performance is not necessarily a good indicator of future performance [20]. Calculating variance and covariance depends upon past data or computing hypothetical past data. This is undesirable. Using expected returns and a given risk-free return is far more satisfactory.

| | |
|---|---|
| $\beta_i = 1$ | Equal volatility to market |
| $\beta_i > 1$ | More volatility than market |
| $0 < \beta_i < 1$ | Less volatility than market |
| $\beta_i = 0$ | Not correlated to market |
| $\beta_i < 0$ | Inversely correlated to market |

*Table 4.1*: The relationship of a security or portfolio $\mathbb{W}$, indicated by all possible values of its Beta factor.

# Chapter 5

# EXTENDING MODERN PORTFOLIO THEORY INTO LITERATURE

## 5.1 A mathematical theory of poetry

Having described the essential elements of modern portfolio theory mathematically, it is time to turn to the original question. Is it possible to analyze a poem using modern portfolio theory and produce results that are usable in literary criticism? The first part requires showing that a poem and a portfolio are in a sense the kind of mathematical object. The second part is a reformulation of the goal of all interdisciplinary research. To satisfy the needs of two distinct disciplines is easier to contemplate than accomplish. The standards of proof and the nature of argument are different from necessity. This is particularly apparent between two fields that are completely unrelated like literature and mathematics.

Mathematics uses very specific language to describe clearly defined concepts that are universally consistent. It is of no consequence if an idea in mathematics has any example in nature. Furthermore, mathematics seeks to be independent not just of physical reality but even of visualization [28, 10]. Literary criticism couldn't be more antithetical. The study of literature is grounded firmly in the human experience. More specifically literary criticism describes the human experience of the aesthetics of the written word [37]. Whereas mathematics is logically consistent whether it is conceivable or not, the ideas in literature are conceivable regardless of their logical consistency.

The problems of an interdisciplinary study might seem compounded here because economics and (if the *Aeneid* is used as an example) Classics are added into the mix. Rather than only combining two fields, the goal requires dabbling in four. This, nevertheless, is not the case. Economics is a field that explains human behavior and human preferences using mathematical models. Far from confusing the issue economics provides models that can be manipulated to explain literary behavior. Likewise, using the *Aeneid* as a test case presents questions that are more concrete than taking a piece of contemporary literature. If one wanted to discover the process of composition used by Simon Armitage in his poetic translation of *Sir Gawain and The Green Knight*, it is necessary to search no further than YouTube. In that vast repository there are countless lectures where the Poet Laureate describes his work on his most famous epic. Similarly, comparable questions about any piece of literature written after 1500 can be satisfactorily answered by historical research. Mathematical modeling of human behavior to understand the actions of an ancient poet can be helpful.

However, before extending modern portfolio theory to explore the *Aeneid* two things are essential. First, a mathematical definition of a poem that allows for this model to be used to analyze a poem must be given. Without a definition of a mathematical object that preserves the essential qualities of a physical poem and has the requisite elements needed to be examined by modern portfolio theory, the two things must remain separate. Secondly, it is necessary to demonstrate that the results given by modern portfolio theory can interpret the actions of a poet.

Here is a definition of a poem.

**Definition 5.1.1.** (Mathematical definition of a poem). A poem $\mathbb{G}$ is an experiment that generates a probability space $(\Omega, \mathcal{F}, P)$.

Just like a security, a poem is an experiment with a probability space. Similarly, a longer poem divided into multiple books, like the *Aeneid*, is a poem composed of poems. It is a portfolio. Mathematically, all three objects, securities, portfolios, and poems are the same. The mathematics behind modern portfolio theory only requires a probabilistic experiment that has functions that map onto the real number line. To postulate that the computations in modern portfolio theory are suddenly impossible because an experiment under consideration is not a security or a portfolio would restrict mathematics to the physical world. The fact that mathematics is not limited in such a manner is what G. H. Hardy credited as its ultimate strength [28]. So long as an experiment has real valued functions, it can possess any arbitrary set of outcomes $\Omega$—including the scribbling of a poet.

There are important distinctions between poems and securities in practice. The $\Omega$ of a poem is not the set of all possible market forces. Poems tend to be the work of a small number of authors (frequently one) while market forces are the combined will of a collection of all buyers and sellers in the market—a large number of people. The human desires expressed in poetry and in a market are also very different. However, for continuity with securities let the $\Omega$ of a poem be the collection of possible poetic influences, corresponding as closely as possible to market forces. This would include poetic styles, decisions of meter, preferences of the poet, narrative elements, and linguistic rules, among other things.

The second question requires a little more nuance. Modern portfolio theory depends upon the assumption that an investor (i.e. the person making the decisions about how to build the portfolio) wants to maximize expected return while minimizing return variance. This is a rational choice for financial modeling. No one consciously wants to lose money.

Accounting for opportunity costs, assuming any other optimization would mean just that. Yet, in practice investors are hardly so rational [3].

Optimization in poetry is a more complicated problem. If it is presumed that the number of lines in a verse paragraph in a poem is analogous to a price for a security, clearly it cannot be taken for granted that minimizing return variance and maximizing expected return are optimal for all poets. Expected return of number of lines is not necessarily a good thing, and the corresponding variance might not be bad. Optimization is poet dependent. Thus, to encompass all poetry it would be essential to develop modern portfolio theory to have a variety of optimizations.

This is where the *Aeneid* becomes a very useful example. It is possible to argue that optimization for Virgil corresponds to conventional optimization in modern portfolio theory. The grammarian tradition records that Virgil sought the most dynamism within a strictly ordered framework [63, 52]. Both Virgil's supporters and detractors make special note of Virgil's attention to detail [51, 26, 23]. It can be assumed from this consensus that the minimum variance portfolio would have an expected return and return variance comparable with what Virgil wanted to achieve in each of the books of the *Aeneid*.

### 5.2 Using modern portfolio theory to analyze the *Aeneid*

**Example 5.2.1.** Let $\mathcal{E}_i$ be an experiment for the $i$-th book of the *Aeneid* generating the probability space $(\Omega_i, \mathcal{F}, P_i)$. On the probability space there is a function $Q_i : \Omega_i \to \mathbb{R}_{\geq 0}$ mapping the partitions of the verse paragraphs onto a line number. The sequence $\langle Q_i(\omega) \mid \forall \omega \in \Omega_i \rangle$ is monotonically increasing. The verse paragraphs are numbered $1, 2, 3, \ldots, n_i$ in this book. The verse paragraph partitions used in this paper are taken from the Oxford Classical Text critical edition of the Latin text of Virgil's *Aeneid* [64].

The number of lines of the $t$-th verse paragraph in $\mathcal{E}_i$ is defined by the function $S_i : \Omega \to \mathbb{R}_{\geq 0}$

$$S_i(t) := Q_i(t) - Q_i(t-1)$$

where $Q_i(0) = 0$.

The return for the $t$-th verse section in $\mathcal{E}_i$ is defined by the function $K_i : \Omega \to \mathbb{R}_{\geq 0}$

$$K_i(t) := \frac{|S_i(t) - S_i(t-1)|}{S_i(t-1)}$$

where $S_i(1)$ is undefined.

It might seem convoluted to have a return function for a poem. A return fits more naturally with a security or a portfolio. Return of investments is a common topic of

| $t$ | $Q_1$ | $Q_2$ | $Q_3$ | $Q_4$ | $Q_5$ | $Q_6$ | $Q_7$ | $Q_8$ | $Q_9$ | $Q_{10}$ | $Q_{11}$ | $Q_{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 7 | 20 | 12 | 30 | 34 | 13 | 24 | 17 | 24 | 15 | 28 | 17 |
| 2 | 11 | 39 | 18 | 39 | 41 | 41 | 36 | 35 | 46 | 61 | 58 | 53 |
| 3 | 33 | 56 | 48 | 53 | 71 | 76 | 80 | 65 | 76 | 95 | 99 | 80 |
| 4 | 49 | 76 | 68 | 89 | 103 | 97 | 106 | 80 | 106 | 145 | 121 | 106 |
| 5 | 64 | 104 | 83 | 104 | 113 | 123 | 134 | 101 | 167 | 162 | 138 | 112 |
| 6 | 75 | 144 | 89 | 128 | 123 | 155 | 147 | 125 | 175 | 165 | 181 | 133 |
| 7 | 80 | 194 | 120 | 159 | 150 | 182 | 169 | 151 | 223 | 184 | 202 | 160 |
| 8 | 101 | 233 | 134 | 172 | 182 | 211 | 191 | 174 | 245 | 197 | 224 | 174 |
| 9 | 123 | 249 | 146 | 197 | 224 | 235 | 211 | 183 | 313 | 214 | 242 | 194 |
| 10 | 131 | 267 | 191 | 218 | 243 | 263 | 248 | 218 | 366 | 259 | 295 | 215 |
| 11 | 141 | 297 | 257 | 237 | 285 | 267 | 285 | 261 | 419 | 275 | 301 | 237 |
| 12 | 156 | 317 | 277 | 278 | 314 | 281 | 322 | 279 | 445 | 286 | 335 | 256 |
| 13 | 179 | 346 | 293 | 295 | 347 | 294 | 340 | 305 | 449 | 307 | 375 | 286 |
| 14 | 197 | 369 | 319 | 330 | 361 | 336 | 372 | 336 | 458 | 344 | 396 | 310 |
| 15 | 207 | 401 | 355 | 361 | 386 | 383 | 405 | 369 | 472 | 361 | 444 | 345 |
| 16 | 222 | 437 | 373 | 392 | 420 | 416 | 434 | 406 | 502 | 379 | 467 | 382 |
| 17 | 253 | 452 | 440 | 415 | 460 | 425 | 444 | 423 | 524 | 398 | 497 | 410 |
| 18 | 271 | 468 | 462 | 436 | 484 | 476 | 474 | 453 | 529 | 425 | 521 | 467 |
| 19 | 296 | 505 | 471 | 449 | 518 | 493 | 510 | 469 | 589 | 438 | 531 | 499 |
| 20 | 304 | 525 | 505 | 473 | 544 | 534 | 539 | 519 | 620 | 509 | 556 | 528 |
| 21 | 324 | 558 | 520 | 503 | 574 | 575 | 571 | 540 | 637 | 542 | 596 | 553 |
| 22 | 334 | 566 | 547 | 521 | 603 | 627 | 600 | 553 | 671 | 574 | 617 | 592 |
| 23 | 371 | 587 | 569 | 552 | 622 | 636 | 640 | 584 | 690 | 605 | 647 | 613 |
| 24 | 386 | 623 | 587 | 570 | 663 | 678 | 646 | 607 | 716 | 632 | 663 | 649 |
| 25 | 401 | 633 | 612 | 583 | 679 | 702 | 654 | 629 | 755 | 652 | 689 | 696 |
| 26 | 417 | 649 | 654 | 629 | 699 | 723 | 669 | 670 | 777 | 688 | 724 | 745 |
| 27 | 440 | 670 | 674 | 658 | 718 | 751 | 677 | 728 | 818 | 718 | 767 | 765 |
| 28 | 463 | 678 | 691 | 670 | 745 | 755 | 690 | 731 | | 746 | 793 | 790 |
| 29 | 493 | 691 | 715 | 678 | 761 | 800 | 704 | | | 754 | 835 | 842 |
| 30 | 519 | 751 | | 692 | 778 | 853 | 722 | | | 768 | 867 | 868 |
| 31 | 543 | 804 | | 705 | 798 | 892 | 732 | | | 790 | 895 | 886 |
| 32 | 560 | | | | 826 | 901 | 743 | | | 832 | 915 | 918 |
| 33 | 578 | | | | 851 | | 749 | | | 872 | | 952 |
| 34 | 612 | | | | 871 | | 760 | | | 908 | | |
| 35 | 642 | | | | | | 782 | | | | | |
| 36 | 656 | | | | | | 802 | | | | | |
| 37 | 694 | | | | | | 817 | | | | | |
| 38 | 722 | | | | | | | | | | | |
| 39 | 756 | | | | | | | | | | | |

*Table 5.1*: Table of $Q$ functions for the *Aeneid*.

conversation in economics, while no one has bothered to think about the returns of the line counts of verse paragraphs of a poem before this paper. Usage doesn't make a concept more or less meaningful. This is a materialistic bias. Mathematically, securities, portfolios, and

| $t$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ | $S_6$ | $S_7$ | $S_8$ | $S_9$ | $S_{10}$ | $S_{11}$ | $S_{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 7 | 20 | 12 | 30 | 34 | 13 | 24 | 17 | 24 | 15 | 28 | 17 |
| 2 | 4 | 19 | 6 | 9 | 7 | 28 | 12 | 18 | 22 | 46 | 30 | 36 |
| 3 | 22 | 17 | 30 | 14 | 30 | 35 | 44 | 30 | 30 | 34 | 41 | 27 |
| 4 | 16 | 20 | 20 | 36 | 32 | 21 | 26 | 15 | 30 | 50 | 22 | 26 |
| 5 | 15 | 28 | 15 | 15 | 10 | 26 | 28 | 21 | 61 | 17 | 17 | 6 |
| 6 | 11 | 40 | 6 | 24 | 10 | 32 | 13 | 24 | 8 | 3 | 43 | 21 |
| 7 | 5 | 50 | 31 | 31 | 27 | 27 | 22 | 26 | 48 | 19 | 21 | 27 |
| 8 | 21 | 39 | 14 | 13 | 32 | 29 | 22 | 23 | 22 | 13 | 22 | 14 |
| 9 | 22 | 16 | 12 | 25 | 42 | 24 | 20 | 9 | 68 | 17 | 18 | 20 |
| 10 | 8 | 18 | 45 | 21 | 19 | 28 | 37 | 35 | 53 | 45 | 53 | 21 |
| 11 | 10 | 30 | 66 | 19 | 42 | 4 | 37 | 43 | 53 | 16 | 6 | 22 |
| 12 | 15 | 20 | 20 | 41 | 29 | 14 | 37 | 18 | 26 | 11 | 34 | 19 |
| 13 | 23 | 29 | 16 | 17 | 33 | 13 | 18 | 26 | 4 | 21 | 40 | 30 |
| 14 | 18 | 23 | 26 | 35 | 14 | 42 | 32 | 31 | 9 | 37 | 21 | 24 |
| 15 | 10 | 32 | 36 | 31 | 25 | 47 | 33 | 33 | 14 | 17 | 48 | 35 |
| 16 | 15 | 36 | 18 | 31 | 34 | 33 | 29 | 37 | 30 | 18 | 23 | 37 |
| 17 | 31 | 15 | 67 | 23 | 40 | 9 | 10 | 17 | 22 | 19 | 30 | 28 |
| 18 | 18 | 16 | 22 | 21 | 24 | 51 | 30 | 30 | 5 | 27 | 24 | 57 |
| 19 | 25 | 37 | 9 | 13 | 34 | 17 | 36 | 16 | 60 | 13 | 10 | 32 |
| 20 | 8 | 20 | 34 | 24 | 26 | 41 | 29 | 50 | 31 | 71 | 25 | 29 |
| 21 | 20 | 33 | 15 | 30 | 30 | 41 | 32 | 21 | 17 | 33 | 40 | 25 |
| 22 | 10 | 8 | 27 | 18 | 29 | 52 | 29 | 13 | 34 | 32 | 21 | 39 |
| 23 | 37 | 21 | 22 | 31 | 19 | 9 | 40 | 31 | 19 | 31 | 30 | 21 |
| 24 | 15 | 36 | 18 | 18 | 41 | 42 | 6 | 23 | 26 | 27 | 16 | 36 |
| 25 | 15 | 10 | 25 | 13 | 16 | 24 | 8 | 22 | 39 | 20 | 26 | 47 |
| 26 | 16 | 16 | 42 | 46 | 20 | 21 | 15 | 41 | 22 | 36 | 35 | 49 |
| 27 | 23 | 21 | 20 | 29 | 19 | 28 | 8 | 58 | 41 | 30 | 43 | 20 |
| 28 | 23 | 8 | 17 | 12 | 27 | 4 | 13 | 3 | | 28 | 26 | 25 |
| 29 | 30 | 13 | 24 | 8 | 16 | 45 | 14 | | | 8 | 42 | 52 |
| 30 | 26 | 60 | | 14 | 17 | 53 | 18 | | | 14 | 32 | 26 |
| 31 | 24 | 53 | | 13 | 20 | 39 | 10 | | | 22 | 28 | 18 |
| 32 | 17 | | | | 28 | 9 | 11 | | | 42 | 20 | 32 |
| 33 | 18 | | | | 25 | | 6 | | | 40 | | 34 |
| 34 | 34 | | | | 20 | | 11 | | | 36 | | |
| 35 | 30 | | | | | | 22 | | | | | |
| 36 | 14 | | | | | | 20 | | | | | |
| 37 | 38 | | | | | | 15 | | | | | |
| 38 | 28 | | | | | | | | | | | |
| 39 | 34 | | | | | | | | | | | |

*Table 5.2*: Table of $S$ functions for the *Aeneid*.

poems are identical. They are all experiments that generate a probability space. It makes no more sense to talk about a return of a security that it does to talk about the return of a verse paragraph. The mathematics of the model must remain the focus, not the utility (or lack thereof) of functions being used therein.

| $t$ | $K_1$ | $K_2$ | $K_3$ | $K_4$ | $K_5$ | $K_6$ | $K_7$ | $K_8$ | $K_9$ | $K_{10}$ | $K_{11}$ | $K_{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.8447 | 0.2624 | 0.7671 | 0.6506 | 0.7283 | 0.0869 | 0.5342 | 0.3013 | 0.1460 | 0.7857 | 0.1646 | 0.3975 |
| 2 | 0.1460 | 0.3401 | 0.1646 | 0.4565 | 0.1646 | 0.3587 | 0.7081 | 0.1646 | 0.1646 | 0.3199 | 0.5916 | 0.0481 |
| 3 | 0.3789 | 0.2236 | 0.2236 | 0.3975 | 0.2422 | 0.1848 | 0.0093 | 0.4177 | 0.1646 | 0.9410 | 0.1460 | 0.0093 |
| 4 | 0.4177 | 0.0869 | 0.4177 | 0.4177 | 0.6118 | 0.0093 | 0.0869 | 0.1848 | 1.3680 | 0.3401 | 0.3401 | 0.7671 |
| 5 | 0.5730 | 0.5528 | 0.7671 | 0.0683 | 0.6118 | 0.2422 | 0.4953 | 0.0683 | 0.6894 | 0.8835 | 0.6692 | 0.1848 |
| 6 | 0.8059 | 0.9410 | 0.2034 | 0.2034 | 0.0481 | 0.0481 | 0.1460 | 0.0093 | 0.8633 | 0.2624 | 0.1848 | 0.0481 |
| 7 | 0.1848 | 0.5140 | 0.4565 | 0.4953 | 0.2422 | 0.1258 | 0.1460 | 0.1072 | 0.1460 | 0.4953 | 0.1460 | 0.4565 |
| 8 | 0.1460 | 0.3789 | 0.5342 | 0.0295 | 0.6304 | 0.0683 | 0.2236 | 0.6506 | 1.6397 | 0.3401 | 0.3013 | 0.2236 |
| 9 | 0.6894 | 0.3013 | 0.7469 | 0.1848 | 0.2624 | 0.0869 | 0.4363 | 0.3587 | 1.0574 | 0.7469 | 1.0574 | 0.1848 |
| 10 | 0.6118 | 0.1646 | 1.5621 | 0.2624 | 0.6304 | 0.8447 | 0.4363 | 0.6692 | 1.0574 | 0.3789 | 0.7671 | 0.1460 |
| 11 | 0.4177 | 0.2236 | 0.2236 | 0.5916 | 0.1258 | 0.4565 | 0.4363 | 0.3013 | 0.0093 | 0.5730 | 0.3199 | 0.2624 |
| 12 | 0.1072 | 0.1258 | 0.3789 | 0.3401 | 0.2810 | 0.4953 | 0.3013 | 0.0093 | 0.8447 | 0.1848 | 0.5528 | 0.1646 |
| 13 | 0.3013 | 0.1072 | 0.0093 | 0.3587 | 0.4565 | 0.6304 | 0.2422 | 0.2034 | 0.6506 | 0.4363 | 0.1848 | 0.0683 |
| 14 | 0.6118 | 0.2422 | 0.3975 | 0.2034 | 0.0295 | 0.8245 | 0.2810 | 0.2810 | 0.4565 | 0.3401 | 0.8633 | 0.3587 |
| 15 | 0.4177 | 0.3975 | 0.3013 | 0.2034 | 0.3199 | 0.2810 | 0.1258 | 0.4363 | 0.1646 | 0.3013 | 0.1072 | 0.4363 |
| 16 | 0.2034 | 0.4177 | 1.6009 | 0.1072 | 0.5528 | 0.6506 | 0.6118 | 0.3401 | 0.1460 | 0.2624 | 0.1646 | 0.0869 |
| 17 | 0.3013 | 0.3789 | 0.1460 | 0.1848 | 0.0683 | 0.9798 | 0.1646 | 0.1646 | 0.8059 | 0.0481 | 0.0683 | 1.2127 |
| 18 | 0.0295 | 0.4363 | 0.6506 | 0.4953 | 0.3199 | 0.3401 | 0.3975 | 0.3789 | 1.3292 | 0.4953 | 0.6118 | 0.2422 |
| 19 | 0.6894 | 0.2236 | 0.3199 | 0.0683 | 0.0093 | 0.5916 | 0.1258 | 0.9410 | 0.2034 | 1.7562 | 0.0295 | 0.1258 |
| 20 | 0.2236 | 0.2810 | 0.4177 | 0.1646 | 0.1646 | 0.5916 | 0.2422 | 0.1848 | 0.3401 | 0.2810 | 0.5528 | 0.0295 |
| 21 | 0.6118 | 0.6894 | 0.0481 | 0.3013 | 0.1258 | 1.0186 | 0.1258 | 0.4953 | 0.3199 | 0.2422 | 0.1848 | 0.5140 |
| 22 | 0.4363 | 0.1848 | 0.1460 | 0.2034 | 0.2624 | 0.6506 | 0.5528 | 0.2034 | 0.2624 | 0.2034 | 0.1646 | 0.1848 |
| 23 | 0.4177 | 0.3975 | 0.3013 | 0.3013 | 0.5916 | 0.6304 | 0.7671 | 0.1072 | 0.0093 | 0.0481 | 0.3789 | 0.3975 |
| 24 | 0.4177 | 0.6118 | 0.0295 | 0.4953 | 0.3789 | 0.0683 | 0.6894 | 0.1460 | 0.5140 | 0.2236 | 0.0093 | 0.8245 |
| 25 | 0.3789 | 0.3789 | 0.6304 | 0.7857 | 0.2236 | 0.1848 | 0.4177 | 0.5916 | 0.1460 | 0.3975 | 0.3587 | 0.9022 |
| 26 | 0.1072 | 0.1848 | 0.2236 | 0.1258 | 0.2624 | 0.0869 | 0.6894 | 1.2515 | 0.5916 | 0.1646 | 0.6692 | 0.2236 |
| 27 | 0.1072 | 0.6894 | 0.3401 | 0.5342 | 0.0481 | 0.8447 | 0.4953 | 0.8835 | | 0.0869 | 0.0093 | 0.0295 |
| 28 | 0.1646 | 0.4953 | 0.0683 | 0.6894 | 0.3789 | 0.7469 | 0.4565 | | | 0.6894 | 0.6304 | 1.0186 |
| 29 | 0.0093 | 1.3292 | | 0.4565 | 0.3401 | 1.0574 | 0.3013 | | | 0.4565 | 0.2422 | 0.0093 |
| 30 | 0.0683 | 1.0574 | | 0.4953 | 0.2236 | 0.5140 | 0.6118 | | | 0.1460 | 0.0869 | 0.3013 |
| 31 | 0.3401 | | | | 0.0869 | 0.6506 | 0.5730 | | | 0.6304 | 0.2236 | 0.2422 |
| 32 | 0.3013 | | | | 0.0295 | | 0.7671 | | | 0.5528 | | 0.3199 |
| 33 | 0.3199 | | | | 0.2236 | | 0.5730 | | | 0.3975 | | |
| 34 | 0.1646 | | | | | | 0.1460 | | | | | |
| 35 | 0.4565 | | | | | | 0.2236 | | | | | |
| 36 | 0.4751 | | | | | | 0.4177 | | | | | |
| 37 | 0.0869 | | | | | | | | | | | |
| 38 | 0.3199 | | | | | | | | | | | |

*Table 5.3*: Table of $K$ functions for the *Aeneid*.

The reason for having a return function for a poem is that modern portfolio theory analyzes returns. Just as some numerical methods require normalization, modern portfolio theory requires the initial data to pass through a return function. To understand why modern portfolio theory needs a return function, it is necessary to first explain what financial profit is. To make a long story short, financial profit is error between price and value [3, 70, 48]. A return function is a composition of relative error. Mapping standard deviation of return and expected return on a Cartesian plane rather than expected profit or standard deviation of profit focuses on market behavior (or in the case of a poem—poetic preferences) rather than the chaos of the whole picture.

This return function is different than the one given in Chapter 2. Return functions are experiment dependent. The return function given here is the relative error between $S_i(t)$ and

| Book | $\mu$ | $\sigma^2$ |
|---|---|---|
| Book 1 | 0.3496 | 0.0478 |
| Book 2 | 0.4206 | 0.0830 |
| Book 3 | 0.4313 | 0.1542 |
| Book 4 | 0.3424 | 0.0399 |
| Book 5 | 0.2932 | 0.0427 |
| Book 6 | 0.4629 | 0.1040 |
| Book 7 | 0.3877 | 0.0461 |
| Book 8 | 0.3648 | 0.0912 |
| Book 9 | 0.5419 | 0.2101 |
| Book 10 | 0.4367 | 0.1090 |
| Book 11 | 0.3478 | 0.0757 |
| Book 12 | 0.3256 | 0.0958 |

*Table 5.4*: Table of expected return and variance for each book.

$S_i(t-1)$. The simple return function for a security is the relative error between to adjacent prices with a sign function corresponding to the direction of movement. There is no reason to believe that whether adjacent verse paragraphs are longer or shorter would have any psychological impact on the poet where as it does on a market; thus the sign function is dropped here.

The expected return of the $i$-th book is the simple mean of all returns

$$\mu_i = \frac{1}{n_i}\sum_{t}^{n_i} K_i(t),$$

and the variance is

$$\sigma_i^2 = \frac{1}{n_i}\sum_{t}^{n_i} (K_i(t)-\mu_i)^2.$$

The list of $\mu$ and $\sigma^2$ for all of the books can be found on Table 5.2.1. Likewise, a plot of the $(\sigma,\mu)$ tuples for each of the books can be seen on Figure 5.1.

The attainable field for the *Aeneid* is $\mathfrak{A} = (\prod_{i=1}^{12}\Omega_i, \bigotimes_{i=1}^{12}\mathcal{F}_i, \mathcal{P})$. If $C$ is the covariance matrix for the *Aeneid*, the vector of weights of the minimum variance portfolio is

$$\mathbf{w}_{min} = \frac{C^{-1}\mathbf{1}_{12}}{\mathbf{1}_{12}^T C^{-1}\mathbf{1}_{12}}.$$

The values of this weight vector are in Table 5.2.1. The $(\sigma,\mu)$ tuple for $\mathfrak{A}_{min}$ is $(0.0647, 0.3723)$.

To compute the minimum variance line $\mathfrak{A}_{MVL}$, assume that there is a risk-free return

| **Book** | **w** |
|---|---|
| Book 1 | 0.1876 |
| Book 2 | 0.0844 |
| Book 3 | -0.0520 |
| Book 4 | 0.1549 |
| Book 5 | 0.1630 |
| Book 6 | 0.1028 |
| Book 7 | 0.1111 |
| Book 8 | 0.0795 |
| Book 9 | 0.0548 |
| Book 10 | 0.0461 |
| Book 11 | 0.0559 |
| Book 12 | 0.0119 |

*Table 5.5*: Table of $\mathbf{w}_{min}$.

| **Book** | **w** |
|---|---|
| Book 1 | 0.1571 |
| Book 2 | -0.0063 |
| Book 3 | -0.0948 |
| Book 4 | 0.1937 |
| Book 5 | -0.3452 |
| Book 6 | 0.2330 |
| Book 7 | 0.4486 |
| Book 8 | -0.1049 |
| Book 9 | 0.3332 |
| Book 10 | 0.2451 |
| Book 11 | -0.2337 |
| Book 12 | -0.0153 |

*Table 5.6*: Table of $\mathbf{w}_{MP}$.

$R = 0.9(\mu_{min})$. The vector of weights of market portfolio $\mathfrak{A}_{MP}$ is

$$\mathbf{w}_m = \frac{C^{-1}(\boldsymbol{\mu} - \mathbf{1}_{12}R)}{\mathbf{1}_{12}^T C^{-1}(\boldsymbol{\mu} - \mathbf{1}_{12}R)}.$$

The values of this weight vector are in Table 5.2.1. The $(\sigma, \mu)$ tuple for $\mathfrak{A}_{min}$ is $(0.1375, 0.5031)$. Having $\mathfrak{A}_{min}$ and $\mathfrak{A}_{MP}$, from Corollary 4.4.6, the weights of every portfolio on $\mathfrak{A}_{MVL}$ can be found. Figure 5.1 has the $(\sigma, \mu)$ graph of $\mathfrak{A}_{MVL}$.

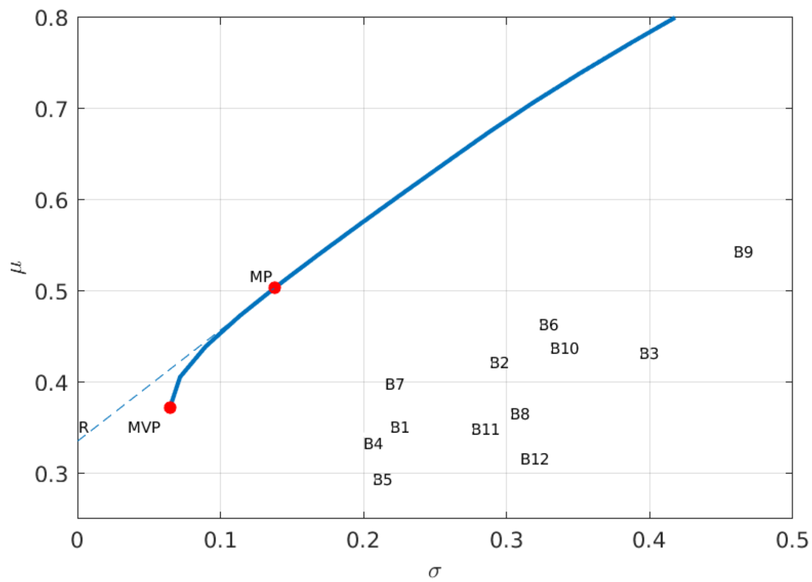


*Figure 5.1*: $(\sigma,\mu)$ plot of each book in the *Aeneid* with efficient frontier, minimum variance portfolio and hypothetical market portfolio.

The Beta factor of the $i$-th book relative to $\mathfrak{A}_{min}$ can be computed

$$\beta_i = \frac{\mu_i}{\mu_{min}}.$$

The risk-free return setting the minimum variance portfolio as base comparison is 0. The risk-free return $R$ is not used here because it was only constructed to find a second portfolio in $\mathfrak{A}_{MVL}$. A list of the Beta factors for each of the books can be found in Table 5.2.1.

All of the books have Beta factors that are reasonably close to 1. This implies that all of the books tend to behaved like the ideal. Virgil may have wanted to do more work, but he had been writing this poem for eleven years [52, 63, 26]. These numbers reflect that. However, there are a few that stand out as abnormally distant. Books 9, 5, 12, and 6 have the Beta factors furthest away from 1. These are the most incomplete books. Likewise, the Beta factors closest to 1 are Books 7,8,1, and 11. This is set of the most complete books.

Finding the Beta factors of the books of the *Aeneid* is an important step. This quantifies completeness. However, to add an element to the great of mass of research that has been done on the *Aeneid* without relating it to earlier research is counterproductive. New research must complement the old [38]. This is where mathematical research must complement literary criticism, and correspondingly abandon some of the rigor of mathematics.

**Example 5.2.2.** A good test case to see how the set of Beta factors can be used in literary

| Book | $\beta$ |
|---|---|
| Book 1 | 0.9390 |
| Book 2 | 1.1297 |
| Book 3 | 1.1585 |
| Book 4 | 0.9197 |
| Book 5 | 0.7875 |
| Book 6 | 1.2434 |
| Book 7 | 1.0414 |
| Book 8 | 0.9800 |
| Book 9 | 1.4556 |
| Book 10 | 1.1730 |
| Book 11 | 0.9341 |
| Book 12 | 0.8747 |

*Table 5.7*: Table of the Beta factors for each book.

criticism would be to compare them with the *hemistich*[1] in the *Aeneid*. There are 57 lines that are *hemistich* in the epic [50]. Debate has raged for the last twenty centuries over whether or not they are a residue of the poem being incomplete or intended by the poet [63, 21].

One might want to run a co-integration test comparing the series of Beta factors and with the series of *hemistich* to see if they are related. Unfortunately, this is not possible. Cointegration test like the Engle-Granger or the Johansen Test require at least 20 elements in each vector to give accurate results [12]. Thus, it is necessary to simply look at the two data sets and construct a plausible argument. In short, it is necessary to engage in literary criticism.

For simplicity the discussion of the relationship between completeness and *hemistich* will be limited. In Figure 5.2 the number of *hemistich* in each book is compared with the Beta factors minus 1 for each book. The value $\beta - 1$ is used rather than $\beta$ to have 0 as the neutral value rather than 1. Looking at Figure 5.2 there is no apparent relationship between the distance of the Beta factor from 1 and the number of *hemistich* the books. The lines traced by both series do follow similar paths at the beginning and the end, but the middle of both of these lines differ considerably. It is reasonable to assume, since there is no pattern between completeness and *hemistich*, that some of the *hemistich* were intended and some not.

Looking at Table 5.2.2 it can be seen that the most complete book, viz. Book 6 has a large number of *hemistich*. This alone suggests that Virgil intended some *hemistich*. However, the

[1]Hemistich are incomplete hexameter verses.

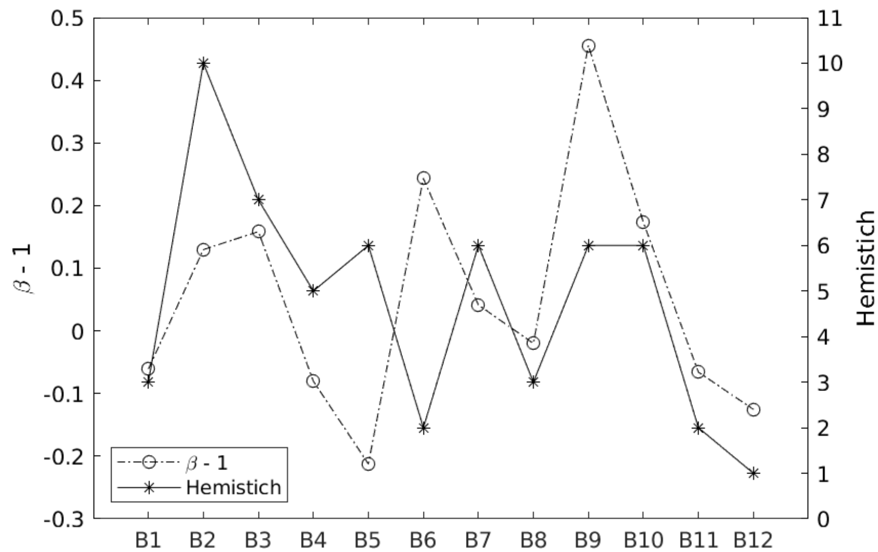


*Figure 5.2*: The Beta factors compared with the number of hemistich in each book.

| Book | $\beta$ | Hemistich |
|---|---|---|
| Book 1 | 0.9390 | 3 |
| Book 2 | 1.1297 | 10 |
| Book 3 | 1.1585 | 7 |
| Book 4 | 0.9197 | 5 |
| Book 5 | 0.7875 | 6 |
| Book 6 | 1.2434 | 2 |
| Book 7 | 1.0414 | 6 |
| Book 8 | 0.9800 | 3 |
| Book 9 | 1.4556 | 6 |
| Book 10 | 1.1730 | 6 |
| Book 11 | 0.9341 | 2 |
| Book 12 | 0.8747 | 1 |

*Table 5.8*: Table enumerating the Beta factor for all of the book and number of *hemistich* in each.

other books with the most completeness have a low level of *hemistich* adding some doubt to this conclusion. The books with the most *hemistich*, Books 2 and 3, have a middling level of completeness. It is possible that the *hemistich* are a intermediary stage of the poets revision process.

# Chapter 6

# CONCLUSION

The strength of Greek geometry over primitive arithmetic was the power of abstraction. The Greek mathematician knew that there existed a number $c$ such that for any $a, b$ where $a^2 + b^2 = c^2$. His Egyptian or Sumerian predecessor would have only known of a finite number of Pythagorean triples. In a similar fashion Hilbert was able to ask his 23 problems because of the trend towards axiomatization and generalization that was sweeping mathematics in his era. Abstraction is not by any stretch of the imagination the only quality of mathematics that makes it uncommonly well dispositioned to describe the universe, but it is a very potent one that tends to be overlooked except by mathematicians.

It may seem intuitive to a mathematician that an experiment is an experiment regardless of whether its elements are the lines of a poem, corporate decisions, or a finite number of coin tosses. That is the way mathematical definitions work. An object is such and such or an object is not such and such. But to the uninitiated in mathematics, it is inconceivable to think of a poem and a stock as two examples of the same object. Most people want mathematics to play the same role that arithmetic did for their primitive ancestors. They want it to provide computation and calculation. This is the greatest barrier towards non-mathematicians applying mathematics.

Mathematics is not about computation and calculation. Mathematics is about well-defined abstract collections and their relationships with each other. Computation is simply a pleasant by product. Of course, aspects of literature can be described mathematically. Literature can never be as dependent upon mathematical laws as a subject like physics or mechanics because of the amorphous nature of the study. Applying mathematics to literature simply takes clearly defining what mathematical objects are seen in literature; the most important relationships have mostly likely already been found.

# Appendix A

# EXTRA DEFINITIONS AND THEOREMS AND PROOFS

## A.1 From Chapter 3

**Definition A.1.1.** (Simple function). Let $(E, \mathcal{R}, m)$ be a measure space and the set $A \subseteq E$ be $\mathcal{R}$-measurable. Assume that $\mathbf{1}_{A_i}$ are the indicator functions of the sets $A_i \subset A$ for $i = 1, 2, \ldots, k$ where $k \geq 0$, i.e.

$$\mathbf{1}_E(x) = \begin{cases} 1 & \text{if} \quad x \in A_i \\ 0 & \text{if} \quad x \notin A_i \end{cases}$$

The function $f : A \to R \subseteq \mathbb{R}$ is a *simple function* if and only if

$$f = \sum_{i}^{k} c_i \mathbf{1}_{A_i}$$

where $c_1, c_2, \ldots, c_k \in \mathbb{R}$ [60]

For a Lebesgue measurable set $B \subseteq E$ the integral with respect to the measure $m$ of the indicator function $\mathbf{1}_B$ is

$$\int_E \mathbf{1}_B dm = m(B).$$

**Definition A.1.2.** (Integral of an unsigned simple function). Let $(E, \mathcal{R}, m)$ be a measure space and the set $A$ be a subset or equal to the set $E$. Assume that $\mathbf{1}_A$ is the indicator function of the $\mathcal{R}$-measurable sets $A_i \subset A$ for $i = 1, 2, \ldots, k$ where $k \geq 0$, and let the function $f : A \to R \subseteq \mathbb{R}$ be an unsigned simple function such that $f = \sum_{i=1}^{k} c_i \mathbf{1}_{A_i}$ where $c_1, c_2, \ldots, c_k \in \mathbb{R}$. The integral of $f$ is defined as

$$\int_E f \, dm = \sum_{i=1}^{k} c_i m(A_i)$$

[60].

The above definition implies that the integral of a simple function will always take values in $[0, \infty]$ because each element $c_i m(E_i)$ for $i = 1, 2, \ldots, k$ are in the same interval. These definitions allow simple function to be measured by decomposing them into a set of scalars and a corresponding set of indicator functions.

**Definition A.1.3.** (Unsigned Measurable Function). Let $(E, \mathcal{R}, m)$ be a measure space. The unsigned function $f : E \to R \subseteq \mathbb{R}$ is a $\mathcal{R}$-measurable function if there exists a sequence of simple functions $f_1, f_2, f_3, \cdots : E \to [0, \infty]$ where $f(x) = \lim_{i \in \mathbb{N}} f_i(x)$ for every $x \in E$ [60].

Now, it is possible to define a Lebesgue integral.

**Definition A.1.4.** (Abstract unsigned Lebesgue integral). Let $(E, \mathcal{R}, m)$ be a measure space. Likewise, assume $f : E \to R \subseteq \mathbb{R}$ is an unsigned $\mathcal{R}$-measurable function. Then the Lebesgue integral on $E$ of $f$ is

$$\int_E f \, dm = \lim_{i \to \infty} \int_E f_i \, dm.$$

where $f_1, f_2, f_3, \cdots : E \to R \subseteq \mathbb{R}$ is a sequence of simple functions [60].

Formulating these definitions for unsigned functions poses no difficulty for measuring signed functions. Let $f$ be a signed function on the measure space $(E, \mathcal{R}, m)$. The absolute value $|f|$ is an unsigned function on the same measure space that has the same magnitude as $f$. Likewise,

$$f = \text{sgn}(f)|f|.$$

**Definition A.1.5.** (Abstract Lebesgue integral). Let $(E, \mathcal{R}, m)$ be a measure space. Likewise, assume $f : E \to R \subseteq \mathbb{R}$ is an $\mathcal{R}$-measurable function. Assume that $f_+$ is an unsigned $\mathcal{R}$-measurable function that point-wise convergent on every point $f(x) \geq 0$ for all $x \in E$. Correspondingly assume that $f_-$ is an unsigned $\mathcal{R}$-measurable function such that $\text{sgn}(f)f_-$ is point-wise convergent on every point $f(x) < 0$ for all $x \in E$. The Lebesgue integral on $E$ of $f$ is

$$\int_E f \, dm = \int_E f_+(x) \, dm(x) - \int_E f_- \, dm$$

[60].

Before leaving measure theory one last concept shall be touched upon. The definition of a measurable function requires it to be point-wise convergent with a sequence of step functions. Reading the given definition of integration might make it seem that integration requires the same. This is not so. The equivalence class of a measurable function extends the utility of integration.

**Definition A.1.6.** (Equivalence Class). Let $(E, \mathcal{R}, m)$ be a measure space. Assume that the functions $f : E \to [0, \infty]$ and $f_0 : E \to [0, \infty]$ are $\mathcal{R}$-measurable. If the symmetric difference[1]

[1] Let $A$ and $B$ be two sets. The symmetric difference of $A$ and $B$ is defined as

$$A \triangle B := (A \backslash B) \cup (B \backslash A).$$

of the $f(E)$ and $f_0(E)$ has a measure $m(f(E) \triangle f_0(E)) = 0$, then the functions $f$ and $f_0$ are equal almost everywhere. This is written as

$$f = f_0 \quad (a.e.).$$

The equivalence class of $f$ is the set of all $\mathcal{R}$-measurable functions $g : D \to [0, \infty]$ such that $f$ equals $g$ almost everywhere, or, in notation,

$$[f] = \{g : f = g \quad (a.e.)\}$$

[60].

## A.2 From Chapter 4

**Theorem A.2.1.** *(Tobin two security theorem). Let $\mathbb{S}_1$ and $\mathbb{S}_2$ be two securities. If $\mu_1 \neq \mu_2$ and $-1 < \rho_{12} < 1$ then the $(\sigma, \mu)$ plot of the attainable field $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mathcal{P})$ will lie on a hyperbola that has a center on the $\mu$-axis [9].*

*Proof.* The equation of a hyperbola is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \tag{A.1}$$

where $(h, k)$ are the coordinates of the center, $(x, y)$ are the coordinates of points on the hyperbola, and $a, b$ are coefficients. A hyperbola with a center on the horizontal axis is

$$\frac{(x)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$

Assume that $\mu_1 - \mu_2 > 0$. Let $x = \sigma_{port}$ and $y = \mu_{port}$. From the equations of expected return and return variance,

$$x^2 = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_{12} \tag{A.2}$$

$$y = w\mu_1 + (1-w)\mu_2. \tag{A.3}$$

Solving (A.3) for $w$

$$\begin{aligned} y &= w\mu_1 + (1-w)\mu_2 \\ &= w\mu_1 + \mu_2 - w\mu_2 \\ y - \mu_2 &= w(\mu_1 - \mu_2) \\ w &= \frac{y - \mu_2}{\mu_1 - \mu_2}. \end{aligned} \tag{A.4}$$

Substituting (A.4) into $(w-1)$ simplifies.

$$
\begin{aligned}
(1-w)^2 &= \left(1-\frac{y-\mu_2}{\mu_1-\mu_2}\right)\left(1-\frac{y-\mu_2}{\mu_1-\mu_2}\right) \\
&= \left(\frac{1}{\mu_1-\mu_2}\right)^2\left((\mu_1-\mu_2)^2-2(y-\mu_2)(\mu_1-\mu_2)+(y-\mu_2)^2\right) \\
&= \left(\frac{1}{\mu_1-\mu_2}\right)^2\left(\mu_1^2-2\mu_1\mu_2+\mu_2^2-2y\mu_1+2y\mu_2+2\mu_1\mu_2-2\mu_2^2+y^2-2y\mu_2+\mu_2^2\right) \\
&= \left(\frac{\mu_1-y}{\mu_1-\mu_2}\right)^2. \qquad \text{(A.5)}
\end{aligned}
$$

Taking (A.4), (A.5), and (A.2) produces

$$
\begin{aligned}
x^2 &= \left(\frac{y-\mu_2}{\mu_1-\mu_2}\right)^2\sigma_1^2+\left(\frac{\mu_1-y}{\mu_1-\mu_2}\right)^2\sigma_2^2+2\left(\frac{y-\mu_2}{\mu_1-\mu_2}\right)\left(\frac{\mu_1-y}{\mu_1-\mu_2}\right)\sigma_{12} \\
&= \left(\frac{1}{(\mu_1-\mu_2)^2}\right)\left(\left(\frac{y-\mu_2}{\mu_1-\mu_2}\right)^2\sigma_1^2+\left(\frac{\mu_1-y}{\mu_1-\mu_2}\right)^2\sigma_2^2+2\left(\frac{y-\mu_2}{\mu_1-\mu_2}\right)\left(\frac{\mu_1-y}{\mu_1-\mu_2}\right)\sigma_{12}\right) \\
&= \left(\frac{1}{(\mu_1-\mu_2)^2}\right)\left(\left(y^2-2y\mu_2+\mu_2^2\right)\sigma_1^2+\left(y^2-2y\mu_1+\mu_1^2\right)\sigma_2^2\right. \\
&\qquad \left.+2\left(y^2+y\mu_1+y\mu_2-\mu_1\mu_2\right)\sigma_{12}\right) \qquad \text{(A.6)}
\end{aligned}
$$

Let

$$A = (\mu_1-\mu_2)^2, \qquad \text{(A.7)}$$

$$B = \sigma_1^2+\sigma_2^2-2\sigma_{12}, \qquad \text{(A.8)}$$

$$C = \sigma_1^2\mu_2+\sigma_2^2\mu_1-\sigma_{12}(\mu_1+\mu_2), \qquad \text{(A.9)}$$

$$D = \sigma_1^2\mu_2^2+\sigma_2^2\mu_1^2-\sigma_{12}\mu_1\mu_2. \qquad \text{(A.10)}$$

Thus, from (A.6), (A.7), (A.8), (A.9), and (A.10)

$$
\begin{aligned}
x^2 &= \frac{1}{A}\left(By^2-2Cy+D\right) \\
&= \frac{B}{A}\left(y^2-\frac{2C}{B}y+\frac{D}{B}\right) \\
&= \frac{B}{A}\left(\left(y-\frac{C}{B}\right)^2-\frac{C^2}{B^2}+\frac{D}{B}\right) \qquad \text{(A.11)}
\end{aligned}
$$

Let

$$k = \frac{C}{B} \qquad \text{(A.12)}$$

$$\ell = \frac{1}{B}(BD-C^2). \qquad \text{(A.13)}$$

Applying (A.12) and (A.13) to (A.11)

$$
\begin{aligned}
x^2 &= \frac{1}{A}\left(B\,(y-k)^2+\ell\right) \\
&= \frac{B\,(y-k)^2}{A}+\frac{\ell}{A} \\
x^2-\frac{B\,(y-k)^2}{A} &= \frac{\ell}{A} \\
\frac{Ax^2}{\ell}-\frac{B\,(y-k)^2}{\ell} &= 1 \\
\frac{x^2}{\frac{\ell}{A}}-\frac{(y-k)^2}{\frac{\ell}{B}} &= 1 \qquad \text{(A.14)}
\end{aligned}
$$

Equation (A.14) has the same form as (A.1) with the $x$-coordinate of the center point being 0. The proof is nearing completion. However, it must also be shown that $\ell \neq 0$ under any circumstances. If that were to happen both fractions would have 0 as a denominator making the equation undefined. Using (A.7), (A.8), (A.9), and (A.10) in (A.13)

$$
\begin{aligned}
\ell &= \frac{1}{B}(BD-C^2) \\
B\ell &= \left(\sigma_1^2+\sigma_2^2-2\sigma_{12}\right)\left(\sigma_1^2\mu_2^2+\sigma_2^2\mu_1^2-\sigma_{12}\mu_1\mu_2\right)-\left(\sigma_1^2\mu_2+\sigma_2^2\mu_1-\sigma_{12}(\mu_1+\mu_2)\right)^2
\end{aligned}
$$

This simplifies to

$$
\begin{aligned}
B\ell &= (\mu_1+\mu_2)^2(\sigma_1^2\sigma_2^2-\sigma_{12}^2) \\
&= A\sigma_1^2\sigma_2^2(1-\rho_{12}^2) \\
\ell &= \frac{A}{B}\sigma_1^2\sigma_2^2(1-\rho_{12}^2).
\end{aligned}
$$

It has been established that $-1<\rho_{12}<1$. Likewise $A>0$ and from $-1<\rho_{12}<1$ and the AM-GM Inequality $B>0$.

$$
\begin{aligned}
B &= \sigma_1^2+\sigma_2^2-2\sigma_{12} \\
&= \sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho_{12} \\
&> \sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2 \geq 0.
\end{aligned}
$$

Hence,

$$
\ell = \frac{A}{B}\sigma_1^2\sigma_2^2(1-\rho_{12}^2) > 0.
$$

□

**Theorem A.2.2.** *(Minimum variance portfolio, two securities). Let $\mathbb{S}_1$ and $\mathbb{S}_2$ be two securities generating the attainable field $\mathfrak{A}$. Allowing for short-selling,*[2] *the minimum variance portfolio $\mathfrak{A}_{\min}$ has the weights $\mathbf{w}_{min} = [w_1, w_2]$ where*

$$w_1 = \frac{a}{a+b} \qquad\qquad w_2 = \frac{b}{a+b}$$

*for*

$$\begin{aligned} a &= \sigma_2^2 - \rho_{12}\sigma_1\sigma_2 \\ b &= \sigma_1^2 - \rho_{12}\sigma_1\sigma_2. \end{aligned}$$

*except when both $\rho_{12} = 1$ and $\sigma_1 = \sigma_2$ [9].*

*Proof.* From (4.2) the variance of $\mathfrak{A}_{\min}$ is

$$\sigma_{min}^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2.$$

Changing the weight variables $w_1 = w$ and $w_2 = 1-w$ for the equation above becomes

$$\sigma_{min}^2 = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\rho_{12}\sigma_1\sigma_2.$$

The right hand side of this equation is a quadratic polynomial of $w$. The roots of $w$ can be found by setting $\sigma_{min}^2 = 0$.

$$\begin{aligned} 0 &= w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\rho_{12}\sigma_1\sigma_2 \\ \frac{d}{dw}(0) &= \frac{d}{dw}\left(w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\rho_{12}\sigma_1\sigma_2\right) \\ &= 2w\sigma_1^2 - 2(1-w)\sigma_2^2 + 2(1-w)\rho_{12}\sigma_1\sigma_2 - 2w\rho_{12}\sigma_1\sigma_2 \\ &= w(\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2) - \sigma_2^2 + \rho_{12}\sigma_1\sigma_2 \\ w &= \frac{\sigma_2^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2} = \frac{a}{a+b}. \end{aligned}$$

Solving for $1-w$

$$\begin{aligned} w &= \frac{a}{a+b} \\ 1-w &= 1 - \frac{a}{a+b} \\ &= \frac{a+b-a}{a+b} \\ &= \frac{b}{a+b} \end{aligned}$$

[2] Short-selling is the legal obligation of a seller to provide securities to a buyer that are not owned by the seller with the intent of converting the obligation to money. A simple way to think of short-selling is as the inverse of buying-long.

If $\rho_{12} = 1$

$$\begin{aligned} w &= \frac{\sigma_2^2 - \sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2} \\ &= \frac{\sigma_2(\sigma_2 - \sigma_1)}{(\sigma_1 - \sigma_2)^2} \\ &= \frac{\sigma_2}{\sigma_1 - \sigma_2} \end{aligned}$$

causing $w$ to be undefined.

Taking the second derivative of the right hand side of the variance equation with respect to $w$ gives

$$\frac{d}{dw}\left(2w\sigma_1^2 - 2(1-w)\sigma_2^2 + 2(1-w)\rho_{12}\sigma_1\sigma_2 - 2w\rho_{12}\sigma_1\sigma_2\right) = 2\sigma_1^2 - 2\sigma_2^2 - 4\rho_{12}\sigma_1\sigma_2.$$

This value is positive because

$$\begin{aligned} \frac{d}{dw}\left(2w\sigma_1^2 - 2(1-w)\sigma_2^2 + 2(1-w)\rho_{12}\sigma_1\sigma_2 - 2w\rho_{12}\sigma_1\sigma_2\right) &= 2\sigma_1^2 - 2\sigma_2^2 - 4\rho_{12}\sigma_1\sigma_2 \\ 2\sigma_1^2 - 2\sigma_2^2 - 4\rho_{12}\sigma_1\sigma_2 &\geq 2\sigma_1^2 - 2\sigma_2^2 - 4\sigma_1\sigma_2 \\ 2\sigma_1^2 - 2\sigma_2^2 - 4\sigma_1\sigma_2 &= 2(\sigma_1 - \sigma_2)^2 \geq 0. \end{aligned}$$

Hence, there exists a portfolio $\mathfrak{A}_{\text{min}}$ and its weights are

$$w_1 = \frac{a}{a+b} \qquad\qquad w_2 = \frac{b}{a+b}$$

for

$$\begin{aligned} a &= \sigma_2^2 - \rho_{12}\sigma_1\sigma_2 \\ b &= \sigma_1^2 - \rho_{12}\sigma_1\sigma_2, \end{aligned}$$

except when both $\rho_{12} = 1$ and $\sigma_1 = \sigma_2$. □

**Corollary A.2.3.** *(Weights minimum variance portfolio). Using the same givens as the above theorem, for portfolio $\mathfrak{A}_{\text{min}}$ the vector of weights $\mathbf{w}_{min}$ is*

$$\mathbf{w}_{min} = \frac{C^{-1}\mathbf{1}_2}{\mathbf{1}_2^T C^{-1}\mathbf{1}_2}$$

*where $C$ is the covariance matrix for the security and $\mathbf{1}_2$ is a vector of 1's with length 2 [9].*

*Proof.* The covariance matrix for $\mathfrak{A}_{\text{min}}$ is

$$C = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$$

This is an invertable matrix, and by Cramer's rule its inverse is

$$C^{-1} = \frac{1}{|C|}\begin{bmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1^2. \end{bmatrix}$$

Multiplying $C^{-1}$ by $\mathbf{1}_2$

$$\begin{aligned} C^{-1}\mathbf{1}_2 &= \frac{1}{|C|}\begin{bmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1^2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{|C|}\begin{bmatrix} \sigma_2^2 - \sigma_{12} \\ \sigma_1^2 - \sigma_{12} \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}. \end{aligned} \tag{A.15}$$

Likewise

$$\begin{aligned} \mathbf{1}_2^T C^{-1}\mathbf{1}_2 &= \frac{1}{|C|}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \\ &= \frac{1}{|C|}\begin{bmatrix} a+b \end{bmatrix} \end{aligned} \tag{A.16}$$

Combining (A.15) and (A.16)

$$\frac{C^{-1}\mathbf{1}_2}{\mathbf{1}_2^T C^{-1}\mathbf{1}_2} = \begin{bmatrix} \frac{a}{a+b} \\ \frac{b}{a+b} \end{bmatrix} = \mathbf{w}_{min}.$$

□

**Lemma A.2.4.** *(Covariance of two portfolios). Given any two portfolios $\mathbb{W}_A$ and $\mathbb{W}_B$ of $n$ securities, the covariance*

$$\text{Cov}(K_A, K_B) = \mathbf{w}_A^T C \mathbf{w}_B^T$$

*[9].*

*Proof.* Using sigma notation

$$\begin{aligned} \text{Cov}(K_A, K_B) &= \text{Cov}\left(\sum_{i=1}^{n} w_{A_i} K_i, \sum_{j=1}^{n} w_{B_j} K_j\right) \\ &= \sum_{i,j=1}^{n} w_{A_i} w_{B_j} \sigma_{ij} \\ &= \mathbf{w}_A^T C \mathbf{w}_B^T. \end{aligned}$$

□

**Lemma A.2.5.** *(Hessian 2C). Given portfolio* $\mathbb{W}$ *of n securities, the hessian matrix of* $\mathbf{w}^T C\mathbf{w}$ *is* $2C$*. Furthermore, the following gradients with respect to* $\mathbf{w}$ *are*

$$\nabla(\mathbf{w}^T \boldsymbol{\mu}) = \boldsymbol{\mu} \tag{A.17}$$

$$\nabla(\mathbf{w}^T \mathbf{1}_n) = \mathbf{1}_n \tag{A.18}$$

$$\nabla(\mathbf{w}^T C\mathbf{w}) = 2C\mathbf{w} \tag{A.19}$$

[9].

*Proof.* From (4.2) the Hessian of $\mathbf{w}^T C\mathbf{w}$ is

$$\begin{aligned}\mathbf{H}(\mathbf{w}^T C\mathbf{w}) &= \mathbf{H}\left(\sum_{i,j=1}^{n} w_i w_j \sigma_{ij}\right)\\ &= \left(\frac{\partial^2}{\partial w_k w_\ell} w_k w_\ell \sigma_{ij}\right)_{k,\ell \le n}\\ &= 2C.\end{aligned}$$

Also from above (A.19) is

$$\begin{aligned}2C\mathbf{w} &= \left(\frac{\partial}{\partial w_k w_\ell} w_k w_\ell \sigma_{ij}\right)_{k,\ell \le n}\\ &= \nabla\left(\mathbf{w}^T C\mathbf{w}\right)\end{aligned}$$

The right hand side of A.17 is

$$\nabla(\mathbf{w}^T \boldsymbol{\mu}) = \begin{bmatrix} \frac{\partial}{\partial w_1}(\mathbf{w}^T \boldsymbol{\mu}) \\ \frac{\partial}{\partial w_2}(\mathbf{w}^T \boldsymbol{\mu}) \\ \vdots \\ \frac{\partial}{\partial w_n}(\mathbf{w}^T \boldsymbol{\mu}) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix} = \boldsymbol{\mu}.$$

The right hand side of A.18 is

$$\nabla(\mathbf{w}^T \mathbf{1}_n) = \begin{bmatrix} \frac{\partial}{\partial w_1}(\mathbf{w}^T \mathbf{1}_n) \\ \frac{\partial}{\partial w_2}(\mathbf{w}^T \mathbf{1}_n) \\ \vdots \\ \frac{\partial}{\partial w_n}(\mathbf{w}^T \mathbf{1}_n) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \mathbf{1}_n.$$

□

**Theorem A.2.6.** *(General minimum variance portfolio). Given an attainable field* $\mathfrak{A} = (\prod_{i=1}^{n} \Omega_i, \bigotimes_{i=1}^{n} \mathcal{F}_i, \mathcal{P})$ *with n securities, the minimum variance portfolio* $\mathfrak{A}_{\min}$ *has the vector of weights*

$$\mathbf{w}_{min} = \frac{C^{-1}\mathbf{1}_n}{\mathbf{1}_n^T C^{-1}\mathbf{1}_n}.$$

*where C is the covariance matrix for the security and* $\mathbf{1}_2$ *is a vector of 1's with length n [9].*

*Proof.* Using the method of Lagrange multipliers is an easy way find the minima of the function generating the weights for the given attainable field. Let the desired Lagrangian function be

$$\mathcal{L}_{\mathbf{w}} = \nabla(\mathbf{w}^T C\mathbf{w}) - \nabla(\lambda(\mathbf{w}^T \mathbf{1}_n - 1))$$

where $\lambda$ is the Lagrangian multiplier.

From 4.4.2

$$\mathcal{L}_{\mathbf{w}} = 2C\mathbf{w} - \lambda\mathbf{1}$$

Setting $\mathcal{L}_{\mathbf{w}} = 0$ it follows that

$$\mathbf{w}_{min} = \frac{1}{\lambda} C^{-1}\mathbf{1}_n. \tag{A.20}$$

From (A.18) and (A.20)

$$\begin{aligned} 1 &= \mathbf{w}^T \mathbf{1}_n \\ &= \mathbf{1}_n^T \mathbf{w} \\ &= \frac{1}{\lambda}\mathbf{1}_n^T C^{-1}\mathbf{1}_n. \end{aligned} \tag{A.21}$$

Hence,

$$\mathbf{w}_{min} = \frac{C^{-1}\mathbf{1}_n}{\mathbf{1}_n^T C^{-1}\mathbf{1}_n}. \tag{A.22}$$

□

**Theorem A.2.7.** *Given an attainable field of n securities $\mathfrak{A}$, let M be a matrix*

$$M = \begin{bmatrix} \boldsymbol{\mu}^T C^{-1}\boldsymbol{\mu} & \boldsymbol{\mu}^T C^{-1}\mathbf{1}_n \\ \boldsymbol{\mu}^T C^{-1}\mathbf{1}_n & \mathbf{1}_n^T C^{-1}\mathbf{1}_n \end{bmatrix}.$$

*If both C and M are invertible, the weight vector of a portfolio on the minimum variance line is*

$$\mathbf{w} = \frac{1}{|M|} C^{-1}\left(|M_1|\boldsymbol{\mu} + |M_2|\mathbf{1}_n\right)$$

*where*

$$M_1 = \begin{bmatrix} \mu & \boldsymbol{\mu}^T C^{-1}\mathbf{1}_n \\ 1 & \mathbf{1}_n^T C^{-1}\mathbf{1}_n \end{bmatrix}, \qquad M_2 = \begin{bmatrix} \boldsymbol{\mu}^T C^{-1}\mathbf{1}_n & \mu \\ \mathbf{1}_n^T C^{-1}\mathbf{1}_n & 1 \end{bmatrix}$$

*[9].*

*Proof.* Let the Lagrangian function be

$$\mathcal{L}(\mathbf{w}) = \nabla(\mathbf{w}^T C\mathbf{w}) - \lambda_1 \nabla(\mathbf{w}\boldsymbol{\mu} - \mu) + \lambda_2 \nabla(\mathbf{w}^T \mathbf{1}_n - 1) = 0.$$

From 4.4.2

$$\mathcal{L}(\mathbf{w}) = 2C\mathbf{w} - \lambda_1 \boldsymbol{\mu} - \lambda_2 \mathbf{1}_n = 0.$$

Solving for $\mathbf{w}$

$$\mathbf{w} = \frac{1}{2}\lambda_1 C^{-1}\boldsymbol{\mu} + \frac{1}{2}\lambda_2 C^{-1}\mathbf{1}_n. \tag{A.23}$$

Taking (A.23), (A.17), and (A.18)

$$\frac{1}{2}\lambda_1 \boldsymbol{\mu}^T C^{-1}\boldsymbol{\mu} + \frac{1}{2}\lambda_2 \boldsymbol{\mu}^T C^{-1}\mathbf{1}_n = \mu \tag{A.24}$$

$$\frac{1}{2}\lambda_1 \mathbf{1}_m^T C^{-1}\boldsymbol{\mu} + \frac{1}{2}\lambda_2 \mathbf{1}_m^T C^{-1}\mathbf{1}_n = 1 \tag{A.25}$$

Initially solving (A.25) for $\lambda_1$

$$\begin{aligned}
\frac{1}{2}\lambda_1 \mathbf{1}_m^T C^{-1}\boldsymbol{\mu} + \frac{1}{2}\lambda_2 \mathbf{1}_m^T C^{-1}\mathbf{1}_n &= 1 \\
\frac{1}{2}\lambda_1 \mathbf{1}_m^T C^{-1}\boldsymbol{\mu} &= 1 - \frac{1}{2}\lambda_2 \mathbf{1}_m^T C^{-1}\mathbf{1}_n \\
\lambda_1 &= \frac{2 - \lambda_2 \mathbf{1}_m^T C^{-1}\mathbf{1}_n}{\mathbf{1}_m^T C^{-1}\boldsymbol{\mu}}.
\end{aligned} \tag{A.26}$$

Substituting (A.26) into (A.24) gives

$$\begin{aligned}
\mu &= \frac{1}{2}\left(\frac{2 - \lambda_2 \mathbf{1}_m^T C^{-1}\mathbf{1}_n}{\mathbf{1}_m^T C^{-1}\boldsymbol{\mu}}\right)\boldsymbol{\mu}^T C^{-1}\boldsymbol{\mu} + \frac{1}{2}\lambda_2 \boldsymbol{\mu}^T C^{-1}\mathbf{1}_n \\
\mu \mathbf{1}_m^T C^{-1}\boldsymbol{\mu} - \boldsymbol{\mu}^T C^{-1}\boldsymbol{\mu} &= -\frac{1}{2}\lambda_2 \mathbf{1}_m^T C^{-1}\mathbf{1}_n \boldsymbol{\mu}^T C^{-1}\boldsymbol{\mu} + \frac{1}{2}\lambda_2 \boldsymbol{\mu}^T C^{-1}\mathbf{1}_n \mathbf{1}_m^T C^{-1}\boldsymbol{\mu} \\
\lambda_2 &= 2\frac{\boldsymbol{\mu}^T C^{-1}\boldsymbol{\mu} - \mu\boldsymbol{\mu}^T C^{-1}\mathbf{1}_n}{\boldsymbol{\mu}^T C^{-1}\boldsymbol{\mu}\mathbf{1}_m^T C^{-1}\mathbf{1}_n - \boldsymbol{\mu}^T C^{-1}\mathbf{1}_n \boldsymbol{\mu}^T C^{-1}\mathbf{1}_n}.
\end{aligned} \tag{A.27}$$

$$\begin{aligned}
\lambda_1 &= \frac{2 - \left(2\frac{\boldsymbol{\mu}^T C^{-1}\boldsymbol{\mu} - \mu\boldsymbol{\mu}^T C^{-1}\mathbf{1}_n}{\boldsymbol{\mu}^T C^{-1}\boldsymbol{\mu}\mathbf{1}_n^T C^{-1}\mathbf{1}_n - \boldsymbol{\mu}^T C^{-1}\mathbf{1}_n\boldsymbol{\mu}^T C^{-1}\mathbf{1}_n}\right)\mathbf{1}_n^T C^{-1}\mathbf{1}_n}{\mathbf{1}_n^T C^{-1}\boldsymbol{\mu}} \\
&= \frac{2\left(\boldsymbol{\mu}^T C^{-1}\boldsymbol{\mu}\mathbf{1}_n^T C^{-1}\mathbf{1}_n - \boldsymbol{\mu}^T C^{-1}\mathbf{1}_n\boldsymbol{\mu}^T C^{-1}\mathbf{1}_n\right)\mathbf{1}_n^T C^{-1}\mathbf{1}_n}{\left(\boldsymbol{\mu}^T C^{-1}\boldsymbol{\mu}\mathbf{1}_n^T C^{-1}\mathbf{1}_n - \boldsymbol{\mu}^T C^{-1}\mathbf{1}_n\boldsymbol{\mu}^T C^{-1}\mathbf{1}_n\right)\mathbf{1}_n^T C^{-1}\boldsymbol{\mu}} \\
&\quad - \frac{\left(\boldsymbol{\mu}^T C^{-1}\boldsymbol{\mu} - \mu\boldsymbol{\mu}^T C^{-1}\mathbf{1}_n\right)\mathbf{1}_n^T C^{-1}\mathbf{1}_n}{\left(\boldsymbol{\mu}^T C^{-1}\boldsymbol{\mu}\mathbf{1}_n^T C^{-1}\mathbf{1}_n - \boldsymbol{\mu}^T C^{-1}\mathbf{1}_n\boldsymbol{\mu}^T C^{-1}\mathbf{1}_n\right)\mathbf{1}_n^T C^{-1}\boldsymbol{\mu}} \\
&= \frac{\mu\boldsymbol{\mu}^T C^{-1}\mathbf{1}_n - \boldsymbol{\mu}^T C^{-1}\boldsymbol{\mu}}{\boldsymbol{\mu}^T C^{-1}\boldsymbol{\mu}\mathbf{1}_n^T C^{-1}\mathbf{1}_n - \boldsymbol{\mu}^T C^{-1}\mathbf{1}_n\boldsymbol{\mu}^T C^{-1}\mathbf{1}_n}
\end{aligned} \tag{A.28}$$

From (A.27) and (A.28)

$$\lambda_1 = \frac{2|M_1|}{|M|}, \qquad \lambda_2 = \frac{2|M_2|}{|M|}. \tag{A.29}$$

Hence, merging (A.23) and (A.29) produces

$$\mathbf{w} = \frac{1}{|M|} C^{-1} \left( |M_1| \boldsymbol{\mu} + |M_2| \mathbf{1}_n \right).$$

□

**Corollary A.2.8.** *Assuming the givens of Theorem 4.4.4 there exists two vectors* $\mathbf{a}, \mathbf{b}$ *such that*

$$\mathbf{w} = \mu \mathbf{a} + \mathbf{b}$$

*[9].*

*Proof.* Let

$$\mathbf{a} = \frac{C^{-1}(\mathbf{1}_n^T C^{-1} \mathbf{1}_n)\boldsymbol{\mu} - (\boldsymbol{\mu}^T C^{-1} \mathbf{1}_n)\mathbf{1}_n}{|M|} \tag{A.30}$$

$$\mathbf{b} = \frac{C^{-1}(\boldsymbol{\mu}^T C^{-1} \boldsymbol{\mu})\mathbf{1}_n - (\boldsymbol{\mu}^T C^{-1} \mathbf{1}_n)\mathbf{1}_n}{|M|}. \tag{A.31}$$

From Theorem 4.4.4 it follows that

$$\mathbf{w} = \mu \mathbf{a} + \mathbf{b}.$$

□

**Corollary A.2.9.** *Let* $\mathfrak{A}$ *be an attainable field of n securities with a minimum variance line* $\mathfrak{A}_{MVL}$. *If there are two distinct portfolios* $\mathbb{W}_1, \mathbb{W}_2 \in \mathfrak{A}_{MVL}$, *then the vector of weights* $\mathbf{w}$ *of all portfolios* $\mathbb{W} \in \mathfrak{A}_{MVL}$ *are*

$$\mathbf{w} = \alpha \mathbf{w}_1 + (1 - \alpha)\mathbf{w}_2$$

*for some number* $\alpha \in \mathbb{R}$.

*Proof.* Suppose that $\mathbb{W}$ is a portfolio composed of two assets, the portfolios $\mathbf{W}_1 and \mathbf{W}_2 \in \mathfrak{A}_{MVL}$. From (4.1), for some number $\alpha \in \mathbb{R}$.

$$\mu_w = \alpha \mu_1 + (1 - \alpha)\mu_2$$
$$\alpha = \frac{\mu_w - \mu_1}{\mu_1 - \mu_2}.$$

From Corollary 4.4.5, using the definitions of **a** and **b** from that proposition,

$$\mathbf{w}_1 = \mu_1\mathbf{a} + \mathbf{b},$$
$$\mathbf{w}_2 = \mu_2\mathbf{a} + \mathbf{b}.$$

Thus,

$$\begin{aligned}\alpha\mathbf{w}_1 + (1-\alpha)\mathbf{w}_2 &= (\alpha\mu_1 + (1-\alpha)\mu_2)\mathbf{a} + \mathbf{b} \\ &= \mu_w\mathbf{a} + \mathbf{b} = \mathbf{w}.\end{aligned}$$

□

**Theorem A.2.10.** *(Generalization of Tobin's two security theorem). Given an attainable field $\mathfrak{A}$, its minimum variance line $\mathfrak{A}_{MVL}$ is a segment of a hyperbola with its center on the vertical axis.*

[9, 31]

*Proof.* Assume two distinct portfolios $\mathbb{W}_1, \mathbb{W}_2 \in \mathfrak{A}_{MVL}$. If $\mathbb{W}$ is a portfolio composed of $\mathbb{W}_1, \mathbb{W}_2$ and $\alpha \in [0,1]$ is the proportion of $\mathbb{W}_1$ in this portfolio then from (4.1) and (4.2)

$$\begin{aligned}\mu_w &= \alpha\mu_1 + (1-\alpha)\mu_2 \\ \sigma_w^2 &= \alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2 + 2\alpha(1-\alpha)\sigma_{12}.\end{aligned}$$

Since $\mu_1 \neq \mu_2$, from Theorem 4.2.1 the $(\sigma, \mu)$ tuple of each $\mathbb{W}$ for all possible values of $\alpha$ is an hyperbola.

□

**Theorem A.2.11.** *(Market portfolio weights). Given an attainable field $\mathfrak{A}$ of $n$ securities and a capital market line $m : \sigma \to \mu$ and a risk free rate $R$, the market portfolio $\mathbb{W}_m$ has a weight vector*

$$\mathbf{w}_m = \frac{C^{-1}(\boldsymbol{\mu} - \mathbf{1}_n R)}{\mathbf{1}_n^T C^{-1}(\boldsymbol{\mu} - \mathbf{1}_n R)}$$

*[9].*

*Proof.* The slope of the capital market line $m$ is

$$\frac{\mu_m - R}{\sigma_m} = \frac{\mathbf{w}^T\boldsymbol{\mu} - R}{\sqrt{\mathbf{w}^T C\mathbf{w}}}$$

Introducing the Lagrangian

$$\mathcal{L}(\mathbf{w}) = \nabla\left(\frac{\mathbf{w}^T\boldsymbol{\mu} - R}{\sqrt{\mathbf{w}^T C\mathbf{w}}}\right) - \lambda\nabla(\mathbf{w}^T\mathbf{1}_n - 1).$$

By setting $\mathcal{L}(\mathbf{w}) = 0$ it is possible to find the maximum slope,

$$\mathcal{L}(\mathbf{w}) = \frac{\boldsymbol{\mu}\sqrt{\mathbf{w}^T C\mathbf{w}} - (\mathbf{w}^T\boldsymbol{\mu} - R)\frac{1}{2\sqrt{\mathbf{w}^T C\mathbf{w}}}2C\mathbf{w}}{\mathbf{w}^T C\mathbf{w}} - \lambda\mathbf{1}_n = 0$$
$$\boldsymbol{\mu}\sigma_m - (\mu_m - R)\frac{C\mathbf{w}}{\sigma_m} - \lambda\sigma_m^2\mathbf{1}_n = 0.$$

Hence,

$$\frac{\mu_m - R}{\sigma_m^2}C\mathbf{w} = \boldsymbol{\mu} - \lambda\sigma_m\mathbf{1}_n \tag{A.32}$$
$$\frac{\mu_m - R}{\sigma_m^2}\mathbf{w}^T C\mathbf{w} = \mu_m - \lambda\sigma_m$$
$$\lambda\sigma_m = \mu_m - \frac{\mu_m R}{\sigma_m^2}\sigma_m^2$$
$$\lambda = \frac{R}{\sigma_m} \tag{A.33}$$

From (A.32) and (A.33)

$$\zeta C\mathbf{w} = \boldsymbol{\mu} - R\mathbf{1}_n$$

where $\zeta = \frac{\mu_m - R}{\sigma_m^2}$. Therefore,

$$\zeta\mathbf{w} = C^{-1}\boldsymbol{\mu} - R\mathbf{1}_n \tag{A.34}$$
$$\zeta = \mathbf{1}_n^T C^{-1}\boldsymbol{\mu} - R\mathbf{1}_n. \tag{A.35}$$

From (A.34) and (A.35)

$$\mathbf{w}_m = \frac{C^{-1}(\boldsymbol{\mu} - \mathbf{1}_n R)}{\mathbf{1}_n^T C^{-1}(\boldsymbol{\mu} - \mathbf{1}_n R)}.$$

□

**Theorem A.2.12.** *(Fundamental theorem of CAPM). Given an attainable field $\mathcal{A}$ of $n$ securities and a risk-free return $R$ the expected return of the $i$-th security is*

$$\mu_i = R + \beta_i(\mu_m - R)$$

*[9].*

*Proof.* Assume that there is a portfolio $\mathbb{W}_s$ that is composed of security $\mathbb{W}_i$ and the market portfolio $\mathbb{W}_m$ with the respective weights $\alpha$ and $(1-\alpha)$. Thus,

$$\begin{aligned}
\mu_s &= \alpha\mu_i + (1-\alpha)\mu_m \\
\sigma_s^2 &= \alpha^2\sigma_i^2 + (1-\alpha)\sigma_m^2 + 2\alpha(1-\alpha)\sigma_{im}.
\end{aligned}$$

At $\alpha = 0$ the slope of the tangent line to the hyperbola of the attainable field for the given securities is equal to the slope of capital market line. Hence,

$$\begin{aligned}
\frac{\mu_m - R}{\sigma_m} &= \frac{\left.\frac{\partial \mu_s}{\partial \alpha}\right|_{\alpha=0}}{\left.\frac{\partial \sigma_s}{\partial \alpha}\right|_{\alpha=0}} \\
&= \frac{\mu_i - \mu_m}{\frac{\sigma_{im} - \sigma_m^2}{\sigma_m}} \\
\mu_i - \mu_m &= \left(\frac{\sigma_{im} - \sigma_m^2}{\sigma_m}\right)\left(\frac{\mu_m - R}{\sigma_m}\right) \\
\mu_i &= R + \frac{\sigma_{im}}{\sigma_m^2}(\mu_m - R)
\end{aligned}$$

Therefore,

$$\mu_i = R + \beta_i(\mu_m - R).$$

□

**Corollary A.2.13.** *(Alternative formula for Beta). Given the assumptions of Theorem 4.5.1, the Beta factor for the i-th security (or portfolio) is*

$$\beta = \frac{\mu_i - R}{\mu_m - R}$$

*[9].*

*Proof.* Since,

$$\mu_i = R + \beta_i(\mu_m - R)$$

it follows that

$$\begin{aligned}
\beta_i(\mu_m - R) &= \mu_i - R \\
\beta_i &= \frac{\mu_i - R}{\mu_m - R}
\end{aligned}$$

□

# Appendix B

# MATLAB CODE

## B.1 Code for Section 4.3

### Contents



### Two security theorem in practice Script

```
clear; close all; clc;
format bank
```

### Names and Prices

This information is given.

```
% Security names
stock = {'AAPL','FB'};
holding = [ 1000 ; 1000 ];

% Current prices and future prices
cprice = [313.05 ; 210.18];
fprice = [400 , 300; 350, 250; 350, 250; 190 , 178];
```

### Expected Value and Variance

```
% Of the future prices
exfprice = [ sum(fprice(:,1).*0.25); sum(fprice(:,2).*0.25)];
% varfprice = [ sum(fprice(:,3).* (fprice(:,1) - exfprice(1)).^2);...
%    sum(fprice(:,3).* (fprice(:,2) - exfprice(2)).^2) ];
```

```
varfprice = var(fprice(:,1:2))';

% Of return
K = [ (fprice(:,1) - cprice(1)) / cprice(1) ,...
    (fprice(:,2) - cprice(2)) / cprice(2) ];
mu = (exfprice - cprice)./ cprice;
sigma2 = varfprice./(cprice.^2);
sigma = sqrt(sigma2);
C = cov(K(:,1),K(:,2));

% Label
strsm = {'\sigma','\mu'};
```

**Attainable Set**

```
% returns and variance for all portfolios
w = [linspace(1,0)',linspace(0,1)'];
muall = zeros(100,1);
sigma2all = zeros(100,1);
for i = 1:100
    muall(i) = w(i,:)*mu;
    sigma2all(i) = w(i,:)*C*w(i,:)';
end
sigmaall = sqrt(sigma2all);

% Plot of attainable field
figure01 = figure(1);
plot(sigmaall,muall,'k')
hold on
plot(sigma,mu,'ko')
text(sigma + 0.003, mu, stock)
hold off
axis([.2 .35 0 0.2])
xlabel(strsm(1))
ylabel(strsm(2))
%title('APPL,FB Attainable Field')
```

```
figure01.Units='inches';
figure01.Position=[0 0 6 4];
figure01.Visible='off';

saveas(figure01,...
    ['/media/web/Lexar/memo/work/usm/thesis/20200225/pics/figure01.png'])
```

**Sample Hyperbola**

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

This is an approximation of the APPL,FB hypberbola.

```
% initial conditions
a = .101;
b = .25;
h = 0;
k = .7;
x = linspace(a,1,1000);
xa = linspace(-1,1);

% hyperbola and asymptotes
hypb = @(x) sqrt( (b^2 .* x.^2)./a.^2 - b.^2);
y1 = hypb(x) + k;
y2 = -hypb(x) + k;
y3 = hypb(-x) + k;
y4 = -hypb(-x) + k;
aspt1 = k + (b/a).*xa;
aspt2 = k -(b/a).*xa;

% figure of attainable field with hyperbola
figure02 = figure(2);
plot(x,y1,'k-.',x,y2,'k-.')
hold on
plot(sigmaall,muall,'k','LineWidth',1.4)
```

```
plot(sigma,mu,'ko')
plot(-x,y3,'k-.',-x,y4,'k-.')
plot(xa, aspt1,'k')
plot(xa, aspt2,'k')
text(sigma + 0.02, mu + 0.02, stock)
hold off
axis([-0.4 0.4 -0.4 1.8])
%title('APPL,FB Attainable Field on a Hyperbola')
xlabel(strsm(1))
ylabel(strsm(2))
figure02.Units='inches';
figure02.Position=[0 0 6 4];
figure02.Visible='off';

saveas(figure02,...
    ['/media/web/Lexar/memo/work/usm/thesis/20200225/pics/figure02.png'])
```

## B.2 Code for Chapter 5

### Contents



### Modern Portfolio Theory Analysis of Aeneid.

```
clear; close all; clc;
format bank
```

## Load Section Line Numbers

```
book01=importdata('book01.txt');
book02=importdata('book02.txt');
book03=importdata('book03.txt');
book04=importdata('book04.txt');
book05=importdata('book05.txt');
book06=importdata('book06.txt');
book07=importdata('book07.txt');
book08=importdata('book08.txt');
book09=importdata('book09.txt');
book10=importdata('book10.txt');
book11=importdata('book11.txt');
book12=importdata('book12.txt');

% Book Matrix
ml0=max([length(book01),length(book02),length(book03),length(book04),...
    length(book05),length(book06),length(book07),length(book08),...
    length(book09),length(book10),length(book11),length(book12)]);

bookmat = nan(ml0,12);

bookmat(:,1)=[book01;nan(length(bookmat)-length(book01),1)];
bookmat(:,2)=[book02;nan(length(bookmat)-length(book02),1)];
bookmat(:,3)=[book03;nan(length(bookmat)-length(book03),1)];
bookmat(:,4)=[book04;nan(length(bookmat)-length(book04),1)];
bookmat(:,5)=[book05;nan(length(bookmat)-length(book05),1)];
bookmat(:,6)=[book06;nan(length(bookmat)-length(book06),1)];
bookmat(:,7)=[book07;nan(length(bookmat)-length(book07),1)];
bookmat(:,8)=[book08;nan(length(bookmat)-length(book08),1)];
bookmat(:,9)=[book09;nan(length(bookmat)-length(book09),1)];
bookmat(:,10)=[book10;nan(length(bookmat)-length(book10),1)];
bookmat(:,11)=[book11;nan(length(bookmat)-length(book11),1)];
bookmat(:,12)=[book12;nan(length(bookmat)-length(book12),1)];
```

**Number of lines per section**

```
price01=book01(2:end)-book01(1:end-1);
price02=book02(2:end)-book02(1:end-1);
price03=book03(2:end)-book03(1:end-1);
price04=book04(2:end)-book04(1:end-1);
price05=book05(2:end)-book05(1:end-1);
price06=book06(2:end)-book06(1:end-1);
price07=book07(2:end)-book07(1:end-1);
price08=book08(2:end)-book08(1:end-1);
price09=book09(2:end)-book09(1:end-1);
price10=book10(2:end)-book10(1:end-1);
price11=book11(2:end)-book11(1:end-1);
price12=book12(2:end)-book12(1:end-1);

% Price Matrix
ml=max([length(price01),length(price02),length(price03),length(price04),...
    length(price05),length(price06),length(price07),length(price08),...
    length(price09),length(price10),length(price11),length(price12)]);

pricemat = nan(ml,12);

pricemat(:,1)=[price01;nan(length(pricemat)-length(price01),1)];
pricemat(:,2)=[price02;nan(length(pricemat)-length(price02),1)];
pricemat(:,3)=[price03;nan(length(pricemat)-length(price03),1)];
pricemat(:,4)=[price04;nan(length(pricemat)-length(price04),1)];
pricemat(:,5)=[price05;nan(length(pricemat)-length(price05),1)];
pricemat(:,6)=[price06;nan(length(pricemat)-length(price06),1)];
pricemat(:,7)=[price07;nan(length(pricemat)-length(price07),1)];
pricemat(:,8)=[price08;nan(length(pricemat)-length(price08),1)];
pricemat(:,9)=[price09;nan(length(pricemat)-length(price09),1)];
pricemat(:,10)=[price10;nan(length(pricemat)-length(price10),1)];
pricemat(:,11)=[price11;nan(length(pricemat)-length(price11),1)];
pricemat(:,12)=[price12;nan(length(pricemat)-length(price12),1)];
```

```
vars = {'book01','book02','book03','book04','book05','book06',...
    'book07','book08','book09','book10','book11','book12','price01',...
    'price02','price03','price04','price05','price06','price07',...
    'price08','price09','price10','price11','price12'};
clear(vars{:})
clear vars
```

**Return**

```
% retmat = (pricemat(2:end,:) - pricemat(1:end-1,:))./pricemat(1:end-1,:);
% expret = mean(retmat,'omitnan');

aveprice = mean(mean(pricemat,'omitnan'));
retmat = abs(pricemat(2:end,:) - aveprice)./aveprice;
% retmat = (pricemat(2:end,:) - aveprice)./aveprice;
```

**Risk**

```
sigma2 = var(retmat,'omitnan');
sigma = std(retmat,'omitnan');
covmat = cov(retmat,'partialrows');
retmean = mean(retmat,'omitnan');
```

**Save Variables**

```
save('variables')
```

**Blotter**

```
security = {'B1','B2','B3','B4','B5','B6','B7','B8','B9','B10','B11','B12'};

Blotter = dataset({mean(pricemat,'omitnan')', 'AveLPS'}, 'obsnames',security);

%disp(Blotter);
```

**Portfolio**

```
p = Portfolio('Name', 'Books of the Aeneid', 'AssetList', security );

p = setDefaultConstraints(p);
```

```
p = setBounds(p, ones(12,1)*-1);

p = setAssetMoments(p, retmean, covmat);
[wgt,~, ~] = estimateFrontierByRisk(p, 0);
[wgtm,~, ~] = estimateFrontierByReturn(p, 0.5031);
% clc;

% display(p);
```

```
Warning: One or more target risk values are outside the feasible range [
0.0647398, 5.57464 ].
Will return portfolios associated with endpoints of the range for these
        values.
```

**Bounds**

```
[lb, ub] = estimateBounds(p);
% display([lb, ub]);
```

**Plot**

```
% x = linspace(0,3,30);
% p1 = polyfit(sigma, retmean,1);
% q = polyval(p1,x);


figure03 = figure(3);
[prsk,pret] = plotFrontier(p, 60);
hold on
scatter(estimatePortRisk(p, wgt), estimatePortReturn(p, wgt), 'filled', 'r');
scatter(estimatePortRisk(p, wgtm), estimatePortReturn(p, wgtm), 'filled', 'r');
text(sigma(1), retmean(1), 'B1', 'Fontsize', 7);
text(sigma(2), retmean(2), 'B2', 'Fontsize', 7);
text(sigma(3), retmean(3), 'B3', 'Fontsize', 7);
text(sigma(4), retmean(4)-.01, 'B4', 'Fontsize', 7);
text(sigma(5), retmean(5), 'B5', 'Fontsize', 7);
```

```
text(sigma(6), retmean(6), 'B6', 'Fontsize', 7);
text(sigma(7), retmean(7)+.01, 'B7', 'Fontsize', 7);
text(sigma(8), retmean(8), 'B8', 'Fontsize', 7);
text(sigma(9), retmean(9), 'B9', 'Fontsize', 7);
text(sigma(10), retmean(10), 'B10', 'Fontsize', 7);
text(sigma(11), retmean(11), 'B11', 'Fontsize', 7);
text(sigma(12), retmean(12)-.01, 'B12', 'Fontsize', 7);
plot(sigma, retmean, 'w.');
text(0.035,0.35,'MVP',  'Fontsize', 7);
text(0.12,0.515,'MP',  'Fontsize', 7);
text(0.001,0.35,'R',  'Fontsize', 7);
line([0 0.13754],[0.3351 0.5031],'LineStyle','--');
hold off
axis([0 0.5 .25 .8])
title('')
legend('off')
xlabel('\sigma')
ylabel('\mu')
figure03.Units='inches';
figure03.Position=[0 0 6 4];
figure03.Visible='off';

saveas(figure03,...
    ['/media/web/Lexar/memo/work/usm/thesis/20200305/pics/figure03.png'])
```

### Beta

```
Beta = (retmean./estimatePortReturn(p, wgt))';
halflines=[3;10;7;5;6;2;6;3;6;6;2;1];



Blotter.Beta = Beta;
Blotter.Hemistich = halflines;
display(Blotter)


Blotter =
```

```
        AveLPS    Beta    Hemistich
B1      19.38     0.94     3.00
B2      25.94     1.13    10.00
B3      24.66     1.16     7.00
B4      22.74     0.92     5.00
B5      25.62     0.79     6.00
B6      28.16     1.24     2.00
B7      22.08     1.04     6.00
B8      26.11     0.98     3.00
B9      30.30     1.46     6.00
B10     26.71     1.17     6.00
B11     28.59     0.93     2.00
B12     28.85     0.87     1.00
```

**Plot Beta Hemistich**

```
figure04 = figure(4);
yyaxis left
plot(Beta-1,'ko-.')
ylabel('\beta - 1')
ax = gca;
ax.YColor = 'k';

yyaxis right
plot(halflines,'k*-')
ylabel('Hemistich')
ax = gca;
ax.YColor = 'k';

xticks(1:1:12)
xticklabels(security)
axis([0 13 0 11])
legend({'\beta - 1','Hemistich'},'Location','southwest')
figure04.Units='inches';
```

```
figure04.Position=[0 0 6 4];
figure04.Visible='off';

saveas(figure04,...
    ['/media/web/Lexar/memo/work/usm/thesis/20200305/pics/figure04.png'])
```

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