5-2021

# 2-adic Valuations of Square Spiral Sequences

Minh Nguyen

# 2-adic Valuations of Square Spiral Sequences

by

Minh Nguyen

A Thesis
Submitted to the Honors College of
The University of Southern Mississippi
in Partial Fulfillment
of Honors Requirements

May 2021

Approved by:

Karen Kohl, Ph.D.,
Thesis Advisor,
School of Mathematics and Natural Sciences

Bernd Schroeder, Ph.D.,
Director,
School of Mathematics and Natural Sciences

Ellen Weinauer, Ph.D., Dean
Honors College

## ABSTRACT

The study of $p$-adic valuations is connected to the problem of factorization of integers, an essential question in number theory and computer science. Given a nonzero integer $n$ and prime number $p$, the $p$-adic valuation of $n$, which is commonly denoted as $\nu_p(n)$, is the greatest non-negative integer $\nu$ such that $p^{\nu} \mid n$. In this paper, we analyze the properties of the 2-adic valuations of some integer sequences constructed from Ulam square spirals. Most sequences considered were diagonal sequences of the form $4n^2 + bn + c$ from the Ulam spiral with center value of 1. Other sequences related to various Ulam square spirals were selected from the Online Encyclopedia of Integer Sequences (OEIS). Conjectures of the 2-adic valuations of these sequences were made based on observations of the binary tree representations of their valuations. We found explicit closed forms for some sequences with finitely many valuations. When sequences produced infinitely many valuations, these results were proved using an adaptation of the Hensel's lemma, previously used by Almodovar et al in their study of 2-adic valuation of quadratic polynomials of the form $n^2 + a$. In both of these cases, we classified a number of similar valuation patterns for the diagonal sequences of Ulam spirals.

***Keywords:*** $p$-adic valuations, Ulam square spiral, OEIS, binary tree, Hensel's lemma

## ACKNOWLEDGMENTS

First and foremost, I would like to express my sincerest gratitude and appreciation to my thesis adviser Dr. Karen Kohl. Without her guidance and support, this project would not have been possible. Next, I want to thank the Honors College and Wright W. and Annie Rea Cross Endowment Scholarship for providing me with the necessary funds to carry out this project. And lastly, I would like to thank my parents for always supporting and encouraging me to challenge myself during my academic journey.

## TABLE OF CONTENTS

# LIST OF ILLUSTRATIONS

# CHAPTER I: INTRODUCTION

The Fundamental Theorem of Arithmetic states that every integer greater than or equal to 2 is a prime number itself or can be expressed as the product of prime numbers; furthermore, this factorization is unique. For example, we can factor 2520 as

$$2250 = 2 \times 3 \times 3 \times 5 \times 5 \times 5 = 2 \times 3^2 \times 5^3$$

and obtain the prime 2, 3, and 5 with the degree of 1, 2, and 3, respectively. Since 2, $3^2$, and $5^3$ are factors of 2250, it follows that they also divide 2250.

Given a non-zero integer $n$ and prime number $p$, the $p$-adic valuation of $n$, which is commonly denoted as $\nu_p(n)$, is the greatest non-negative integer $\nu$ such that $p^\nu$ divides $n$. That is, the $p$-adic valuation for $\mathbb{Z}$ is the function $\nu_p : \mathbb{Z} \to \mathbb{N}$ defined by

$$\nu_p(n) = \begin{cases} \max\{\nu \in \mathbb{N} : p^\nu \mid n\} & \text{if } n \neq 0 \\ \infty & \text{if } n = 0 \end{cases}$$

for $n \in \mathbb{Z}$. Using the prior example, we have

$$\nu_2(2250) = 1, \quad \nu_3(2250) = 2, \quad \text{and} \quad \nu_5(2250) = 3.$$

This is very useful for certain computational tasks. For example, an adaptation of the Fürer's integer multiplication algorithm by De et al used modular arithmetic and $p$-adic numbers, which are constructed using $p$-adic valuations, instead of arithmetic over complex numbers to achieve a time complexity of $O(N \cdot \log N \cdot 2^{O(\log^* N)})$ [4].

A natural question to ask at this point is is whether it is possible to find the $p$-adic valuation of sequences of integers for a fixed $p$. We achieve this by taking the $p$-adic valuation of each element of the sequence and generate a sequence of $p$-adic valuations. For example, consider the Fibonacci sequence $F_n = F_{n-1} + F_{n-2}$ where $F_0 = 1$ and $F_1 = 1$. We have $F_n = \{1, 1, 2, 3, 5, 8, \ldots\}$. By taking the 2-adic valuation of each element of this sequence, we generate a sequence of 2-adic valuations: $\nu_2(F_n) = \{0, 0, 1, 0, 0, 3, \ldots\}$.

There are three different cases one could expect when dealing with $p$-adic valuation of integer sequences.

1. **Constant**: This case arises when the valuation of each element is the same. For example, consider sequence $\{x_n\} = \{12, 21, 30, 39, 48, \ldots\}$ where $\nu_3(x_n) = \{1, 1, 1, 1, 1, \ldots\}$. Observe that the 3-adic valuation of each element of $\{x_n\}$ is the same.

2. **Finite**: This case arises when we have a non-constant but finite valuation. That is, there exists a regularity or periodic pattern in the valuation of the sequence. For example, the sequence $\{y_n\} = \{80, 124, 176, 236, 304, 380, \ldots\}$ such that $\nu_2(y_n) = \{4, 2, 4, 2, 4, 2, \ldots\}$. Even though the 2-adic valuation is non-constant, there is a noticeable periodic pattern in the valuation of $\{y_n\}$.

3. **Infinite**: This last case arises when the valuation is neither constant nor finite. Consider the sequence $\{z_n\} = \{5, 25, 125, 625, 3125, 15625, \ldots\}$ where $\nu_5(z_n) = \{1, 2, 3, 4, 5, 6, \ldots\}$. Notice that the 5-adic valuation of $\{z_n\}$ is non-constant since the valuation of each element is different and not finite since the 5-adic valuations are increasing without bound.

Sometimes, the regularity is hard to see when the sequence of $p$-adic valuations is short, so it is important to generate a sequence long enough so that we could spot it. For example, consider the quadratic polynomial $4n^2 + 48n + 16$. It generates the sequence $\{68, 128, 196, 272, 356, 448, \ldots\}$ with 2-adic valuation $\{2, 7, 2, 4, 2, 6, \ldots\}$. Based on observation alone, this 2-adic valuation appears infinite. However, if we were to include more terms in the sequence of 2-adic valuation, the sequence then becomes

$$\left\{\boxed{2,7,2,4,2,6,2,4}, \boxed{2,7,2,4,2,6,2,4}, \boxed{2,7,2,4,2,6,2,4}, \ldots\right\}.$$

Here we see that there is a regularity in the 2-adic valuation of the sequence. Hence, this 2-adic valuation is in fact finite despite the first observation.

From the last example, we see that as the regularity grows more complex, it is difficult to differentiate between periodic and non-periodic $p$-adic valuation. Therefore, we rely on other tools to help us visualize the difference between these two types of valuations. A valuation tree drawing is used to accomplish this task. Since there are three different cases

of $p$-adic valuation of integer sequences, one could expect three different types of valuation trees as well. We expand more on the notation and construction of valuation trees in a later chapter.

# CHAPTER II: LITERATURE REVIEW

The study of $p$-adic valuations is connected to the problem of factorization of integers, an essential question in number theory and computer science. For a fixed value of $p$, the $p$-adic valuation of a sequence of integers gives us a partial factoring and can help us see the structure of the factorization of that sequence. Hence, many studies were conducted to find closed-forms of $p$-adic valuations of different integer sequences. A closed-form of the $p$-adic valuation of the sequence would be very useful when the closed form of the sequence is unknown. Even when the closed form of a sequence is known and relatively simple, the $p$-adic valuation of that sequence can still lead to some interesting results.

For example, Almodovar et al [1] demonstrated that the 2-adic valuation of the simplest class of quadratic polynomials, $\nu_2(n^2+a)$, takes on different forms depending on the choice of the parameter $a$. Their conjecture and proof rely heavily on the analysis of valuation tree drawings of their sequences. They found that the 2-adic valuation of $n^2+a$ using direct computation shows that this sequence admits a simple closed-form for $a \neq 4,7 \mod 8$. The article ends with the proof of the infinite branching in the 2-adic valuation of $n^2+7$.

In another study, Amdeberhan et al. [2] studied the 2-adic valuation of Stirling numbers and provided some approximations of the 2-adic valuation of $S(n,5)$ to support their belief that it is possible to obtain accurate approximations for the 2-adic valuations of Stirling numbers by simple integer combinations of the most basic 2-adic valuations of the integers.

Beyerstedt et. al [3] analyzed the $p$-adic valuations of $A_n$, a sequence of ASM numbers, for arbitrary primes $p$. By summing the recurrence and using $A_1 = 1$, they obtain an analytic formula for the $p$-adic valuation of $A_n$. The proof of their main theorem is achieved by induction on the number of digits in the expansion of $n$ in base $p$.

Sanna [11] studied the $p$-adic valuation of harmonic numbers and show that there exists a subset $S_p$ of the positive integers, with logarithmic density greater than 0.273, and such that for any $n \in S_p$ the $p$-adic valuation of $H_n$ is equal to $-\lfloor \log_b n \rfloor$.

Following these other studies, we set $p=2$; our main motivation for choosing $p=2$ is that it is easier to work with only even and odd cases rather than other remainder possibilities

that would arise with other choice of $p$. This allows us to quickly select integer sequences that are of interest to us.

We focus our attention on sequences related to square spirals. One well known square spiral sequence is the Ulam spiral created by Stein et al [13] for their study of square spiral sequences and distribution of primes of those sequences. It is constructed by writing the positive integers in a counterclockwise spiral pattern on a square lattice. Below is an example of the Ulam spiral sequence that starts at 1 and ends at 121 in a counterclockwise direction.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 101 | 100 | 99 | 98 | 97 | 96 | 95 | 94 | 93 | 92 | 91 |
| 102 | 65 | 64 | 63 | 62 | 61 | 60 | 59 | 58 | 57 | 90 |
| 103 | 66 | 37 | 36 | 35 | 34 | 33 | 32 | 31 | 56 | 89 |
| 104 | 67 | 38 | 17 | 16 | 15 | 14 | 13 | 30 | 55 | 88 |
| 105 | 68 | 39 | 18 | 5 | 4 | 3 | 12 | 29 | 54 | 87 |
| 106 | 69 | 40 | 19 | 6 | 1 | 2 | 11 | 28 | 53 | 86 |
| 107 | 70 | 41 | 20 | 7 | 8 | 9 | 10 | 27 | 52 | 85 |
| 108 | 71 | 42 | 21 | 22 | 23 | 24 | 25 | 26 | 51 | 84 |
| 109 | 72 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 83 |
| 110 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 |
| 111 | 112 | 113 | 114 | 115 | 116 | 117 | 118 | 119 | 120 | 121 |

*Figure 1*: Ulam spiral with some diagonal sequences

Notice the different diagonal sequences formed by the colored numbers on the Ulam spiral. One interesting property that the Ulam spiral possesses is that it is possible to express every diagonal sequence on the Ulam spiral with the quadratic polynomial:

$$4n^2 + bn + c \qquad \text{for } b, c \in \mathbb{Z} \text{ and } n = 0, 1, \ldots$$

For example, choosing $b = 2$ and $c = 4$ yields the quadratic polynomial $4n^2 + 2n + 4$. It generates the red diagonal sequence $\{10, 24, 46, 76, 114, 160, 214, \ldots\}$ and has 2-adic valuation $\{1, 3, 1, 2, 1, 5, 1, \ldots\}$.

A partial proof of the form $4n^2 + bn + c$ was given on Math Stack Exchange [7], and we complete the proof in the next chapter. Note that for certain values of $b$ and $c$, the

polynomial $4n^2+bn+c$ can also produce horizontal and vertical sequences. We selected some sequences of the form $4n^2+bn+c$ for our study. We also selected other sequences related to square spiral as well. Other sequences for this study include:

- OEIS A156859 [8]: $n^2+n+\lfloor (n+1)/2 \rfloor$.

  This sequence is the main column of a version of the counterclockwise square spiral with a center of 0 (Figure 2). It generates the sequence $\{0,3,7,14,22,33,45,60,\ldots\}$ with 2-adic valuation $\{\infty,0,0,1,1,0,0,2,\ldots\}$.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 100 | 99 | 98 | 97 | 96 | 95 | 94 | 93 | 92 | 91 | 90 |
| 101 | 64 | 63 | 62 | 61 | 60 | 59 | 58 | 57 | 56 | 89 |
| 102 | 65 | 36 | 35 | 34 | 33 | 32 | 31 | 30 | 55 | 88 |
| 103 | 66 | 37 | 16 | 15 | 14 | 13 | 12 | 29 | 54 | 87 |
| 104 | 67 | 38 | 17 | 4 | 3 | 2 | 11 | 28 | 53 | 86 |
| 105 | 68 | 39 | 18 | 5 | 0 | 1 | 10 | 27 | 52 | 85 |
| 106 | 69 | 40 | 19 | 6 | 7 | 8 | 9 | 26 | 51 | 84 |
| 107 | 70 | 41 | 20 | 21 | 22 | 23 | 24 | 25 | 50 | 83 |
| 108 | 71 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 82 |
| 109 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 |
| 110 | 111 | 112 | 113 | 114 | 115 | 116 | 117 | 118 | 119 | 120 |

*Figure 2*: Main column of a square spiral with center 0

- OEIS A325958 [9]: $16n^2+4n+4$.

  This sequence is the sum of corners of a $(2n+1)\times(2n+1)$ square spiral (Figure 3). It generates the sequence $\{4,24,76,160,276,424,604,816,\ldots\}$ which has 2-adic valuations $\{3,2,5,2,3,2,4,2,\ldots\}$.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 101 | 100 | 99 | 98 | 97 | 96 | 95 | 94 | 93 | 92 | 91 |
| 102 | 65 | 64 | 63 | 62 | 61 | 60 | 59 | 58 | 57 | 90 |
| 103 | 66 | 37 | 36 | 35 | 34 | 33 | 32 | 31 | 56 | 89 |
| 104 | 67 | 38 | 17 | 16 | 15 | 14 | 13 | 30 | 55 | 88 |
| 105 | 68 | 39 | 18 | 5 | 4 | 3 | 12 | 29 | 54 | 87 |
| 106 | 69 | 40 | 19 | 6 | 1 | 2 | 11 | 28 | 53 | 86 |
| 107 | 70 | 41 | 20 | 7 | 8 | 9 | 10 | 27 | 52 | 85 |
| 108 | 71 | 42 | 21 | 22 | 23 | 24 | 25 | 26 | 51 | 84 |
| 109 | 72 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 83 |
| 110 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 |
| 111 | 112 | 113 | 114 | 115 | 116 | 117 | 118 | 119 | 120 | 121 |

*Figure 3*: OEIS A325958 on a square spiral

- OEIS A001107 [10]: $4n^2 - 3n$.

  This sequence is the 10-gonal (or decagonal) numbers (Figure 4). It generates the sequence $\{0, 1, 10, 27, 52, 85, 126, 175, 232, \ldots\}$ with 2-adic valuation $\{\infty, 0, 1, 0, 2, 0, 1, 0, 3, \ldots\}$.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 100 | 99 | 98 | 97 | 96 | 95 | 94 | 93 | 92 | 91 | 90 |
| 101 | 64 | 63 | 62 | 61 | 60 | 59 | 58 | 57 | 56 | 89 |
| 102 | 65 | 36 | 35 | 34 | 33 | 32 | 31 | 30 | 55 | 88 |
| 103 | 66 | 37 | 16 | 15 | 14 | 13 | 12 | 29 | 54 | 87 |
| 104 | 67 | 38 | 17 | 4 | 3 | 2 | 11 | 28 | 53 | 86 |
| 105 | 68 | 39 | 18 | 5 | 0 | 1 | 10 | 27 | 52 | 85 |
| 106 | 69 | 40 | 19 | 6 | 7 | 8 | 9 | 26 | 51 | 84 |
| 107 | 70 | 41 | 20 | 21 | 22 | 23 | 24 | 25 | 50 | 83 |
| 108 | 71 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 82 |
| 109 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 |
| 110 | 111 | 112 | 113 | 114 | 115 | 116 | 117 | 118 | 119 | 120 |

*Figure 4*: OEIS A001107 on a square spiral with center 0

Our main goal of this study is to analyze the behaviors of the 2-adic valuations of some sequences related to square spirals using the concept of 2-adic valuation trees. This will expands the study of $p$-adic valuation of integer sequences further by providing more insight into the study of 2-adic valuation of different classes of sequences related to square spirals.

# CHAPTER III: METHODOLOGY

In this chapter, we first explain the sequence selection process and list some properties of $p$-adic valuations. Next, we discuss the motivation behind the use of tree structures and detail the construction of a 2-adic valuation tree. And the last section provides a proof of the form $4n^2 + bn + c$ of diagonal sequence of Ulam spiral.

## Sequence Selection

For this study, we restrict ourselves to the choice of $p = 2$ and only pick sequences related to square spirals. Since the 2-adic expansions of sequences with high density of primes are mostly zero; we must ensure that our sequences consist of enough even numbers or relatively few primes so that they are divisible by 2. Appropriate sequences should also contain at least one defining property such as an explicit formula, a recurrence, or a generating function. These defining properties are useful in helping us construct and prove the 2-adic valuations of the sequence. In particular, we picked sequences with interesting valuation trees; that is, the valuation tree has some interesting structure such as symmetry, single direction branching, alternating branching, etc.

## Properties of $p$-adic valuation

There are some helpful properties of $p$-adic valuation that we used extensively throughout the paper. We list these properties in this section. Let $a, b \in \mathbb{Z}$. Then

(a) $\nu_p(a \cdot b) = \nu_p(a) + \nu_p(b)$

(b) If $\nu_p(a) = \nu_p(b)$, then $\nu_p(a+b) \geq \min\{\nu_p(a), \nu_p(b)\}$.

(c) If $\nu_p(a) \neq \nu_p(b)$, then $\nu_p(a+b) = \min\{\nu_p(a), \nu_p(b)\}$.

**Example 3.1.** Let $p = 2$. Since $12 = 4(3) = 2^2(3)$, then $\nu_2(12) = 2$. Let $a = 4$ and $b = 3$. Using property (a), we have

$$\nu_2(12) = \nu_2(4 \cdot 3) = \nu_2(4) + \nu_2(3) = 2 + 0 = 2,$$

which matches the previous solution. Now consider $\nu_2(8)$. Since $8 = 2^3$, then $\nu_2(8) = 3$. Let $a = 4$ and $b = 4$. Then $8 = a + b$ and $\nu_2(a) = \nu_2(b) = 2$. By property (b), we see that

$$\nu_2(8) = \nu_2(4+4) = 3 \geq \min\{\nu_2(4), \nu_2(4)\} = \min\{2,2\} = 2.$$

Lastly, consider $\nu_2(7)$. Note that $\nu_2(7) = 0$. Let $a = 3$ and $b = 4$. Then $7 = a + b$ and $\nu_2(a) = 0$, $\nu_2(b) = 2$ and $\nu_2(a) \neq \nu_2(b)$. By property (c), we have

$$\nu_2(7) = \nu_2(3+4) = \min\{\nu_2(3), \nu_2(4)\} = \min\{0,2\} = 0.$$

**Plots of the 2-adic Valuations of Square Spiral Sequences**

Consider the diagonal sequence $\{68, 128, 196, 272, 356, 448, 548, \ldots\}$. It is generated by the quadratic polynomial $p(n) = 4n^2 + 48n + 16$ and has 2-adic valuation $\{2, 7, 2, 4, 2, 6, 2, \ldots\}$. Figure 5 shows the plot of the 2-adic valuations of $p(n)$ for $n = 1, 2, \ldots, 100$.

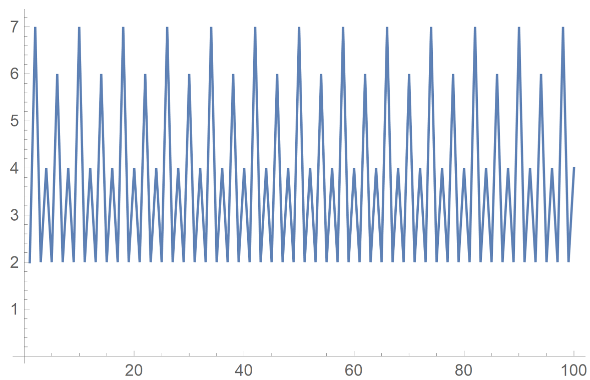



*Figure 5*: 2-adic valuations of $p(n) = 4n^2 + 48n + 16$ for $n = 1, 2, \ldots, 100$

Notice that there exists a regularity in the 2-adic valuation of this sequence. The values on the plot oscillate between the numbers $2, 4, 6$, and $7$, which leaves the impression that the 2-adic valuation of this sequence is finite. To see if this observation holds for larger range of $n$, we generate a plot of the 2-adic valuations of $p(n)$ for $n = 1, 2, \ldots, 1000$ in Figure 6. With a larger range of values, it is difficult to see the oscillation effect. However, observe that the values on the plot do not exceed 7; this indicates that the valuation of the sequence remains at most 7 as $n$ gets larger. Hence, our conjecture is that the 2-adic valuation of the diagonal sequence $p(n) = 4n^2 + 48n + 16$ is finite. The proof of this conjecture is left for a later section.

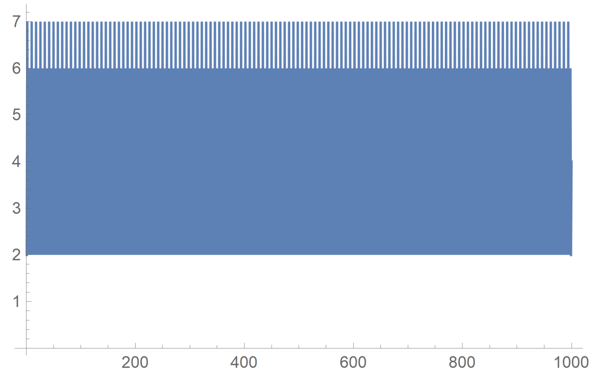



*Figure 6*: 2-adic valuations of $p(n) = 4n^2 + 48n + 16$ for $n = 1, 2, \ldots, 1000$

Now consider the diagonal sequence $\{10, 24, 46, 76, 114, 160, 214, 276, \ldots\}$. This sequence is generated by the quadratic polynomial $p(n) = 4n^2 + 2n + 4$ and has 2-adic valuations $\{1, 3, 1, 2, 1, 5, 1, 2, \ldots\}$. Figure 7 shows the plot of the 2-adic valuation of $p(n)$ for $n = 1, 2, \ldots, 100$.

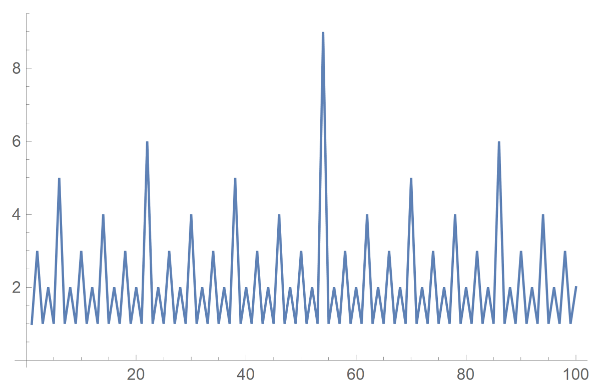



*Figure 7*: 2-adic valuations of $p(n) = 4n^2 + 2n + 4$ for $n = 1, 2, \ldots .100$

The valuation appears non-periodic since it is difficult to see if there exists a regularity in the plot. However, it is easy to see that the values on the plot are at most 9 for $n = 1, \ldots, 100$. To see if this observation holds for $n = 1, 2, \ldots, 1000$, we generate a plot of the 2-adic valuation of $p(n)$ for $n = 1 \ldots 1000$ in Figure 8. We see that the values in this plot are at most 13, a higher value compared to the one observed in Figure 7. Hence, we conjecture that as $n$ increases, the 2-adic valuation of $p(n) = 4n^2 + 2n + 4$ also increases without bound as well, and thus this sequence produces infinitely many valuations.

The question remains as to how we could describe the 2-adic valuation of a sequence that

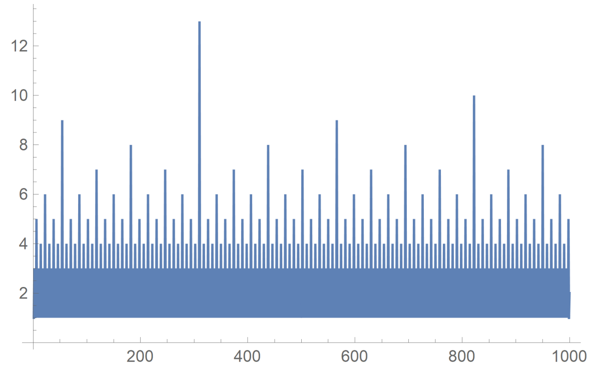



*Figure 8*: 2-adic valuations of $p(n) = 4n^2 + 2n + 4$ for $n = 1, 2, \ldots, 1000$

produces infinitely many valuations. The plots shown in this section are useful for quickly determining if a sequence produces finitely or infinitely many valuations. However, if we want to further study the structure of the 2-adic valuations of sequences, we will quickly find that these plots are no longer sufficient for this task. One way to accomplish this task is to represent the 2-adic valuations of sequences as binary trees. The discussion on 2-adic valuation trees of sequences are provided in the next two sections.

**Numbers associated with $p$-adic tree**

In this section, we expand more on the definition of numbers associated with vertices in 2-adic valuation trees from Almodovara et al [1]. Let $p$ be a prime. Every $p$-adic tree starts with a single root vertex at the zeroth level. The numbers associated with the root vertex are of the form $n$ for all $n \in \mathbb{N}$.

We refine the form $n$ associated with the root vertex to get the numbers associated with vertices of the first level. Let $a_0$ be a remainder of dividing $n$ by $p$ such that $a_0 \in \{0, 1, 2, \ldots, p-1\}$. It follows that the root vertex $v_0$ then splits into $p$ vertices at the first

level. The numbers corresponding to these vertices are defined as

$$n \equiv 0 \pmod{p}$$
$$n \equiv 1 \pmod{p}$$
$$n \equiv 2 \pmod{p}$$
$$\vdots$$
$$n \equiv (p-1) \pmod{p}.$$

Therefore,

$$n \equiv a_0 \pmod{p} \implies n - a_0 = mp \implies n = mp + a_0$$

Then all numbers associated with vertices at level one have the form $mp + a_0$.

Let $a_1$ be a remainder of dividing $m$ by $p$ such that $a_1 \in \{0, 1, 2, \ldots, p-1\}$. Then $m$ could also be written in the form $lp + a_1$ for $a_1 \in \{0, 1, 2 \ldots, p-1\}$ then

$$mp + a_0 = (lp + a_1)p + a_0 = lp^2 + a_1 p + a_0$$

Hence, each vertex at level one is then split into $p$ vertices at level two and the numbers associated with these vertices have the form $lp^2 + a_1 p + a_0$. By a similar procedure, we could also refine the form $lp^2 + a_1 p + a_0$ to get vertices at level three as well.

In general, let $k \in \mathbb{N}$ be the level of the $p$-adic valuation tree and $v_k$ be the vertices at level $k$. Then the numbers associated with $v_{k+1}$ that were split from $v_k$ have the form

$$p^{k+1} n + p^k a_k + b_{k-1}$$

where

$$b_{k-1} = \begin{cases} p^{k-1} a_{k-1} + \ldots + p a_1 + a_0 & \text{if } k-1 \geq 0 \\ 0 & \text{if } k-1 < 0 \end{cases}$$

and $a_{k-1}, \ldots, a_1,$ and $a_0$ have already been determined. Example (3.1) shows a three-level 2-adic tree with numbers associated with each vertex listed.

**Example 3.1.** Let $p = 2$. The root vertex $v_0$ at the zeroth level splits into 2 vertices at level one. The numbers of those vertices have the form

$$2^{0+1} n + 2^0 a_0 + 0 = 2n + a_0 \quad \text{with} \quad a_0 \in \{0, 1\}.$$

Then the numbers associated with the two vertices at level 1 are either even $(2n)$ or odd $(2n+1)$. Figure 9 shows the root vertex and the two vertices at the first level along with the numbers associated with them.

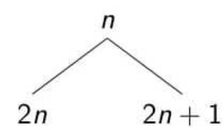



*Figure 9*: The numbers associated with the root vertex and the first two levels

Each $v_1$ then splits into two vertices at level 2, and the numbers associated with the vertices at level 2 are defined as

$$2^{1+1}n + 2^1 a_1 + b_0 = 4n + 2a_1 + a_0 \quad \text{with} \quad a_1 \in \{0,1\}$$

For the vertices associated with $2n$, the value of $a_0$ is already determined to be 0. Then the numbers associated with those vertices have the form

$$4n + 2a_1 = \begin{cases} 4n & \text{if} \quad a_1 = 0 \\ 4n+2 & \text{if} \quad a_1 = 1 \end{cases}$$

For the vertices associated with $2n+1$, the value of $a_0$ is already determined to be 1. Then the numbers associated with those vertices have the form

$$4n + 2a_1 + 1 = \begin{cases} 4n+1 & \text{if} \quad a_1 = 0 \\ 4n+3 & \text{if} \quad a_1 = 1 \end{cases}$$

Figure 10 shows the root vertex and the first two levels along with the numbers associated with them.

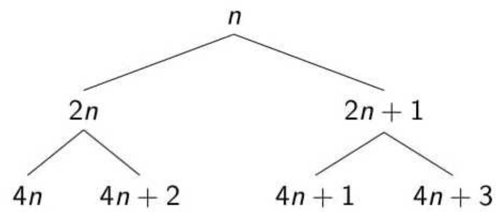



*Figure 10*: The numbers associated with the root vertex and the first two levels

Since the value of $a_1$ and $a_0$ had already been determined at this point, we then could use the same procedure as above to find the numbers associated with vertices branched from each $v_2$. We can choose $a_2 \in \{0,1\}$ such that

$$8n+4a_2+2a_1+a_0 = \begin{cases} 8n & \text{if} \quad a_2=0, \quad a_1=0, \quad a_0=0 \\ 8n+4 & \text{if} \quad a_2=1, \quad a_1=0, \quad a_0=0 \\ 8n+2 & \text{if} \quad a_2=0, \quad a_1=1, \quad a_0=0 \\ 8n+6 & \text{if} \quad a_2=1, \quad a_1=1, \quad a_0=0 \\ 8n+1 & \text{if} \quad a_2=0, \quad a_1=0, \quad a_0=1 \\ 8n+5 & \text{if} \quad a_2=1, \quad a_1=0, \quad a_0=1 \\ 8n+3 & \text{if} \quad a_2=0, \quad a_1=1, \quad a_0=1 \\ 8n+7 & \text{if} \quad a_2=1, \quad a_1=1, \quad a_0=1 \end{cases}$$

Figure 11 shows the root vertex and the first three levels along with the numbers associated with them.

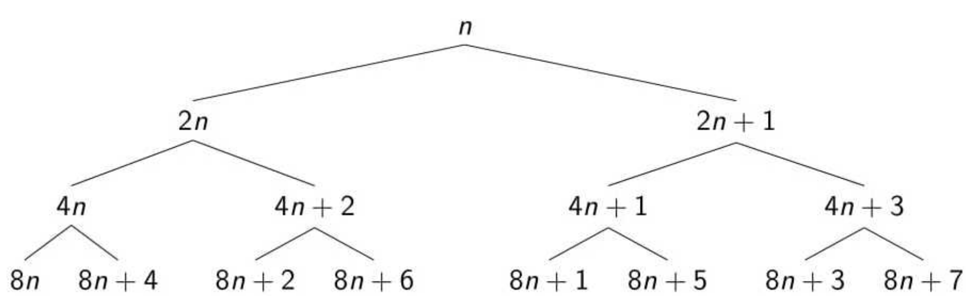



*Figure 11*: The numbers associated with the root vertex and the first three levels

### Construction of a $p$-adic tree

The following construction is an expansion of the construction shown by Almodovara et al [1]. Let $\{x_n\}$ be an integer sequence and $f(x)$ be a polynomial with integer coefficients such that $\{x_n\} = f(pn+a_0)$. Since $pn+a_0 \equiv a_0 \pmod{p}$, it follows that

$$f(pn+a_0) \equiv f(a_0) \pmod{p}. \tag{1}$$

for $a_0 \in \{0, 1, \ldots, p-1\}$.

Assume $f(a_0) \not\equiv 0 \pmod{p}$. It follows from (1) that

$$f(pn+a_0) \not\equiv 0 \pmod{p} \quad \Longrightarrow \quad p \nmid f(pn+a_0).$$

Then there exists no $k \in \mathbb{N}$ satisfying $f(pn+a_0) = pk$. Therefore, we have

$$\nu_p(f(pn+a_0)) = 0 \quad \text{if} \quad f(a_0) \not\equiv 0 \pmod{p} \tag{2}$$

On the other hand, assume $f(a_0) \equiv 0 \pmod{p}$. It follows from (1) that

$$f(pn+a_0) \equiv 0 \pmod{p} \quad \Longrightarrow \quad p \mid f(pn+a_0).$$

Then $f(pn+a_0) = pk$ for some $k \in \mathbb{N}$. Since $k$ could contain other multiples of $p$, we have

$$\nu_p(f(pn+a_0)) \geq 1 \quad \text{if} \quad f(a_0) \equiv 0 \pmod{p}. \tag{3}$$

Since the valuation is non-constant, the $p$-adic valuation of the vertex with numbers $pn+a_0$ is non-terminating but is at least 1 at this level. It follows that this vertex splits into $p$ vertices at the next level. From (2) and (3), the numbers associated with vertices of the next level is defined as $p^2n + pa_1 + a_0$ for $a_1 \in \{0, 1, \ldots, p-1\}$ and $a_0$ is already determined.

Observe from (2) and (3) that $\nu_p(f(pn+a_0))$ depends on whether $p$ divides $f(a_0)$. Hence, we now consider whether $p^2$ divides $f(pa_1+a_0)$ to determine the value of $\nu_p(f(p^2n + pa_1 + a_0))$. Since $p^2n + pa_1 + a_0 \equiv pa_1 + a_0 \pmod{p^2}$, then

$$f(p^2n + pa_1 + a_0) \equiv f(pa_1 + a_0) \pmod{p^2}. \tag{4}$$

By a similar procedure as before, assume $f(pa_1+a_0) \not\equiv 0 \pmod{p^2}$. It follows that

$$f(p^2n+pa_1+a_0) \not\equiv 0 \pmod{p^2} \quad \Longrightarrow \quad p^2 \nmid f(p^2n+pa_1+a_0).$$

Hence,

$$\nu_p(f(p^2n+pa_1+a_0)) = 1 \quad \text{if} \quad f(pa_1+a_0) \not\equiv 0 \pmod{p^2}. \tag{5}$$

On the other hand, assume $f(pa_1+a_0) \equiv 0 \pmod{p^2}$. Then

$$f(p^2n+2a_1+a_0) \equiv 0 \pmod{p^2} \quad \Longrightarrow \quad p^2 \mid f(p^2n+2a_1+a_0).$$

Thus $f(p^2n+pa_1+a_0) = p^2m$ for some $m \in \mathbb{N}$ and $\nu_p(f(p^2m+pa_1+a_0)) \geq 2$. Therefore

$$\nu_p(f(p^2n+pa_1+a_0)) \geq 2 \quad \text{if} \quad f(pa_1+a_0) \equiv 0 \pmod{p^2} \tag{6}$$

and we would repeat the same process again to determine the value of $\nu_p(f(p^3n+p^2a_2+pa_1+a_0))$ for $a_2 \in \{0,1,\ldots,p-1\}$ with $a_1, a_0$ already determined.

In general, if $a_0, a_1, a_2, \ldots, a_{k-1} \in \{0,1,\ldots,p-1\}$ such that

$$\begin{aligned} f(a_0) &\equiv 0 \pmod{p} \\ f(pa_1+a_0) &\equiv 0 \pmod{p^2} \\ f(p^2a_2+pa_1+a_0) &\equiv 0 \pmod{p^3} \\ &\vdots \\ f(p^ka_k+p^{k-1}a_{k-1}+\ldots+pa_1+a_0) &\not\equiv 0 \pmod{p^{k+1}} \end{aligned}$$

Then for any $n \equiv p^ka_k+p^{k-1}a_{k-1}+\ldots+pa_1+a_0 \pmod{p^{k+1}}$, we have

$$\nu_p(f(n)) = k. \tag{7}$$

The process above provides a formulaic way to determine whether a vertex is terminating or non-terminating. However, it is often difficult to predict possible branching with just this process alone. Therefore, we use a $p$-adic tree to visualize the process above. Example (3.2) shows how we can use a $p$-adic tree to describe the valuation of a sequence.

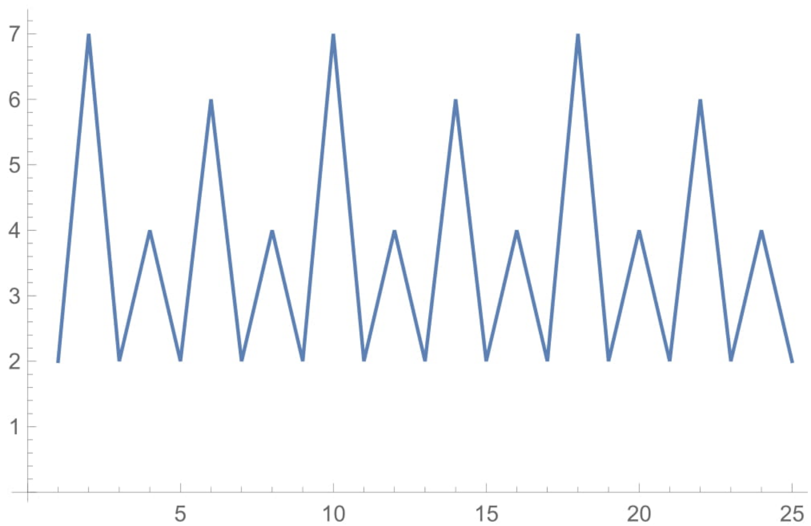
**Example 3.2.** The sequence $\{68, 128, 196, 272, 356, 448, 548, \ldots\}$ is generated by the quadratic function $p(n) = 4n^2 + 48n + 16$ for $n \geq 1$. We use Mathematica to form a conjecture of the 2-adic valuation tree of this sequence. Below are the Mathematica commands used and their purpose:

- `p[n]:= 4n^2 + 48n + 16;` – Define $p(n)$ to be the function $4n^2 + 48n + 16$.
- `Table[IntegerExponent[n,p],{n,a,b}]` – Generate a table of $p$-adic valuations of integers in the range $[a, b]$.
- `ListPlot[Table[IntegerExponent[n,p],{n,a,b}]` – Generate a list-plot of the table of $p$-adic valuations of integers in the range $[a, b]$.

We provide the inputs and outputs obtained from Mathematica below.

```
In[∘]:= p[n_] := p[n] = 4 n^2 + 48 n + 16;
        ListPlot[Table[IntegerExponent[p[n], 2], {n, 1, 25}], Joined → True]
```

Out[∘]=




```
In[∘]:= Table[IntegerExponent[p[n], 2], {n, 1, 25}]
Out[∘]= {2, 7, 2, 4, 2, 6, 2, 4, 2, 7, 2, 4, 2, 6, 2, 4, 2, 7, 2, 4, 2, 6, 2, 4, 2}
```

The list-plot of the first 25 numbers of the diagonal sequence $p(n)$ contains four different 2-adic valuations; the values of these valuations are $2, 4, 6$, and $7$. This observation along with other explorations in Mathematica led to the conjecture that the 2-adic valuation tree

of $p(n)$ only contains the four values $2,4,6,7$, and thus the tree is terminal. We prove this conjecture by constructing a 2-adic tree.

The construction of the 2-adic valuation tree of this sequence begins with the root vertex at level 0, and the numbers associated with the root vertex have the form $n$ for $n \in \mathbb{N}$. Since we can factor $4n^2+48n+16$ as $2^2(n^2+12n+4)$, it follows that

$$\nu_2(4n^2+48n+16) = \nu_2(2^2)+\nu_2(n^2+12n+4) = 2+\nu_2(n^2+12n+4).$$

Since it is possible for $n^2+12n+4$ to contain other multiples of 2, we can only conclude that $\nu_2(4n^2+48n+16) \geq 2$. Therefore, the root vertex is non-constant which implies that it is non-terminating but is at least 2. We use the symbol $(*)$ used by Almodovar et al [1] to denote that a vertex is non-terminating. It follows that the root vertex splits into 2 vertices at the first level.

The numbers associated with the left vertex are even and have the form $2k$. On the other hand, the numbers associated with the right vertex are odd and have the form $2k+1$. We can find the 2-adic valuation of these vertices by substituting $2k$ and $2k+1$ back into $f(n)$. By substituting $2k$, we have $4(2k)^2+48(2k)+16 = 2^4(k^2+6k+1)$. Then

$$\begin{aligned}\nu_2(4(2n)^2+48(2n)+16) &= \nu_2(2^4)+\nu_2(k^2+6k+1)\\ &= 4+\nu_2(k^2+6k+1).\end{aligned}$$

Since $k^2+6k+1$ could contain other multiples of 2, then $\nu_2(4(2k)^2+48(2k)+16) \geq 4$ which implies that the vertex with numbers $2n$ is non-terminating.

By substituting $2k+1$, we have $4(2k+1)^2+48(2k+1)+16 = 2^2(4k^2+28k+17)$ and

$$\begin{aligned}\nu_2(4(2k+1)^2+48(2k+1)+16) &= \nu_2(2^2(4k^2+28k+17)\\ &= \nu_2(2^2)+\nu_2(4k^2+28k+17)\\ &= 2+\nu_2(4k^2+28k+17).\end{aligned}$$

Notice that $4k^2+28k+17$ is odd and contains no other multiple of 2. Therefore, we conclude that $\nu_2(4(2k+1)^2+48(2k+1)+16) = 2$ which implies that the vertex with

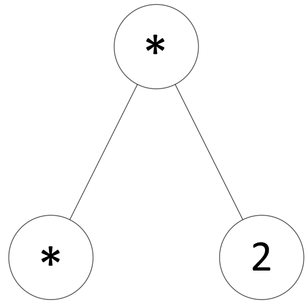


*Figure 12*: 2-adic valuation tree of root vertex and first level

numbers $2k+1$ terminates. This completes the first level of the tree. Figure 12 describes the process above with a 2-adic valuation tree.

Since the even vertex at level 1 is non-terminating, it splits into 2 vertices at level 2. These vertices have the form $4l$ and $4l+2$. We repeat the same substitution process as above. By substituting $4l$, we have

$$\begin{aligned} v_2(4(4l)^2+48(4l)+16) &= v_2(2^4(4l^2+12l+1)) \\ &= v_2(2^4)+v_2(4l^2+12l+1) \\ &= 4+v_2(4l^2+12l+1) = 4 \end{aligned}$$

which implies that the vertex with numbers $4l$ terminates. By substituting $4l+2$,

$$\begin{aligned} v_2(4(4l+2)^2+48(4l+2)+16) &= v_2(2^6(l^2+4l+2)) \\ &= v_2(2^6)+v_2(l^2+4l+2) \\ &= 6+v_2(l^2+4l+2) \geq 6 \end{aligned}$$

which implies that the vertex with numbers $4l+2$ is non-terminating. This completes the second level of the tree. Figure 13 extends Figure 12 to include the second level of the tree.

We repeat the same process again since the vertex with numbers $4l+2$ at level 2 is non-terminating. This vertex splits into 2 vertices at level 3 with the left vertex having numbers of the form $8m+2$ and the right vertex having numbers $8m+6$. Then by substitution, we

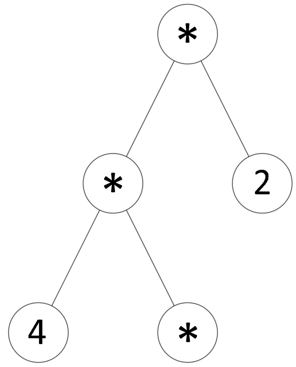



*Figure 13*: 2-adic valuation tree of root vertex and first two levels

have

$$\begin{aligned} \nu_2(4(8m+2)^2+48(8m+2)+16) &= \nu_2(2^7(2m^2+4m+1)) \\ &= \nu_2(2^7)+\nu_2(2m^2+4m+1) \\ &= 7+\nu_2(2m^2+4m+1) = 7 \end{aligned}$$

which implies that the vertex with numbers $8m+2$ terminates. Next, we substitute in $8m+6$,

$$\begin{aligned} \nu_2(4(8m+6)^2+48(8m+6)+16) &= \nu_2(2^6(4m^2+12m+7)) \\ &= \nu_2(2^6)+\nu_2(4m^2+12m+7) \\ &= 6+\nu_2(4m^2+12m+7) = 6 \end{aligned}$$

which implies that the vertex with numbers $8m+6$ also terminates. This completes the third level and the tree since there is no non-terminating vertex left. Figure 14 shows the complete 2-adic valuation tree.

We compare this result to the one generated by a tree drawing package in Mathematica [5]. This tree package takes an integer sequence as input and generates a conjecture 2-adic valuation tree up to the specify depth. Figure 15 shows the 2-adic valuation tree of the diagonal sequences $p(n) = 4n^2+48n+16$ generated in Mathematica. We see that the trees in both Figure 14 and 15 are the same. We will use this tree drawing package to assist in selecting sequences and form conjecture of the 2-adic valuation trees of different square spiral sequences in later chapters.

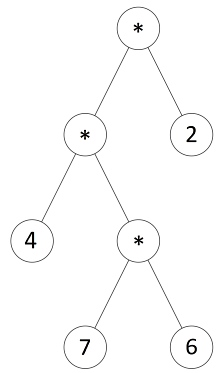



*Figure 14*: Complete 2-adic valuation tree of $4n^2+48n+16$

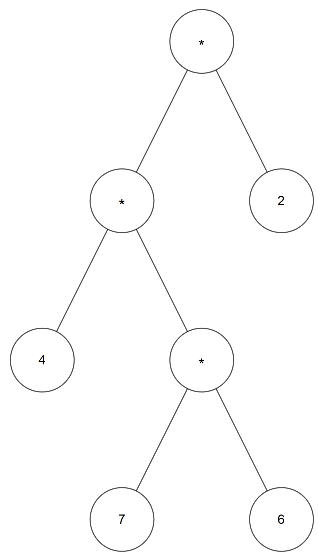



*Figure 15*: 2-adic valuation tree of $4n^2+48n+16$ generated in Mathematica

**Theorem III.5.1 (2-adic valuation of diagonal sequence $\mathbf{4n^2+48n+16}$).**

$$v_2(4n^2+48n+16)=\begin{cases}2 & \textit{if } n\equiv 1 \pmod 2\\ 4 & \textit{if } n\equiv 0 \pmod 4\\ 6 & \textit{if } n\equiv 6 \pmod 8\\ 7 & \textit{if } n\equiv 2 \pmod 8.\end{cases}$$

**Proof of Diagonal Sequences of Ulam spiral**

We show that all diagonal sequences of the Ulam spiral have the form $4n^2+bn+c$. This was partially proved by Myerson on Math Stack Exchange [7]. We extend this partial result to provide a complete proof in this section. First, we make some observations about the diagonal sequences on the Ulam spiral. Figure 16 shows some diagonal sequences on the Ulam spiral.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 101 | 100 | 99 | 98 | 97 | 96 | 95 | 94 | 93 | 92 | 91 |
| 102 | 65 | 64 | 63 | 62 | 61 | 60 | 59 | 58 | 57 | 90 |
| 103 | 66 | 37 | 36 | 35 | 34 | 33 | 32 | 31 | 56 | 89 |
| 104 | 67 | 38 | 17 | 16 | 15 | 14 | 13 | 30 | 55 | 88 |
| 105 | 68 | 39 | 18 | 5 | 4 | 3 | 12 | 29 | 54 | 87 |
| 106 | 69 | 40 | 19 | 6 | 1 | 2 | 11 | 28 | 53 | 86 |
| 107 | 70 | 41 | 20 | 7 | 8 | 9 | 10 | 27 | 52 | 85 |
| 108 | 71 | 42 | 21 | 22 | 23 | 24 | 25 | 26 | 51 | 84 |
| 109 | 72 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 83 |
| 110 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 |
| 111 | 112 | 113 | 114 | 115 | 116 | 117 | 118 | 119 | 120 | 121 |

*Figure 16*: Ulam spiral with some diagonal sequences

Let $\{a_n\}$ be the diagonal sequence $\{3, 13, 31, 57, 91, \ldots\}$ for $n \geq 1$. Observe that

$$a_2 - a_1 = 13 - 3 = 10$$

$$a_3 - a_2 = 31 - 13 = 18$$

$$a_4 - a_3 = 57 - 31 = 26$$

$$a_5 - a_4 = 91 - 57 = 34$$

and

$$(a_3 - a_2) - (a_2 - a_1) = 18 - 10 = 8$$

$$(a_4 - a_3) - (a_3 - a_2) = 26 - 18 = 8$$

$$(a_5 - a_4) - (a_4 - a_3) = 34 - 26 = 8.$$

Therefore, $\{a_n\}$ has the non-homogeneous recurrence relation

$$8 = (a_n - a_{n-1}) - (a_{n-1} - a_{n-2}) = a_n - 2a_{n-1} + a_{n-2}.$$

A quick verification using terms from $\{a_n\}$ shows that

$$31-2(13)+3=8, \quad 57-2(31)+13=8, \quad 91-2(57)+31=8.$$

This non-homogeneous recurrence relation also holds for other diagonals sequence as well.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 101 | 100 | 99 | 98 | 97 | 96 | 95 | 94 | 93 | 92 | 91 |
| 102 | 65 | 64 | 63 | 62 | 61 | 60 | 59 | 58 | 57 | 90 |
| 103 | 66 | 37 | 36 | 35 | 34 | 33 | 32 | 31 | 56 | 89 |
| 104 | 67 | 38 | 17 | 16 | 15 | 14 | 13 | 30 | 55 | 88 |
| 105 | 68 | 39 | 18 | 5 | 4 | 3 | 12 | 29 | 54 | 87 |
| 106 | 69 | 40 | 19 | 6 | 1 | 2 | 11 | 28 | 53 | 86 |
| 107 | 70 | 41 | 20 | 7 | 8 | 9 | 10 | 27 | 52 | 85 |
| 108 | 71 | 42 | 21 | 22 | 23 | 24 | 25 | 26 | 51 | 84 |
| 109 | 72 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 83 |
| 110 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 |
| 111 | 112 | 113 | 114 | 115 | 116 | 117 | 118 | 119 | 120 | 121 |

*Figure 17*: Rings on Ulam spiral

Observe from Figure 17 that the first ring has 1 element, and the second ring has 8 elements. The third ring has 16 elements, then 24, 32, 40, and so on. The numbers of elements on each ring, aside from the first, increase by 8 between each ring. Since each number of a diagonal sequence lies on a different ring, it follows that the second difference of these numbers also increases by 8.

For example, consider the diagonal sequence $\{10, 24, 46, 76, 114, \ldots\}$, we have

$$46-2(25)+10=8, \quad 76-2(46)+24=8, \quad 114-2(75)+46=8.$$

Likewise for the diagonal sequence $\{6, 18, 38, 66, 102, \ldots\}$,

$$38-2(18)+6=8, \quad 66-2(38)+18=8, \quad 102-2(66)+38=8.$$

Therefore, all diagonal sequences of the Ulam spiral are defined by the non-homogeneous recurrence relation $a_n - 2a_{n-1} + a_{n-2} = 8$. We solve this relation for a closed-form.

We shift the index of $a_n - 2a_{n-1} + a_{n-2} = 8$ to get

$$a_{n+2} - 2a_{n+1} + a_n = 8. \tag{8}$$

Then the general solution of (8) is

$$a_n = a_n^{(h)} + a_n^{(p)} \tag{9}$$

where $a_n^{(h)}$ is the solution of the corresponding homogeneous recurrence relation of (8) and $a_n^{(p)}$ is the particular solution of (8).

The corresponding homogeneous recurrence relation have the form $a_{n+2} + a_{n+1} + a_n = 0$ with characteristic polynomial $x^2 - 2x + 1$. The roots of this polynomial is $x = 1$ with a multiplicity of 2. Therefore $a_n^{(h)} = A(1^n) + Bn(1^n) = A + Bn$ is the solution of the corresponding homogeneous recurrence relation.

We now solve for the particular solution $a_n^{(p)}$. Assume $a_n^{(p)} = \alpha n^2 + \beta n + \gamma$. We substitute $a_n^{(p)}$ into (8) and get

$$\begin{aligned}
8 &= [\alpha(n+2)^2 + \beta(n+2) + 8] - 2[\alpha(n+1)^2 + \beta(n+1) + 8] + (\alpha n^2 + \beta n + 8) \\
&= [\alpha(n^2 + 4n + 4) + \beta n + 2\beta + 8] - 2[\alpha(n^2 + 2n + 1) + \beta n + \beta + 8] + (\alpha n^2 + \beta n + 8) \\
&= \alpha n^2 + 4\alpha n + 4\alpha + \beta n + 2\beta + 8 - 2\alpha n^2 - 4\alpha n - 2\alpha - 2\beta n - 2\beta - 16 + \alpha n^2 + \beta n + 8 \\
&= 2\alpha n^2 + 4\alpha n + 4\alpha + 2\beta n + 2\beta + 16 - 2\alpha n^2 - 4\alpha n - 2\alpha - 2\beta n - 2\beta - 16 \\
&= 2\alpha.
\end{aligned}$$

So $8 = 2\alpha$ which implies $\alpha = 4$. Therefore $a_n^{(p)} = 4n^2 + \beta n + \gamma$ with arbitrary $\beta$ and $\gamma$. Then the solution of (9) is

$$a_n = A + Bn + 4n^2 + \beta n + \gamma = 4n^2 + (B + \beta)n + (A + \gamma)$$

Let $b = B + \beta$ and $c = A + \gamma$, then we have

$$a_n = 4n^2 + bn + c \tag{10}$$

as the solution of the non-homogeneous recurrence relation $a_{n+2} + a_{n+1} + a_n = 8$. Therefore, all diagonal sequences of Ulam spiral are defined by (10).

To verify this result, consider the diagonal sequence $\{3, 13, 31, 57, 91, \ldots\}$. We have

$a_1 = 3$ and $a_2 = 13$. By substitution, we have a system of two equations

$$3 = 4 + b + c$$

$$13 = 16 + 2b + c$$

Solving this system yield $b = -2$ and $c = 1$. Then $\{3, 13, 31, 57, 91, \ldots\}$ is defined by the quadratic polynomial $4n^2 - 2n + 1$. We use Mathematica to find the sequence generated by $f(n) = 4n^2 - 2n + 1$ for $n \geq 1$. We found that

$$f(n) = \{3, 13, 31, 57, 91, 133, 183, \ldots\},$$

which matched our sequence.

Likewise, for the diagonal sequence $\{6, 18, 38, 66, 102, \ldots\}$, we have $a_1 = 6$ and $a_2 = 18$. By similar procedure as above, we found that $b = 0$ and $c = 2$. Then $4n^2 + 2$ is the quadratic polynomial that generates $\{6, 18, 38, 66, 102, \ldots\}$. Let $f(n) = 4n^2 + 2$ for $n \geq 1$, then

$$f(n) = \{6, 18, 38, 66, 102, 146, 198, \ldots\},$$

which matched the diagonal sequence.

## CHAPTER IV: PROOFS OF SOME ULAM DIAGONAL SEQUENCES

In this chapter, we state the 2-adic valuations of some diagonal sequences of the Ulam spiral. We prove these statements by showing that their valuations can be represented by 2-adic trees.

### Proof of Diagonal Sequence $4n^2+48n+16$ Valuation

**Theorem IV.1.1 (2-adic valuation of diagonal sequence $4n^2+48n+16$).**

$$v_2(4n^2+48n+16) = \begin{cases} 2 & \textit{if } n \equiv 1 \pmod 2 \\ 4 & \textit{if } n \equiv 0 \pmod 4 \\ 6 & \textit{if } n \equiv 6 \pmod 8 \\ 7 & \textit{if } n \equiv 2 \pmod 8. \end{cases}$$

*Proof.* See Example 3.2 in Chapter 3. □

### Proof of Diagonal Sequence $4n^2+2n+4$ Valuation

The Ulam spiral sequence $\{10, 24, 46, 76, 114, 160, 214, 276, \ldots\}$ is generated by the quadratic polynomial $4n^2+2n+4$ and has 2-adic valuations $\{1, 3, 1, 2, 1, 5, 1, 2, \ldots\}$. Figure 18 shows the listplot of the 2-adic valuations of this sequence while Figure 19 shows the 2-adic tree to a depth of 3.

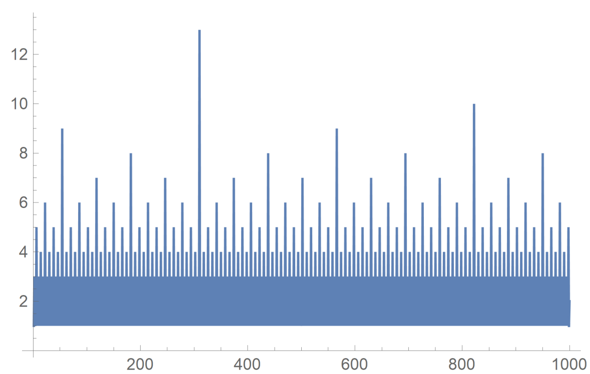



*Figure 18*: 2-adic listplot of $4n^2+2n+4$.

Observe that at each level $k$, the terminating vertex has valuation exactly $k$, and the other non-terminating vertex has valuation at least $k$. Since this 2-adic tree appears to be non-terminating, we rely on the technique used by Almodovar et al [1], which is an adaptation of

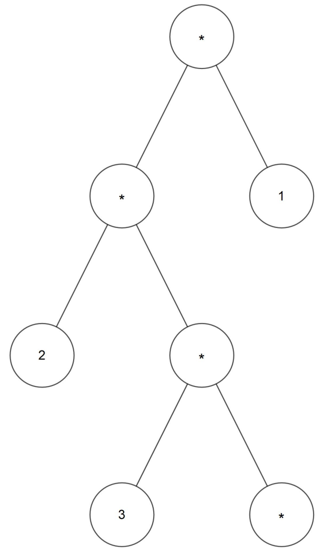



*Figure 19*: 2-adic tree of the sequence $4n^2+2n+4$.

Hensel's Lemma (Theorem 2.20) [12]. Using this lifting technique, we show that the 2-adic valuation of the diagonal sequence $4n^2+2n+4$ is infinite and can be captured by such a tree.

**Theorem IV.2.1.** *Let $v$ be a non-terminating node in the valuation tree of $\nu_2(4n^2+2n+4)$ at the $k$-th level with $k \geq 1$. Then $v$ splits into two vertices at the $(k+1)$-th level. One terminates with valuation $k+1$. The other has valuation of at least $k+2$.*

*Proof.* Let $f(n) = 4n^2+2n+4$. The numbers associated with the vertex $v$ at level $k$ have the form $2^k n + b_{k-1}$, and the numbers associated with vertices at level $k+1$ beneath $v$ have the form

$$N_k = 2^{k+1}n + 2^k a_k + b_{k-1}$$

where

$$b_{k-1} = 2^{k-1}a_{k-1} + \ldots + 2a_1 + a_0$$

has been determined. The induction hypothesis is that

$$\begin{aligned} f(2^k n + b_{k-1}) &\equiv 0 \pmod{2^{k+1}} \\ 2^{2k+2}n^2 + 2^{k+3}nb_{k-1} + 2^{k+1}n + 4b_{k-1}^2 + 2b_{k-1} + 4 &\equiv 0 \pmod{2^{k+1}} \\ 4b_{k-1}^2 + 2b_{k-1} + 4 &\equiv 0 \pmod{2^{k+1}} \end{aligned}$$

so that $\nu_2(4b_{k-1}^2+2b_{k-1}+4)\geq k+1$. Whether the numbers associated with vertices at level $k+1$ beneath $v$ is terminating or not depends on the choice of $a_k=0$ or $a_k=1$. Hence, consider the congruence

$$f(N_k)=4N_k^2+2N_k+4\equiv 0 \pmod{2^{k+2}}$$

that is,

$$\begin{aligned}
4[2^{k+1}n+2^k a_k+b_{k-1}]^2+2[2^{k+1}n+2^k a_k+b_{k-1}]+4 &\equiv 0 \pmod{2^{k+2}}\\
2^{2k+4}n^2+2^{2k+4}na_k+2^{2k+2}a_k^2+2^{k+4}nb_{k-1}+2^{k+3}a_kb_{k-1}+2^{k+2}n+2^{k+1}a_k&\\
+4b_{k-1}^2+2b_{k-1}+4 &\equiv 0 \pmod{2^{k+2}}
\end{aligned}$$

which reduces to

$$4b_{k-1}^2+2^{k+1}a_k+2b_{k-1}+4\equiv 0 \pmod{2^{k+2}}. \tag{11}$$

We solve (11) for the unknown $a_k$. Note that (11) can be rewritten as

$$2^{k+1}a_k\equiv -(4b_{k-1}^2+2b_{k-1}+4) \pmod{2^{k+2}}$$

By the induction hypothesis, $4b_{k-1}^2+2b_{k-1}+4=2^{k+1}m$ for some $m\in\mathbb{Z}$.

$$\begin{aligned}
2^{k+1}a_k &\equiv -(2^{k+1}m) \pmod{2^{k+2}}\\
a_k &\equiv -m \pmod{2}
\end{aligned}$$

as the solution to (4.1). Therefore the vertex descending from $v$ with $a_k\not\equiv -m \pmod 2$ terminates with valuation $k+1$, and the other vertex has valuation at least $k+2$. □

We will discuss the 2-adic valuations of other diagonal sequences more in Chapter 6 where we classify some diagonal sequences based on similar tree patterns.

# CHAPTER V: PROOFS OF SOME NON-DIAGONAL ULAM SEQUENCES

We explore and prove the 2-adic valuation trees of some non-diagonal Ulam sequences in this chapter.

**Proof of A156859 Tree**

From the OEIS, the sequence A156859 is the sequence of the main column of a version of the square spiral with a center of zero (Figure 2). It is defined by the function $f(n) = n^2 + n + \lfloor \frac{n+1}{2} \rfloor$ for $n \in \mathbb{N}$. This function generates the sequence $\{0, 3, 7, 14, 22, 33, 45, 60, \dots\}$ with 2-adic valuations $\{\infty, 0, 0, 1, 1, 0, 0, 2, \dots\}$. Figure 20 shows the listplot of the 2-adic

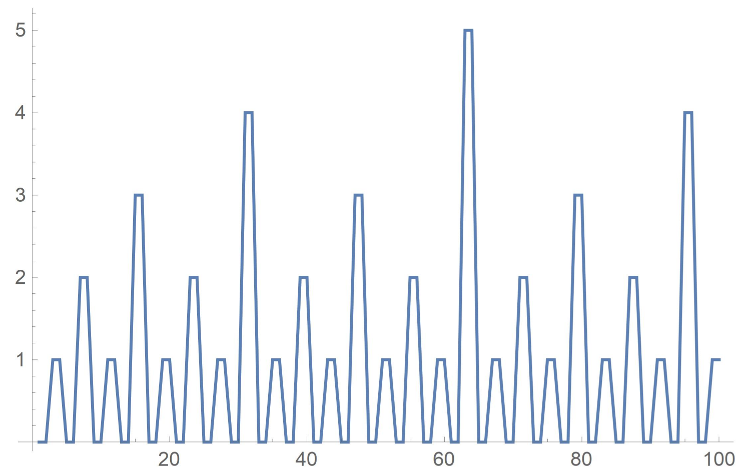



*Figure 20*: 2-adic listplot of A156859.

valuations of this sequence. From Figure 20, it is easy to see a distinct pattern of the 2-adic valuations of this sequence. Figure 21 shows the 2-adic tree of A156859 up to the fifth level.

Again, we use the lifting technique to show that the 2-adic tree of A156859 is infinite and non-terminating.

**Theorem V.1.1 (2-adic tree of A156859).** *Let $v$ be a non-terminating node of $f(n) = n^2 + n + \lfloor \frac{n+1}{2} \rfloor$ at the $k$-th level with $k \geq 1$. Then $v$ splits into two vertices at the $(k+1)$-th level. For even $n$, the one with $a_k = 1$ terminates with valuation $k-1$. The other with $a_k = 0$ has valuation of at least $k$. For odd $n$, the one with $a_k = 0$ terminates with valuation $k-1$. The other with $a_k = 1$ has valuation of at least $k$.*

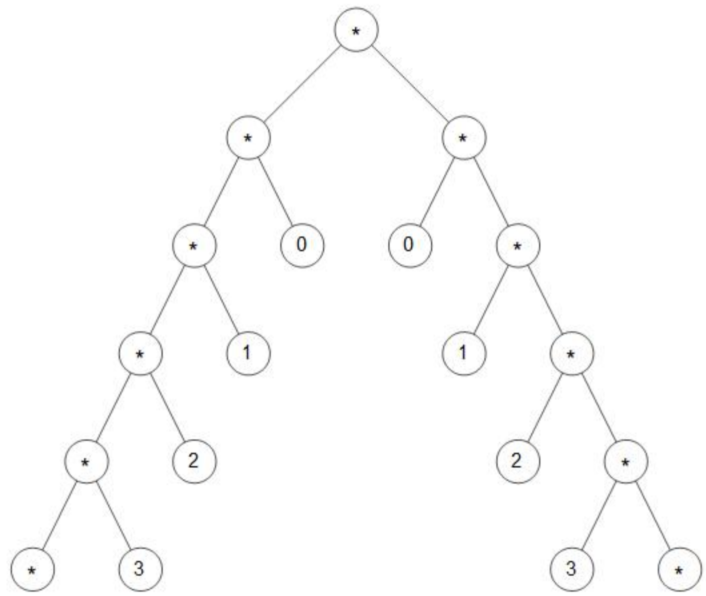



*Figure 21*: 2-adic tree of the sequence A156859.

*Proof.* To prove that the 2-adic valuation tree of $f(n) = n^2 + n + \lfloor \frac{n+1}{2} \rfloor$ is non-terminating, we prove that both the even and odd branch are non-terminating.

We first consider the even branch. Note that the numbers associated with the vertex $v$ at level $k$ have the form $2^k n + b_{k-1}$, and the numbers associated with vertices at level $k+1$ beneath $v$ have the form

$$N_k = 2^{k+1}n + 2^k a_k + b_{k-1}$$

where

$$b_{k-1} = 2^{k-1}a_{k-1} + \ldots + 2a_1 + a_0.$$

Since we are on the even branch, $a_{k-1}, \ldots, a_0$ are already determined to be 0 so that $b_{k-1} = 0$ and $N_k = 2^{k+1}n + 2^k a_k$. The induction hypothesis is that

$$f(2^k n + b_{k-1}) \equiv b_{k-1}^2 + b_{k-1} + \left\lfloor \frac{b_{k-1}+1}{2} \right\rfloor \equiv 0 \pmod{2^{k-1}},$$

so that $\nu_2\left(b_{k-1}^2 + b_{k-1} + \left\lfloor \frac{b_{k-1}+1}{2} \right\rfloor\right) \geq k-1$.

Consider the congruence

$$f(N_k) = N_k^2 + N_k + \left\lfloor \frac{N_k+1}{2} \right\rfloor \equiv 0 \pmod{2^k}$$

that is,

$$(2^{k+1}n+2^ka_k)^2+(2^{k+1}n+2^ka_k)+\left\lfloor\frac{(2^{k+1}n+2^ka_k)+1}{2}\right\rfloor \equiv 0 \pmod{2^k}. \tag{12}$$

We solve (12) for the unknown $a_k$.

$$\begin{aligned}
(2^{k+1}n+2^ka_k)^2+(2^{k+1}n+2^ka_k)+\left\lfloor\frac{(2^{k+1}n+2^ka_k)+1}{2}\right\rfloor &\equiv 0 \pmod{2^k}\\
2^{k+1}n+2^{2k+2}n^2+\left\lfloor\frac{1}{2}(1+2^{k+1}n+2^ka_k)\right\rfloor+2^ka_k+2^{2k+2}na_k+2^{2k}a_k^2 &\equiv 0 \pmod{2^k}\\
\left\lfloor\frac{1}{2}(1+2^{k+1}n+2^ka_k)\right\rfloor &\equiv 0 \pmod{2^k}\\
2^kn+2^{k-1}a_k+\left\lfloor\frac{1}{2}\right\rfloor &\equiv 0 \pmod{2^k}\\
2^kn+2^{k-1}a_k+0 &\equiv 0 \pmod{2^k}\\
a_k &\equiv 0 \pmod{2}
\end{aligned}$$

as the solution for $a_k$. Therefore the vertex descending from $v$ with $a_k = 1$ terminates with valuation $k-1$, and the other vertex with $a_k = 0$ has valuation at least $k$.

We now prove that the odd branch is also non-terminating. The numbers associated with the vertex $v$ at level $k$ have the form $2^kn+b_{k-1}$, and the numbers associated with vertices at level $k+1$ beneath $v$ have the form

$$N_k = 2^{k+1}n+2^ka_k+b_{k-1}$$

where

$$b_{k-1} = 2^{k-1}a_{k-1}+\ldots+2a_1+a_0.$$

Since we are on the odd branch, $a_{k-1},\ldots,a_0$ are already determined to be 1. Hence, $b_{k-1} = 2^k-1$ and $N_k = 2^{k+1}n+2^ka_k+2^k-1 = 2^{k+1}n+2^k(a_k+1)-1$. The induction hypothesis is that

$$f(2^kn+b_{k-1}) \equiv b_{k-1}^2+b_{k-1}+\left\lfloor\frac{b_{k-1}+1}{2}\right\rfloor \equiv 0 \pmod{2^{k-1}},$$

so that $\nu_2\left(b_{k-1}^2+b_{k-1}+\left\lfloor\frac{b_{k-1}+1}{2}\right\rfloor\right) \geq k-1$.

Consider the congruence

$$f(N_k) = N_k^2 + N_k + \left\lfloor \frac{N_k+1}{2} \right\rfloor \equiv 0 \pmod{2^k}$$

that is,

$$\begin{aligned} &(2^{k+1}n + 2^k(a_k+1) - 1)^2 + (2^{k+1}n + 2^k(a_k+1) - 1) \\ &+ \left\lfloor \frac{(2^{k+1}n + 2^k(a_k+1) - 1) + 1}{2} \right\rfloor \equiv 0 \pmod{2^k}. \end{aligned} \tag{13}$$

We solve (13) for the unknown $a_k$.

$$\begin{aligned} 2^k + 2^{2k} - 2^{k+1} + 2^{k+1}n - 2^{k+2}n + 2^{2k+2}n + 2^{2k+2}n + 2^{2k+2}n^2 + \left\lfloor \frac{1}{2}(2^{k+1}n + 2^k(a_k+1)) \right\rfloor& \\ +2^k a_k - 2^{k+1}a_k + 2^{2k+1}a_k + 2^{2k+2}na_k + 2^{2k}a_k^2 \equiv 0 \pmod{2^k}& \\ \left\lfloor \frac{1}{2}(2^{k+1}n + 2^k(a_k+1)) \right\rfloor \equiv 0 \pmod{2^k}& \\ 2^k n + 2^{k-1}(a_k+1) \equiv 0 \pmod{2^k}& \\ a_k + 1 \equiv 0 \pmod{2}& \end{aligned}$$

as the solution for $a_k$. Therefore the vertex descending from $v$ with $a_k = 0$ terminates with valuation $k-1$, and the other vertex with $a_k = 1$ has valuation at least $k$. □

**Proof of A325958 Tree**

The OEIS sequence A325958, a sequence of the sum of corners of $(2n+1)\times(2n+1)$ square (Figure 3), is defined by the quadratic polynomial $16n^2+4n+4$. This polynomial generates the sequence $\{4, 24, 76, 160, 276, 424, 604, 816, \ldots\}$ which has 2-adic valuations $\{3, 2, 5, 2, 3, 2, 4, 2, \ldots\}$. Figure 22 shows the 2-adic tree of this sequence.

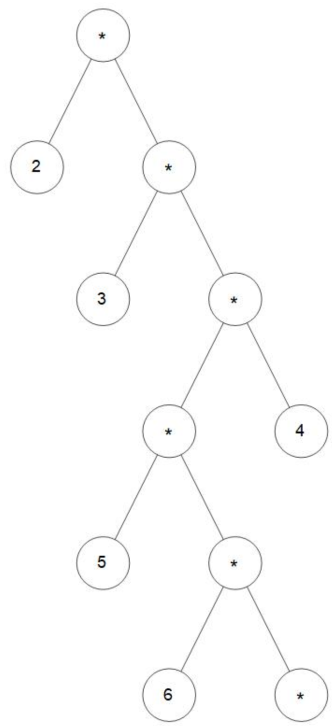



*Figure 22*: 2-adic tree of the sequence A325958.

**Theorem V.2.1 (2-adic tree of A325958.).** *Let $v$ be a non-terminating node of $f(n) = 16n^2+4n+4$ at the k-th level with $k \geq 1$. Then $v$ splits into two vertices at the $(k+1)$-th level. One terminates with valuation $k+2$. The other has valuation of at least $k+3$.*

*Proof.* Note that the numbers associated with the vertex $v$ at level $k$ have the form $2^k n + b_{k-1}$, and the numbers associated with vertices at level $k+1$ beneath $v$ have the form

$$N_k = 2^{k+1}n + 2^k a_k + b_{k-1}$$

where

$$b_{k-1} = 2^{k-1}a_{k-1} + \ldots + 2a_1 + a_0$$

has been determined. We have $f(2^k n+b_{k-1}) = 16[2^k n+b_{k-1}]^2+4[2^k n+b_{k-1}]+4$ which evaluates to

$$2^{2k+4}+2^{k+5}b_{k-1}+2^{k+2}+16b_{k-1}^2+4b_{k-1}+4.$$

The induction hypothesis is that

$$2^{2k+4}+2^{k+5}b_{k-1}+2^{k+2}+16b_{k-1}^2+4b_{k-1}+4 \equiv 0 \pmod{2^{k+2}}$$

which simplifies to

$$16b_{k-1}^2+4b_{k-1}+4 \equiv 0 \pmod{2^{k+2}} \implies \nu_2(16b_{k-1}^2+4b_{k-1}+4) \geq k+2.$$

Now consider the congruence

$$16N_k^2+4N_k+4 \equiv 0 \pmod{2^{k+3}}$$

which expands and simplifies to

$$2^{k+2}a_k+16b_{k-1}^2+4b_{k-1}+4 \equiv 0 \pmod{2^{k+3}}. \tag{14}$$

We solve (14) for the unknown $a_k$. Note that (14) can be rewritten as

$$2^{k+2}a_k \equiv -(16b_{k-1}^2+4b_{k-1}+4) \pmod{2^{k+3}}. \tag{15}$$

By the induction hypothesis, $16b_{k-1}^2+4b_{k-1}+4 = 2^{k+2}m$. Then (15) becomes

$$2^{k+2}a_k \equiv -(2^{k+2}m) \pmod{2^{k+3}}$$

which reduces to

$$a_k \equiv -m \pmod 2$$

as the solution to (14). Therefore, the vertex descending from $v$ with $a_k \not\equiv -m \pmod 2$ terminates with valuation $k$, and the other vertex has valuation at least $k+1$. $\square$

**Proof of A001107 Tree**

The OEIS sequence A001107 is a sequence of decagonal numbers on a square spiral with a center of 0 (Figure 4). It is defined by the quadratic polynomial $4n^2 - 3n$ which generates the sequence $\{1, 10, 27, 52, 85, 126, 175, 232, \ldots\}$. This sequence has 2-adic valuations $\{0, 1, 0, 2, 0, 1, 0, 3, \ldots\}$. Figure 23 shows the 2-adic tree of this sequence.

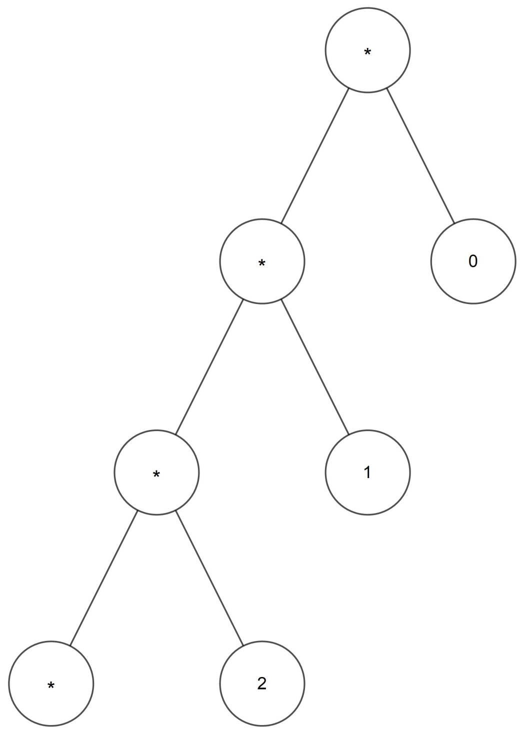



*Figure 23*: 2-adic tree of the sequence A001107.

For this sequence, we present two proofs using different techniques to show that its 2-adic tree is infinite. For the first proof, we rely on the lifting technique, and we use a recurrence technique for the second proof.

**Theorem V.3.1 (2-adic tree of A001107).** *Let $v$ be a non-terminating node of $f(n) = 4n^2 - 3n$ at the k-th level with $k \geq 1$. Then $\nu_2(4n^2 - 3n) = 0$ if $n$ is odd. For even n, v splits into two vertices at the $(k+1)$-th level. The one with $a_k = 1$ terminates with valuation k. The other with $a_k = 0$ has valuation of at least $k+1$.*

*Proof.* We first show that the odd branch terminates with valuation 0. Note that

$$f(2n+1) = 4(2n+1)^2 - 3(2n+1) = 2(8n^2+5n)+1$$

which is odd. Thus, $\nu_2(f(2n+1)) = 0$, and the odd branch terminates with valuation 0.

For the proof of the even branch, note that the numbers associated with the vertex $v$ have the form $2^k n + b_{k-1}$, and the numbers associated with the next level's vertices beneath $v$ have the form

$$N_k = 2^{k+1}n + 2^k a_k + b_{k-1}$$

where

$$b_{k-1} = 2^{k-1}a_{k-1} + \ldots + 2a_1$$

has been determined. But $a_{k-1}, \ldots, a_0 = 0$ since we are on the left branch. Then $b_{k-1} = 0$, and thus $v = 2^k n$ and $N_k = 2^{k+1}n + 2^k a_k$. The induction hypothesis is that

$$f(b_{k-1}) = 4b_{k-1}^2 - 3b_{k-1} \equiv 0 \pmod{2^k}$$

so that $\nu_2(4b_{k-1}^2 - 3b_{k-1}) \geq k$. Consider the congruence

$$f(N_k) = 4N_k^2 - 3N_k \equiv 0 \pmod{2^{k+1}}$$

that is,

$$4[2^{k+1}n + 2^k a_k]^2 - 3[2^{k+1}n + 2^k a_k] \equiv 0 \pmod{2^{k+1}}. \tag{16}$$

for the unknown $a_k$. Then (16) reduces to

$$-3a_k \equiv 0 \pmod 2$$

as the solution for $a_k$. Notice here that if $a_k = 1$, then $-3a_k = -3 \not\equiv 0 \pmod 2$. So the vertex descending from $v$ with $a_k = 1$ terminates with valuation $k$. On the other hand, if $a_k = 0$, then $-3a_k = 0 \equiv 0 \pmod 2$. Thus, the vertex descending from $v$ with $a_k = 0$ has valuation of at least $k+1$. $\square$

The second proof requires a more thorough explanation. We first must find a recurrence for the sequence A001107. To accomplish this, we rely on Mathematica and its packages.

We use the `HolonomicFunctions.m` package [6] to find a recurrence relation for the OEIS sequence A001107. The recurrence turns out to be the ratio of successive terms of the sequence. We defined it as

$$a_n = C(n) \cdot a_{n-1} \tag{17}$$

where

$$C(n) = \frac{n(4n-3)}{(n-1)(4n-7)}.$$

Suppose we are given $a_0$ and want to relate $a_3$ to $a_0$, then

$$a_3 = C(3) \cdot a_2 = C(3) \cdot C(2) \cdot a_1 = C(3) \cdot C(2) \cdot C(1) \cdot a_0. \tag{18}$$

Let $P(n, j)$ be a function where $n$ is the initial subscript and $j$ determines the $j+1$ steps needed to traverse back to our desired destination. We have

$$P(n, j) = \prod_{i=0}^{j} C(n-i)$$

where $j = n-1$ if $n \geq 1$; otherwise, $j = 0$. Then (18) becomes

$$a_3 = P(3, 2) \cdot a_0.$$

The function $P(n, j)$ provides a way for us to relate any two elements of the sequence; however, it is quite cumbersome to work with if we want to compare two elements that are not close. So we present a formula for that product in the following lemma.

**Lemma V.3.2 (Solution of A001107 recurrence relation).** *Let*

$$F(n, j) = \frac{n(4n-3)}{(n-j-1)(4n-4j-7)} \tag{19}$$

*for* $n, j \in \mathbb{N}$*, where* $j = n-1$ *if* $n \geq 1$*; otherwise,* $j = 0$*. Then* $P(n, j) = F(n, j)$ *for all* $j \geq 0$*.*

*Proof.* We prove by induction. For $j = 0$, we have

$$P(n,0) = C(n-0) = C(n) = \frac{n(4n-3)}{(n-1)(4n-7)}$$

and

$$F(n,0) = \frac{n(4n-3)}{(n-0-1)(4n-4\cdot 0-7)} = \frac{n(4n-3)}{(n-1)(4n-7)}$$

which implies $P(n,0) = F(n,0)$. Hence, assume $P(n,j) = F(n,j)$ for $j \geq 0$. By the induction hypothesis, we have

$$\begin{aligned} P(n,j+1) &= \prod_{i=0}^{j+1} C(n-i) = \left(\prod_{i=0}^{j} C(n-i)\right) \cdot C(n-(j+1)) \\ &= \frac{n(4n-3)}{(n-j-1)(4n-4j-7)} \cdot \frac{(n-j-1)(4(n-j-1)-3)}{(n-j-1-1)(4(n-j-1)-7)} \\ &= \frac{n(4n-3)}{(n-j-1)(4n-4j-7)} \cdot \frac{(n-j-1)(4n-4j-7)}{(n-j-2)(4n-4j-11)} \\ &= \frac{n(4n-3)}{(n-j-2)(4n-4j-11)}. \end{aligned}$$

We also have

$$F(n,j+1) = \frac{n(4n-3)}{(n-(j+1)-1)(4n-4(j+1)-7)} = \frac{n(4n-3)}{(n-j-2)(4n-4j-11)}.$$

It follows that $P(n,j+1) = F(n,j+1)$. By induction, the claim $P(n,j) = F(n,j)$ is true for all $j \geq 0$. $\square$

With Lemma 4.3.2, we now have all the necessary tools to show that the 2-adic tree of the sequence A00107 has the properties from Theorem 4.4.1.

*Proof.* We first show that the odd branch terminates with valuation 0. Note that

$$f(2n+1) = 4(2n+1)^2 - 3(2n+1) = 2(8n^2+5n)+1$$

which is odd. Thus, $\nu_2(f(2n+1)) = 0$, and the odd branch terminates with valuation 0.

For the even branch, we assume that $\nu_2(n) \geq k$ and try to relate the vertex with $a_k = 0$ back to $n$. That is, we want to relate $f(2n)$ and $f(n)$. We accomplish this by using (17) and (19). To move from $2n$ back to $n$, we must take $n$ steps backward. Using Mathematica, $F(2n,n-1)$ simplifies to

$$F(2n,n-1) = \frac{2(8n-2)}{4n-3}. \tag{20}$$

By (17), we have

$$f(2n) = \frac{2(8n-2)}{4n-3} \cdot f(n)$$

which implies that $\nu_2(f(2n)) \geq k+1$ since (20) contains a multiple of 2. Hence, this vertex is non-terminating and has valuation at least $k+1$.

For the vertex with $a_k = 1$ where $k$ is the level of vertex we step back from, we relate $f(2^{k-1}(2n+1))$ and $f(2^{k-1}n)$ by taking $2^{k-1}(n+1)$ steps backward. Using Mathematica, we have

$$F[2^{k-1}(2n+1), 2^{k-1}(n+1)-1] = \frac{(2n+1)(2^{k+2}n+2^{k+1}-3)}{n(2^{k+1}n-3)}. \tag{21}$$

By (17), it follows that

$$f(2^{k-1}(2n+1)) = \frac{(2n+1)(2^{k+2}n+2^{k+1}-3)}{n(2^{k+1}n-3)} \cdot f(2^{k-1}n)$$

Note that (21) does not contains any multiple of 2. Therefore, $\nu_2[f(2^{k-1}(2n+1))] = k$ and so this vertex terminates with valuation $k$. □

### Proof of the Class of Sequences of the Form $4n^2 + (2t+1)n + c$

The non-diagonal sequence $4n^2 - 3n$ discussed in the previous section belongs to the class of square spiral sequences of the form $4n^2 + (2t+1)n + c$. The 2-adic trees for some of the sequences generated by the class $4n^2 + (2t+1)n + c$ are shown in Figure 24. Notice that these trees are infinite and single branching on either the left or right branch and has a starting valuation of 0 that increases by a constant 1 on each level. We prove that these properties hold for all 2-adic trees of the class of sequences of the form $4n^2 + (2t+1)n + c$.

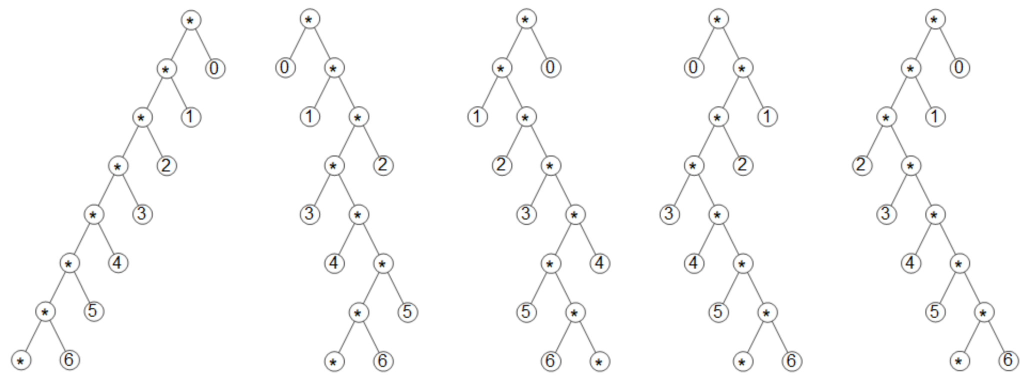


*Figure 24*: 2-adic trees of $4n^2 + (2t+1)n + c$.

**Theorem V.4.1 (2-adic tree of $\mathbf{4n^2 + (2t+1)n + c}$).** *Let $v$ be a non-terminating node of $f(n) = 4n^2 + (2t+1)n + c$ at the $k$-th level with $k \geq 1$. Then $v$ splits into two vertices at the $(k+1)$-th level. One terminates with valuation $k$. The other has valuation of at least $k+1$.*

*Proof.* The numbers associated with the vertex $v$ at level $k$ have the form $2^k n + b_{k-1}$, and the numbers associated with vertices at level $k+1$ beneath $v$ have the form

$$N_k = 2^{k+1} n + 2^k a_k + b_{k-1}$$

where

$$b_{k-1} = 2^{k-1} a_{k-1} + \ldots + 2a_1 + a_0$$

has been determined. The induction hypothesis is that

$$f(b_{k-1}) = 4b_{k-1}^2 + (2t+1)b_{k-1} + c \equiv 0 \pmod{2^k}$$

so that $v_2(4b_{k-1}^2+(2t+1)b_{k-1}+4) \geq k$. Consider the congruence

$$f(N_k) = 4N_k^2+(2t+1)N_k+c \equiv 0 \pmod{2^{k+1}}$$

which expands and simplifies to

$$2^k a_k + 4b_{k-1}^2 + (2t+1)b_{k-1} + c \equiv 0 \pmod{2^{k+1}}. \tag{22}$$

We solve (22) for the unknown $a_k$. Note that (22) can be rewritten as

$$2^k a_k \equiv -[4b_{k-1}^2 + (2t+1)b_{k-1} + c] \pmod{2^{k+1}}.$$

By the induction hypothesis, $4b_{k-1}^2+(2t+1)b_{k-1}+c = 2^k m$ for some $m \in \mathbb{Z}$. Then (22) becomes

$$2^k a_k \equiv -2^k m \pmod{2^{k+1}}$$

which reduces to

$$a_k \equiv -m \pmod{2}$$

as the solution to (22). Therefore, the vertex descending from $v$ with $a_k \not\equiv -m \pmod{2}$ terminates with valuation $k$, and the other vertex has valuation of at least $k+1$. □

# CHAPTER VI: OBSERVATIONS OF THE 2-ADIC TREES OF SOME DIAGONAL SEQUENCES

In this chapter, we discuss the characteristics of the valuation trees of some classes of diagonal sequences. These characteristics are constant, finite, single infinite, and double infinite valuation. Due to the seemingly infinite numbers of different tree patterns, we only provide a partial classification of Ulam diagonal sequences of the form $4n^2+bn+c$.

## Constant Valuation

In this section, we prove the explicit formulas of the 2-adic valuations of some classes of diagonal sequences with constant valuation using simple factorization.

**Lemma VI.1.1.** *For all $n \in \mathbb{N}$, if $b \equiv 0 \pmod 2$ and $c \not\equiv 0 \pmod 2$, then*

$$\nu_2(4n^2+bn+c) = 0.$$

*Proof.* We write $b = 2k$ and $c = 2l+1$. Then,

$$4n^2+bn+c = 4n^2+(2k)n+(2l+1) = 2(2n^2+kn+l)+1.$$

Note that for all $n \in \mathbb{N}$, $2(2n^2+kn+l)+1$ is odd. Hence, $\nu_2(4n^2+bn+c) = 0$. $\square$

**Lemma VI.1.2.** *For all $n \in \mathbb{N}$, if $b \equiv 4 \pmod 8$ and $c \equiv 4 \pmod 8$, then*

$$\nu_2(4n^2+bn+c) = 2.$$

*Proof.* Write $b = 8k+4$ and $c = 8l+4$. Then,

$$4n^2+bn+c = 4n^2+(8k+4)n+(8l+4) = 2^2(n^2+2kn+n+2l+1).$$

Note that for all $n \in \mathbb{N}$, $2n^2+4kn+2l+1$ is odd. Therefore, $\nu_2(4n^2+bn+c) = 2$. $\square$

### Finite and Non-Constant Valuation

Similar to the case where the valuation is finite, if the valuation is non-constant but finite, an explicit formula can be achieved by simple factorization. We provide the explicit formula of some classes of diagonal sequences with non-constant but finite valuation in this section.

**Lemma VI.2.1.** *For all $n \in \mathbb{N}$, if $b \equiv 0 \pmod{32}$ and $c \equiv 12 \pmod{32}$, then*

$$\nu_p(4n^2 + bn + c) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod{2} \\ 4 & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

*Proof.* Write $b = 32k$ and $c = 32l + 12$. If $n$ is even, say $n = 2m$, then

$$4n^2 + bn + c = 2^2(4m^2 + 16km + 8l + 3)$$

and thus $\nu_2(4n^2 + bn + c) = 2$. If $n$ is odd, say $n = 2m + 1$, then

$$4n^2 + bn + c = 2^4(m^2 + 4km + m + 2k + 2l + 1)$$

and so $\nu_2(4n^2 + bn + c) = 4$. □

**Lemma VI.2.2.** *For all $n \in \mathbb{N}$, if $b \equiv 8 \pmod{16}$ and $c \equiv 16 \pmod{32}$, then*

$$\nu_p(4n^2 + bn + c) = \begin{cases} 4 & \text{if } n \equiv 0 \pmod{2} \\ 2 & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

*Proof.* Write $b = 16k + 8$ and $c = 32l + 16$. If $n$ is even, say $n = 2m$, then

$$4n^2 + bn + c = 2^4(m^2 + 2km + m + 2l + 1)$$

and thus $\nu_2(4n^2 + bn + c) = 4$. If $n$ is odd, say $n = 2m + 1$, then

$$4n^2 + bn + c = 2^2(4m^2 + 8km + 8m + 8l + 4k + 7)$$

and so $\nu_2(4n^2 + bn + c) = 2$. □

**Lemma VI.2.3.** *For all $n \in \mathbb{N}$, if $b \equiv 0 \pmod{16}$ and $c \equiv 4 \pmod{16}$, then*

$$\nu_p(4n^2+bn+c) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod{2} \\ 3 & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

*Proof.* Write $b = 16k$ and $c = 16l+4$. If $n$ is even, say $n = 2m$, then

$$4n^2+bn+c = 2^2(4m^2+8km+4l+1)$$

and thus $\nu_2(4n^2+bn+c) = 2$. If $n$ is odd, say $n = 2m+1$, then

$$4n^2+bn+c = 2^3(2m^2+4km+2m+2l+2k+1)$$

and so $\nu_2(4n^2+bn+c) = 3$. □

**Lemma VI.2.4.** *For all $n \in \mathbb{N}$, if $b \equiv 0 \pmod{24}$ and $c \equiv 8 \pmod{16}$, then*

$$\nu_p(4n^2+bn+c) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod{2} \\ 2 & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

*Proof.* Write $b = 24k$ and $c = 16l+8$. If $n$ is even, say $n = 2m$, then

$$4n^2+bn+c = 2^3(2m^2+6km+2l+1)$$

and thus $\nu_2(4n^2+bn+c) = 3$. If $n$ is odd, say $n = 2m+1$, then

$$4n^2+bn+c = 2^2(4m^2+12km+4m+4l+6k+3)$$

and so $\nu_2(4n^2+bn+c) = 2$. □

**Lemma VI.2.5.** *For all $n \in \mathbb{N}$, if $b \equiv 0 \pmod{32}$ and $c \equiv 16 \pmod{64}$, then*

$$\nu_p(4n^2+bn+c) = \begin{cases} 2 & \text{if } n \equiv 1 \pmod{2} \\ 4 & \text{if } n \equiv 0 \pmod{4} \\ 5 & \text{if } n \equiv 2 \pmod{4} \end{cases}$$

*Proof.* Write $b = 32k$ and $c = 64l + 16$. If $n$ is odd, say $n = 2m+1$, then

$$4n^2 + bn + c = 2^2(4m^2 + 16km + 4m + 16l + 8k + 5)$$

and thus $\nu_2(4n^2 + bn + c) = 2$. If $n = 4m$, then

$$4n^2 + bn + c = 2^4(4m^2 + 4l + 8km + 1)$$

and thus $\nu_2(4n^2 + bn + c) = 4$. If $n = 4m+2$, then

$$4n^2 + bn + c = 2^5(2m^2 + 4km + 2m + 2l + 2k + 1)$$

and thus $\nu_2(4n^2 + bn + c) = 5$. $\square$

The list of different classes of diagonal sequences discussed in the previous two sections is not exhaustive. In fact, there appears to be trees with finite valuations at every depth.

**Conjecture VI.2.1**: *For any depth $d \in \mathbb{N}$, there exists a finite tree of depth $d$.*

### Infinite and Single Branching

In this section, we provide a proof for the class of diagonal sequences of the form $4n^2 + (4t+2)n + 2s$. This class generates infinite and single branching 2-adic valuation trees. Trees that belong to this class appear much more frequently than other single infinite branching trees; thus, we can analyze the characteristics of trees belonging to these two classes in much more detail. There are other classes of single infinite branching trees such as the class of diagonal sequences of the form $4n^2 + 4n + 4s$. However, their valuation trees are spaced very far apart, making the analysis of their characteristics very difficult.

Figure 25 shows some 2-adic trees produced by $4n^2 + (4t+2)n + 2s$. Observe that these trees are infinite and single branching on either the left or right branch. In addition, the 2-adic valuations increase by a constant of 1 on each level. One important thing to note here is that valuations are at least one; our induction hypothesis is based on this observation. We now prove that these properties hold for all 2-adic trees of this class of diagonal sequences.

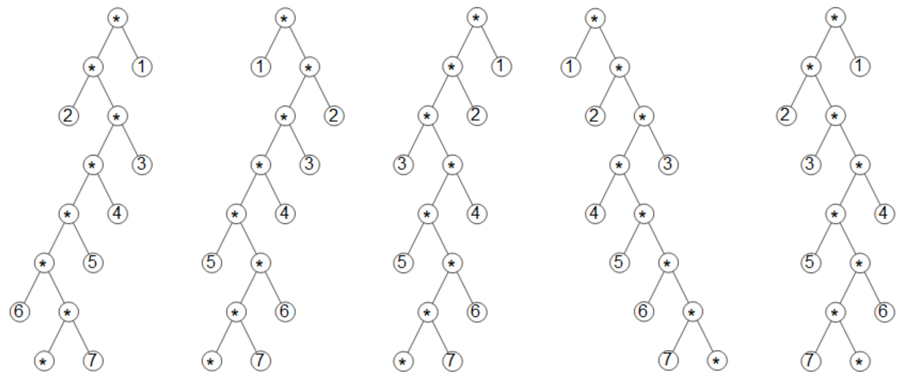


*Figure 25*: 2-adic trees of $4n^2+(4t+2)n+2s$.

**Theorem VI.3.1 (2-adic tree of $\mathbf{4n^2+(4t+2)n+2s}$).** *Let $v$ be a non-terminating node of $f(n)=4n^2+(4t+2)n+2s)$ at the k-th level with $k\geq 1$. Then $v$ splits into two vertices at the $(k+1)$-th level. One terminates with valuation $k+1$. The other has valuation of at least $k+2$.*

*Proof.* The numbers associated with the vertex $v$ at level $k$ have the form $2^kn+b_{k-1}$, and the numbers associated with vertices at level $k+1$ beneath $v$ have the form

$$N_k=2^{k+1}n+2^ka_k+b_{k-1}$$

where

$$b_{k-1}=2^{k-1}a_{k-1}+\ldots+2a_1+a_0$$

has been determined. The induction hypothesis is that

$$f(b_{k-1})=4b_{k-1}^2+(4t+2)b_{k-1}+4s\equiv 0 \pmod{2^{k+1}}$$

so that $v_2(4b_{k-1}^2+(4t+2)b_{k-1}+4)\geq k+1$. Consider the congruence

$$f(N_k)=4N_k^2+(4t+2)N_k+4s\equiv 0 \pmod{2^{k+2}}$$

which expands and simplifies to

$$2^{k+1}a_k+4b_{k+1}^2+(4t+2)b_{k-1}+2s\equiv 0 \pmod{2^{k+2}}. \tag{23}$$

We solve (23) for the unknown $a_k$. Note that (23) can be rewritten as

$$2^{k+1}a_k \equiv -[4b_{k+1}^2 + (4t+2)b_{k-1} + 2s] \pmod{2^{k+2}}.$$

By the induction hypothesis, $4b_{k+1}^2 + (4t+2)b_{k-1} + 2s = 2^{k+1}m$ for some $m \in \mathbb{Z}$. Then (23) becomes

$$2^{k+1}a_k \equiv -2^{k+1}m \pmod{2^{k+2}}$$

which reduces to

$$a_k \equiv -m \pmod{2}$$

as the solution to (23). Therefore, the vertex descending from $v$ with $a_k \not\equiv -m \pmod{2}$ terminates with valuation $k+1$, and the other vertex has valuation of at least $k+2$. □

**Infinite and Double Branching**

Let $f(n,t) = 4n^2 + (32t+16)n$ for non-negative integer $t$. This particular class of Ulam diagonals contains some interesting properties that are worth discussing. When $t = 0$, we have $f(n,0) = 4n^2 + 16n$. Figure 26 shows the partial 2-adic valuation tree of this sequence. This 2-adic tree appears to be non-terminating (or infinite) on the left branch and terminates on the right branch. Furthermore, the left-branch splits into two non-terminating branches at the vertex associated with numbers of the form $4n$.

When $t = 1$, then $f(n,1) = 4n^2 + 48$. Figure 27 shows the the 2-adic valuation tree of this sequence. Notice the similarities between Figure 26 and 27. The 2-adic tree of $f(n,1)$ is left-branching and terminates on the right branch. The left branch then splits at the vertex $4n$ into two non-terminating branches. In addition, each vertex on the leftmost branch of both trees have the same 2-adic valuation. The only difference between Figure 26 and 27 is the behavior of the non-terminating right branch that was split off at vertex $4n$. In Figure 26, this specific branch always branches to the right side and terminates on the left side. On the other hand, the branch at the same position in Figure 27 terminates on the right side at first and branches left, but afterward it starts to branch right for the next few vertices. In fact, the observations presented above holds true for all 2-adic trees of the form $f(n,t) = 4n^2 + (32t+16)n$.

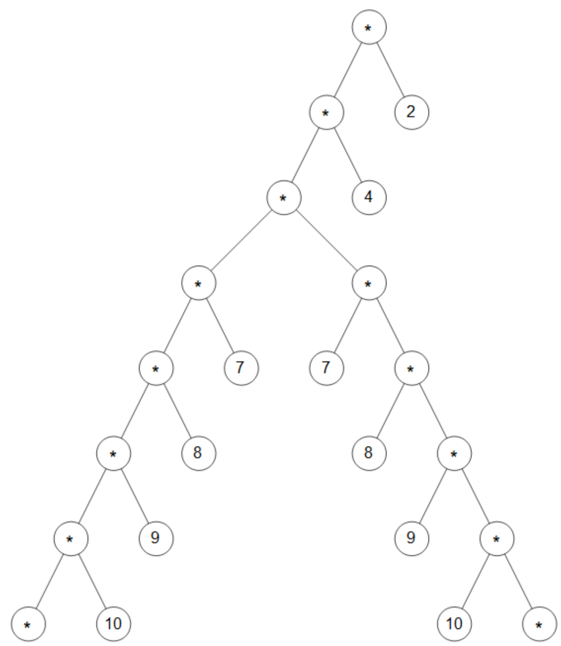



*Figure 26*: 2-adic tree of $4n^2 + 16n$.

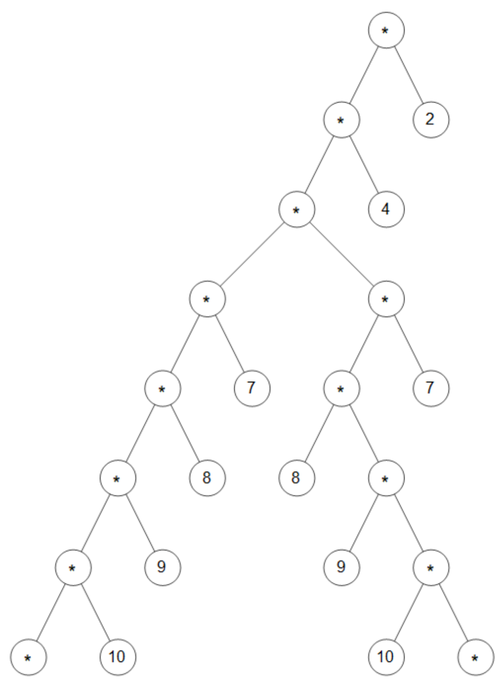



*Figure 27*: 2-adic tree of $4n^2 + 48n$.

**Theorem VI.4.1 (2-adic tree of $\mathbf{4n^2 + (32t+16)n}$).** *Let $v_k$ be the non-terminating node at level k. The 2-adic trees of the diagonal class $4n^2 + (32t+16)n$ has the following properties:*

*(i) Let $k = 0$ or 1. Then the vertex beneath $v_k$ with $a_k = 1$ terminates with valuation*

$2k+2$. *The other vertex with* $a_k = 0$ *has valuation of at least* $2k+4$.

*(ii) Let* $k = 2$. *Then both vertices descending from* $v_2$ *are non-terminating and have valuation of at least* $7$.

*(iii) Let* $k > 2$ *and* $v_k$ *be a non-terminating vertex descending from* $v_2$ *with* $a_2, a_1, a_0 = 0$. *Then* $v_k$ *splits into two vertices at the* $(k+1)$*-th level. The one with* $a_k = 1$ *terminates with valuation* $k+4$. *The other with* $a_k = 0$ *has valuation of at least* $k+4$.

*(iv) Let* $k > 2$ *and* $v_k$ *be a non-terminating vertex beneath* $v_2$ *with* $a_2 = 1$ *and* $a_1, a_0 = 0$. *Then* $v_k$ *splits into two vertices at the* $(k+1)$*-th level. One terminates with valuation* $k+4$. *The other has valuation of at least* $k+4$.

*Proof.* We prove that all 2-adic trees of the diagonal class $f(n,t) = 4n^2 + (32t+16)n$ satisfy properties (i), (ii), (iii) , and (iv).

(i) Suppose $k = 0$. The numbers associated with $v_0$ have the form $n$, and the numbers associated with the vertex descending from $v_0$ with $a_0 = 1$ have the form $2n+1$. Then

$$f(2n+1,t) = 2^2(2n+1)(8t+2n+5) \implies \nu_2(f(2n+1,t)) = 2 = 2(0)+2.$$

On the other hand, the numbers associated with the vertex descending from $v_0$ with $a_0 = 0$ have the form $2n$. Then

$$f(2n,t) = 2^4 n(n+4t+2) \implies \nu_2(f(2n,t)) \geq 4 = 2(0)+4$$

Now consider $k = 1$. The numbers associated with $v_1$ have the form $2n$, and the numbers associated with the vertex descending from $v_1$ with $a_1 = 1$ have the form $4n+2$. Then

$$f(4n+2,t) = 2^4(2n+1)(2n+4t+3) \implies \nu_2(f(4n+2)) = 4 = 2(1)+2.$$

Meanwhile, the numbers associated with the vertex descending from $v_1$ with $a_1 = 0$ have the form $4n$. Then

$$f(4n,t) = 2^6 n(n+2t+1) \implies \nu_2(f(4n,t)) \geq 6 = 2(1)+4.$$

(ii) Assume $k = 2$. The numbers associated with $v_2$ have the form $4n$ since $a_0, a_1$ are already determined to be 0. Then the vertex descending from $v_2$ with $a_2 = 0$ is associated with the numbers the form $8n$. Then

$$f(8n,t) = 2^7 n(2n+2t+1) \implies \nu_2(f(8n,t)) \geq 7$$

since $n(2n+2t+1)$ is even for all $n \in \mathbb{Z}$. The other vertex with $a_2 = 1$ is associated with the numbers of the form $8n+4$. We see that

$$f(8n+4,t) = 2^7(2n+1)(n+t+1) \implies \nu_2(f(8n+4,t)) \geq 7$$

since $(2n+1)(n+t+1)$ could be either even or odd depending on the choices of $n$ and $t$. Thus, both vertices descending from $v_2$ are non-terminating.

(iii) Let $k > 2$ and $v_k$ be a non-terminating vertex descending from $v_2$ with $a_2, a_1, a_0 = 0$. Then $v_k$ has numbers of the form $2^k n + b_{k-1}$, and the numbers associated with vertices at level $k+1$ beneath $v$ have the form

$$N_k = 2^{k+1}n + 2^k a_k + b_{k-1},$$

where

$$b_{k-1} = 2^{k-1}a_{k-1} + \ldots + 2a_1 + a_0.$$

Since we are on the left branch, $a_{k-1}, \ldots a_0$ are already determined to be 0. Thus, we have $b_{k-1} = 0$ and $N_k = 2^{k+1}n + 2^k a_k$. The induction hypothesis states that

$$f(b_{k-1},t) = 4b_{k-1}^2 + (32t+16)b_{k-1} \equiv 0 \pmod{2^{k+4}}$$

so that $\nu_2(f(b_{k-1},t)) \geq k+4$. Consider the congruence

$$f(N_k,t) = 4N_k^2 + (32t+16)N_k \equiv 0 \pmod{2^{k+5}}$$

that is,

$$4(2^{k+1}n + 2^k a_k)^2 + (32t+16)(2^{k+1}n + 2^k a_k) \equiv 0 \pmod{2^{k+5}} \tag{24}$$

We solve (24) for the unknown $a_k$.

$$2^{k+5}n+2^{2k+4}n^2+2^{k+6}nt+2^{k+4}a_k+2^{2k+4}na_k+2^{k+5}ta_k+2^{2k+2}a_k^2 \equiv 0 \pmod{2^{k+5}}$$
$$2^{k+4}a_k \equiv 0 \pmod{2^{k+5}}$$
$$a_k \equiv 0 \pmod{2}$$

as the solution for $a_k$. Therefore, the vertex $a_k = 1$ terminates with valuation $k+5$, and the other vertex with $a_k = 0$ has valuation of at least $k+5$.

(iv) Let $k > 2$ and $v_k$ be a non-terminating vertex descending from $v_2$ with $a_2 = 1$, and $a_1, a_0 = 0$. Then $v_k$ has the form $2^k n + b_{k-1}$, and the numbers associated with vertices at level $k+1$ beneath $v$ have the form

$$N_k = 2^{k+1}n + 2^k a_k + b_{k-1}$$

where

$$b_{k-1} = 2^{k-1}a_{k-1} + \ldots + 2a_1 + a_0.$$

Since $a_0$, $a_1$, and $a_2$ are already determined to be $a_0, a_1 = 0$, and $a_2 = 1$, then

$$b_{k-1} = 2^{k-1}a_{k-1} + \ldots + 2^3 a_3 + 4. \tag{25}$$

The induction hypothesis states that

$$f(b_{k-1},t) = 4b_{k-1}^2 + (32t+16)b_{k-1} \equiv 0 \pmod{2^{k+4}}$$

so that $\nu_2(f(b_{k-1},t)) \geq k+4$. Consider the congruence

$$f(N_k,t) = 4N_k^2 + (32t+16)N_k \equiv 0 \pmod{2^{k+5}}$$

which expands and simplifies to

$$2^{k+4}a_k + 2^{k+4}nb_{k-1} + 2^{k+3}a_k b_{k-1} + 4b_{k-1}^2 + 32tb_{k-1} + 16b_{k-1} \equiv 0 \pmod{2^{k+5}}. \tag{26}$$

We solve (26) for the unknown $a_k$. Note that (26) can be rewritten as

$$2^{k+4}a_k + 2^{k+4}nb_{k-1} + 2^{k+3}a_k b_{k-1} \equiv -(4b_{k-1}^2 + 32tb_{k-1} + 16b_{k-1}) \pmod{2^{k+5}}. \tag{27}$$

By the induction hypothesis, $4b_{k-1}^2 + 32tb_{k-1} + 16b_{k-1} = 2^{k+4}m$ for some $m \in \mathbb{Z}$. From (25), we have $2^{k+4}nb_{k-1} = 2^{k+6}np$ and $2^{k+3}a_k b_{k-1} = 2^{k+5}q$ for some $p, q \in \mathbb{Z}$. Then (27) becomes

$$2^{k+4}a_k + 2^{k+6}np + 2^{k+5}a_k q \equiv -(2^{k+4}m) \pmod{2^{k+5}}$$

which reduces to

$$a_k \equiv -m \pmod{2}$$

as the solution to (26). It follows that the vertex descending from $v$ with $a_k \not\equiv -m \pmod{2}$ terminates with valuation $k+5$, and the other vertex has valuation of at least $k+5$.

□

The double-infinite branching class discussed above is only a small subset of the list of double-infinite branching trees of Ulam diagonal sequences. However, the vertices in which the double-branching pattern occurs in these trees are inconsistent, making the analysis of their valuation patterns difficult.

# CHAPTER VII: CONCLUSION

In this paper, we showed that the 2-adic valuations of several sequences associated with Ulam square spirals could be expressed as binary trees. For sequences that produced finitely many valuations, explicit closed forms were easily achieved using simple factorization. Although we were unable to find explicit closed forms for sequences with infinitely many valuations, we proved by induction that the valuation trees of these sequences are non-terminating using an adaptation of the Hensel's lemma. Our observations revealed that there are some characteristics that are shared among different trees and thus different classes of trees are formed. We selected some classes and proved that all trees in these classes have the same characteristics.

There are still many unsolved questions regarding the 2-adic valuations of Ulam square spirals that could be used for future work. We listed some of these questions below:

- Are there infinitely many distinct classes of 2-adic valuation trees of Ulam diagonal sequences?
- Could the recurrence technique shown in Chapter 5 be generalized as a tool to prove arbitrary valuation trees?
- We have shown that some trees always branch left or right; however, this is not the case for every tree. Suppose we are at a non-terminating vertex at level $k$ of a tree, is it possible to explicitly predict whether the vertex will branch left or right?
- All trees used in this study were constructed using the explicit formulas of their sequences. If a sequence does not have an explicit formula, how might we construct its valuation tree?

## REFERENCES

[1] Almodovar, L., Byrnes, A. N., Fink, J., Guan, X., Kesarwani, A., Lavigne, G., . . . Yuan, A. (n.d.). A closed-form solution might be given by a tree. Valuations of quadratic polynomials. , 18.

[2] Amdeberhan, T., Manna, D., and Moll, V. H. (2008, January). The 2-adic Valuation of Stirling Numbers. *Experimental Mathematics*, 17 (1), 69–82.

[3] Beyerstedt, E. and Moll, V. H. and Sun, X. (n.d.). The p-adic Valuation of the ASM Numbers. , 13.

[4] De, Anindya and Kurur, Piyush P and Saha, Chandan and Saptharishi, Ramprasad (2008). Fast integer multiplication using modular arithmetic. *Proceedings of the fortieth annual ACM symposium on Theory of computing*, 499-506.

[5] Kohl, K. http://karentkohl.org/papers/padic_trees.m (version: 2017-9-4)

[6] Koutschan, C. (2009). Advanced Applications of the Holonomic Systems Approach, Ph.D. thesis, RISC, J. Kepler University, Linz.

[7] Myerson, G. Quadratic polynomials describe the diagonal lines in the Ulam-Spiral, URL (version: 2015-07-07): https://math.stackexchange.com/q/1352547

[8] OEIS. https://oeis.org/A156859

[9] OEIS. https://oeis.org/A325958

[10] OEIS. https://oeis.org/A001107

[11] Sanna, C. (2016, September). On the p-adic valuation of harmonic numbers. *Journal of Number Theory*, 166, 41-46.

[12] Schumer, P. D. (1996). *Introduction to number theory*, 52-54. Brooks/Cole.

[13] Stein, M. L., Stanislaw M. U., and Mark B. W. (1964, May). A visual display of some properties of the distribution of primes. *The American Mathematical Monthly*, 5, 516.