1-1-2014

Uniform Asymptotics For the Tail Probability of Weighted Sums With Heavy Tails

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Abstract. This paper studies the tail probability of weighted sums of the form $\sum_{i=1}^n c_i X_i$, where random variables X_i's are either independent or pairwise quasi-asymptotical independent with heavy tails. Using h-insensitive function, the uniform asymptotic equivalence of the tail probabilities of $\sum_{i=1}^n c_i X_i$, $\max_{1\le k\le n}\sum_{i=1}^k c_i X_i$ and $\sum_{i=1}^n c_i X_i^+$ is established, where X_i's are independent and follow the long-tailed distribution, and c_i's take value in a broad interval. Some further uniform asymptotic results for the weighted sums of X_i's with dominated varying tails are obtained. An application to the ruin probability in a discrete-time insurance risk model is presented.

MSC: 41A60; 62P05; 62E20; 91B30

Keywords: h-insensitive function, long-tailed distribution, consistently varying tail, dominated variation, quasi-asymptotical independence

1. Introduction

In this paper, all asymptotic and limit relations are taken as $x \to \infty$ unless otherwise stated. For independently and identically distributed (iid) subexponential random variables $X_i, i \ge 1$, it is well-known that, for any $n \ge 2$,

$$P\left(\sum_{i=1}^n X_i > x\right) \sim P\left(\max_{1\le k\le n}\sum_{i=1}^k X_i > x\right) \sim P\left(\sum_{i=1}^n X_i^+ > x\right) \sim \sum_{i=1}^n P(X_i > x), \tag{1}$$

where $x^+ = \max\{x, 0\}$. There are quite a few ways to generalize these asymptotic relations. One way is to consider some broader classes of heavy-tailed distributions, see, e.g., Ng et al. [18]. Another way is to study the randomly stopped sums, see, e.g., Denisov et al. [6]. Allowing some dependence of X_i's, similar results can be obtained for different classes of heavy-tailed distributions, see Wang and Tang [22], Geluk and Ng [11], Tang [20] , Geluk and Tang [12], and references therein.

A more general way is to work on the weighted sums of form $\sum_{i=1}^n c_i X_i$, where weights c_i's are real numbers. If X_i's are iid subexponential random variables, Tang and Tsitsiashvili [21] proved that for any $0 < a \le b < \infty$, the asymptotic relation

$$P\left(\sum_{i=1}^n c_i X_i > x\right) \sim \sum_{i=1}^n P(c_i X_i > x), \tag{2}$$

holds uniformly for $a \le c_i \le b, 1 \le i \le n$, in the sense that

$$\lim_{x\to\infty}\sup_{a\le c_i\le b, 1\le i\le n}\left|\frac{P(\sum_{i=1}^n c_i X_i > x)}{\sum_{i=1}^n P(c_i X_i > x)} - 1\right| = 0.$$

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Recently, Liu et al. [16] and Li [14] established the same asymptotic relation for some dependent X_i's.

Chen et al. [3] showed that for any fixed $0 < a \le b < \infty$ it holds that uniformly for $a \le c_i \le b$, $1 \le i \le n$,

$$P\Big(\sum_{i=1}^{n} c_i X_i > x\Big) \sim P\Big(\max_{1\le k\le n} \sum_{i=1}^{k} c_i X_i > x\Big) \sim P\Big(\sum_{i=1}^{n} c_i X_i^+ > x\Big), \tag{3}$$

where X_i's are independent, not necessarily identically distributed, random variables with long-tailed distributions. This result is extended by substituting b with any positive function $b(x)$ such that $h(x) \nearrow \infty$ and $b(x) = o(x)$ in this paper.

Replacing the constant weights c_i's with random weights θ_i's, the asymptotic relation (2) and (3) still hold if the weights θ_i's, independent of X_i's, are uniformly bounded away from zero and infinity. Then it is very natural to consider the randomly weighted sum of form $\sum_{i=1}^{n} \theta_i X_i$. Wang and Tang [23] obtained $P\big(\sum_{i=1}^{n} \theta_i X_i > x\big) \sim P\big(\max_{1\le k\le n} \sum_{i=1}^{k} \theta_i X_i > x\big) \sim P\big(\sum_{i=1}^{n} \theta_i X_i^+ > x\big)$ for the case that the random weights are not necessarily bounded and X_i's are independently random variables with common distribution belonging to a smaller class than the class of subexponential distributions. Furthermore, Zhang et al. [24], Chen and Yuen [4] established the same results for dependent X_i's, where the dependence structures of X_i's are essentially same for proof of their results.

The rest of this paper is organized as follows. Section 2 reviews some important classes of heavy-tailed distributions. Section 3 states the main results along with some corollaries. Section 4 gives an application of the main results to the ruin probability in a discrete-time insurance risk model. The proof of the main results and some lemmas are presented in Section 5.

2. Classes of Heavy-Tailed Distributions

A random variable X or its distribution F is said to be heavy-tailed to the right or have a heavy (right) tail if the corresponding moment generate function does not exist on the positive real line, i.e., $Ee^{tX} = \int_{-\infty}^{\infty} e^{tx} dF(x) = \infty$ for any $t > 0$. The most important class of heavy-tailed distributions is the class of subexponential distributions, denoted by $\mathcal{S}$. Write the tail distribution by $\overline{F}(x) = 1 - F(x)$ for any distribution F. Let F^{*n} denote the n-fold convolution of F. A distribution F concentrated on $[0, \infty)$ is subexponential if

$$\overline{F^{*n}}(x) \sim n\overline{F}(x)$$

for some or, equivalently, for all $n \ge 2$. More generally, a distribution F on $(-\infty, \infty)$ belongs to the subexponential class if $F^+(x) = F(x) I_{\{x\ge 0\}}$ does.

Closely related to the subexponential class $\mathcal{S}$, the class $\mathcal{D}$ of dominated varying distributions consists of distributions satisfying

$$\limsup_{x\to\infty} \frac{\overline{F}(yx)}{\overline{F}(x)} < \infty$$

for some or, equivalently, for all $0 < y < 1$. A slightly smaller class of $\mathcal{D}$ is the class of distributions with consistently varying tail, denoted by $\mathcal{C}$. Say that a distribution F belongs to the class $\mathcal{C}$ if

$$\lim_{y \searrow 1} \liminf_{x\to\infty} \frac{\overline{F}(yx)}{\overline{F}(x)} = 1 \text{ or, equivalently, } \lim_{y \nearrow 1} \limsup_{x\to\infty} \frac{\overline{F}(yx)}{\overline{F}(x)} = 1.$$

A distribution F belongs to the class $\mathcal{L}$ of long-tailed distributions if

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1$$

for some or, equivalently, for all y. A tail distribution $\overline{F}$ is called h-insensitive if $\overline{F}(x+y) \sim \overline{F}(x)$ holds uniformly for all $|y| \le h(x)$, where $h(x)$ is a positive nondecreasing function and $\lim_{x\to\infty} h(x) = \infty$. The concept of h-insensitive function is extensively used in the monograph of Foss et al. [9]. For any distribution $F \in \mathcal{L}$, it can be shown that $\overline{F}$ is h-insensitive for some positive nondecreasing function $h(x) := h_F(x)$ such that $h(x) \nearrow \infty$ and $h(x) = o(x)$, see, e.g., Lemma 5.1 in Section 5, Section 2 in Foss and Zachary [10], Lemma 4.1 of Li et al. [15]. Consequently, $\overline{F}$ is ch-insensitive for any fixed positive real number c.

It is known that the proper inclusion relations

$$\mathcal{C} \subset \mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L}$$

hold, see, e.g., Embrechts et al. [8], Foss et al. [9].

3. Main Results

Throughout the rest of this paper $X_i, i \ge 1$, are random variables with distribution $F_i, i \ge 1$, respectively. Adopt the notation M_cF and $*_{1\le i\le n} M_{c_i}F_i$ in Barbe and McCormick [1]. For $X \sim F$ and $c > 0$, let $M_cF(x) = F(x/c)$ be the distribution of cX. The distribution of $\sum_{i=1}^n c_iX_i$ is $*_{1\le i\le n} M_{c_i}F_i$, where $X_i, 1 \le i \le n$, are independent random variables and $*_{1\le i\le n} M_{c_i}F_i$ is the convolution of $M_{c_i}F_i, 1 \le i \le n$.

The first main result generalizes Lemma 4.1 of Chen et al. [3] with different approach in two ways. First, it increases the upper bound of the weights and decreases the lower bound of the weights. Second, the fixed shift term A in Lemma 4.1 of Chen et al. [3] is enlarged to some unbounded function, which is irrespective of the upper bound of the weights.

Theorem 3.1. *If $X_i \sim F_i \in \mathcal{L}, 1 \le i \le n$, are independent random variables, there exists a positive nondecreasing function $h(x) := h(x; F_1, \cdots, F_n)$ satisfying $h(x) \nearrow \infty$ such that $*_{1\le i\le n} M_{c_i}F_i$ is uniformly $h(x)$-long-tailed for $a(x) \le c_i \le b(x), 1 \le i \le n$, in the sense that*

$$P\left(\sum_{i=1}^n c_iX_i > x \pm h(x)\right) \sim P\left(\sum_{i=1}^n c_iX_i > x\right)$$

holds uniformly for $a(x) \le c_i \le b(x), 1 \le i \le n$, i.e.,

$$\lim_{x\to\infty} \sup_{a(x)\le c_i\le b(x), 1\le i\le n} \left| \frac{\overline{*_{1\le i\le n} M_{c_i}F_i}(x \pm h(x))}{\overline{*_{1\le i\le n} M_{c_i}F_i}(x)} - 1 \right| = 0, \tag{4}$$

where the positive function $b(x)$ satisfies $b(x) \nearrow \infty$ and $b(x) = o(x)$, $h(x)$ is irrespective of $b(x)$, $a(x) = h^{-\delta}(x) \searrow 0$ for some $\delta > 0$.

Remark 3.1. Considering the case of Weibull distribution $F_1(x) = 1 - e^{-cx^\tau} \in \mathcal{S} \subset \mathcal{L}$ with $0 < \tau < 1$, it indicates that the restriction on $a(x)$ can not be weakened in general.

It is known that the class $\mathcal{L}$ is closed under convolution (see, e.g., Theorem 3 of Embrechts and Goldie [7], Corollary 2.42 of Foss et al. [9]), which can be also derived directly from Theorem 3.1.

Corollary 3.1. *If $X_i \sim F_i \in \mathcal{L}, 1 \le i \le n$, are independent random variables, then the distribution of $\sum_{i=1}^n c_i X_i > x)$ is long-tailed for any fixed $c_i > 0, 1 \le i \le n$. Consequently, the class $\mathcal{L}$ of long-tailed distributions is closed under convolution.*

Theorem 3.2. *If $X_i \sim F_i \in \mathcal{L}, 1 \le i \le n$, are independent random variables, there exist positive functions $a(x)$ and $b(x)$ satisfying $a(x) \searrow 0$ and $b(x) \nearrow \infty$ such that the asymptotic relations (3) hold uniformly for $a(x) \le c_i \le b(x)$, $1 \le i \le n$.*

The following result can be also founded in Lemma 3.4 of Foss et al. [9].

Corollary 3.2. *A distribution $F \in \mathcal{S}$ iff $F \in \mathcal{L}$ and $\overline{F * F}(x) \sim 2\overline{F}(x)$.*

Random variables $X_i, i \ge 1$, are pairwise strong quasi-asymptotically independent (pSQAI) if, for any $i \ne j$,

$$\lim_{\min\{x_i, x_j\} \to \infty} P\left(|X_i| > x_i | X_j > x_j\right) = 0,$$

which was used in Geluk and Tang [12], Liu et al. [16] and Li [14], and related to what is called asymptotic independence; see e.g. Resnick [17].

Theorem 3.3. *If $X_i \sim F_i \in \mathcal{C}, 1 \le i \le n$, are pSQAI random variables and $b(x)$ is an arbitrary fixed positive function satisfying $b(x) \nearrow \infty$ and $b(x) = o(x)$, then it holds that, uniformly for any $0 < c_i \le b(x), 1 \le i \le n$,*

$$P\left(\sum_{i=1}^n c_i X_i > x\right) \sim P\left(\max_{1 \le k \le n} \sum_{i=1}^k c_i X_i > x\right) \sim P\left(\sum_{i=1}^n c_i X_i^+ > x\right) \sim \sum_{i=1}^n P(c_i X_i > x). \quad (5)$$

Corollary 3.3. *Under assumption of Theorem 3.3, the above result still holds for $0 \le c_i \le b(x), 1 \le i \le n$, and $\min_{1 \le i \le n} c_i > 0$.*

The next theorem extends Lemma 2.1 of Liu et al [16] and Theorem 2.1 of Li [14] with a different proof, which is based on Theorem 3.1.

Theorem 3.4. *If $X_i \sim F_i \in \mathcal{D} \cap \mathcal{L}, 1 \le i \le n$, are pSQAI random variables, there exist a positive function $a(x) \searrow 0$ and a positive function $b(x) \nearrow \infty$ such that (5) holds uniformly for $a(x) \le c_i \le b(x), 1 \le i \le n$.*

Remark 3.2. Both $a(x)$ and $b(x)$ depend on $h(x)$ in Theorem 3.2 and 3.4, where $h(x) = o(x)$ is given in Theorem 3.1. More specifically, $a(x) = h^{-\delta}(x)$ for some $\delta > 0$ and $b(x) = o(h(x))$, for example, $b(x) = h^{1/2}(x)$.

Remark 3.3. If the constant weights $c_i, 1 \le i \le n$ are replaced by random weights $\theta_i, 1 \le i \le n$, which are independent of $X_i, 1 \le i \le n$, conditioning on the random weights can easily establish the corresponding results for random weights sums.

The proof of Theorem 3.4 gives an extension of Lemma 4.3 of Geluk and Tang [12].

Corollary 3.4. *If $X_i \sim F_i \in \mathcal{L}, 1 \le i \le n$, are pQSAI random variables, it holds that, for some the positive functions $b(x) \nearrow \infty$ and $a(x) \searrow 0$,*

$$\lim_{x\to\infty} \inf_{a(x)\le c_i\le b(x), 1\le i\le n} \frac{P\big(\sum_{i=1}^n c_i X_i > x\big)}{\sum_{i=1}^n P\big(c_i X_i > x\big)} \ge 1. \tag{6}$$

4. Application to Risk Theory

Consider the following discrete-time insurance risk model

$$U_0 = x,\ U_n = U_{n-1}(1 + r_n) - X_n, n \ge 1,$$

where U_n stands an insurer's surplus at the end of period n with a deterministic initial surplus x, r_n represents the constant interest force of an insurer's risk-free investment, and the net loss X_n over period n equals the total amount of claims plus other costs minus the total amount of premiums during period n. It is an interesting and important problem arising from the above discrete-time insurance risk model to study the ruin probabilities of the insurer. See Tang [19] for detailed discussion.

The ruin probability by time n is defined as

$$\psi(x;n) = P\Big(\min_{i=1}^n U_i < 0 \,|\, U_0 = x\Big).$$

It is easy to see that the surplus process is of form

$$U_0 = x,\ U_n = \prod_{i=1}^n (1 + r_i)x - \sum_{i=1}^n \Big(\prod_{j=i+1}^n (1 + r_j)\Big) X_i, n \ge 1.$$

Define the discounted surplus process as follows

$$\widetilde{U}_n = \Big(\prod_{i=1}^n (1 + r_i)\Big)^{-1} U_n = x - \sum_{i=1}^n c_i X_i,$$

where $c_i = \prod_{j=1}^i (1 + r_j)^{-1}$ represents the discount factor from time i to time 0, $1 \le i \le n$. Then the corresponding ruin probability can be written as

$$\psi(x;n) = P\Big(\min_{i=1}^n \widetilde{U}_i < 0 \,|\, \widetilde{U}_0 = x\Big) = P\Big(\max_{1\le i\le k} \sum_{i=1}^k c_i X_i > x\Big).$$

Applying Theorem 3.2 and Theorem 3.4 in Section 3, the following asymptotic results can be obtained.

Corollary 4.1. *Assume that net losses $X_i, i \ge 1$ are independent random variables, which are not necessarily identically distributed, with distribution $F_i, i \ge 1$, respectively. If $F_i \in \mathcal{L}, 1 \le i \le n$, then*

$$\psi(x;n) \sim P\Big(\sum_{i=1}^n c_i X_i > x\Big) \sim P\Big(\sum_{i=1}^n c_i X_i^+ > x\Big).$$

If $F_i \in \mathcal{D} \cap \mathcal{L}, 1 \le i \le n$, then

$$\psi(x;n) \sim P\Big(\sum_{i=1}^n c_i X_i > x\Big) \sim P\Big(\sum_{i=1}^n c_i X_i^+ > x\Big) \sim \sum_{i=1}^n P(c_i X_i > x).$$

5. Proof of Results

A function $h(x)$ is called slowly varying at infinity if $h(xy) \sim h(x)$ for any $y > 0$, It is well-known that $h(x) = o(x^{\delta})$ for any $\delta > 0$ if $h(x)$ is a slowly varying function, see, e.g., Bingham et al. [2]. The following result is crucial for the proof of all theorems in this paper. It shows that any tail distribution of a long-tailed distribution is uniformly h-insensitive for a slowly varying function h.

Lemma 5.1. *If $X \sim F \in \mathcal{L}$, then $\overline{F}$ is h-insensitive for a positive nondecreasing and slowly varying function $h(x) := h(x; F) : (0, \infty) \to (0, \infty)$ satisfying $h(x) \nearrow \infty$, $h(x) \leq ch(\frac{x}{c})$ for all $c \geq 1$, and*

$$\lim_{x\to\infty} \sup_{a(x)\leq c\leq b(x)} \left| \frac{P(cX > x \pm h(x))}{P(cX > x)} - 1 \right| = 0, \tag{7}$$

where $b(x)$ is an arbitrary positive function such that $b(x) \nearrow \infty$ and $b(x) = o(x)$, and $a(x) = h^{-\delta}(x)$ for some $\delta > 0$.

Proof. For any fixed $\delta > 0$, let $\{x_n, n \geq 1\}$ be a sequence of increasing positive real numbers such that $x_{n+1} \geq 2x_n > 0$, $n \geq 1$, and for any $x \geq x_n$,

$$\sup_{|y|\leq n} \left| \frac{\overline{F}(x+y)}{\overline{F}(x)} - 1 \right| \leq \max \left\{ \left| \frac{\overline{F}(x+n^{1+\delta})}{\overline{F}(x)} - 1 \right|, \left| \frac{\overline{F}(x-n^{1+\delta})}{\overline{F}(x)} - 1 \right| \right\} \leq \frac{1}{n}. \tag{8}$$

Borrowing the idea of the proof of Corollary 2.5 in [5], let

$$h(x) = \begin{cases} \frac{2}{x_1}x & x_0 = 0 < x < x_1 \\ n + \frac{x-x_{n-1}}{x_n-x_{n-1}} & x_{n-1} \leq x < x_n, n \geq 2. \end{cases}$$

Clearly, $h(x)$ is a positive nondecreasing, piecewise linear, continuous function and $h(x) \nearrow \infty$. Since $h(x)$ is a nondecreasing function, $h(xy) \sim h(x)$ for any $y > 0$ is equivalent to $h(2x) \sim h(x)$, which follows from the facts that $h(x) \nearrow \infty$ and $h(x) \leq h(2x) < h(x_{n+1}) = n + 2 \leq h(x) + 2$ for any $x_{n-1} \leq x < x_n$.

For any $x \geq x_n$, i.e., $x \in [x_{n+k}, x_{n+k+1})$ for some $k := k(x) \geq 0$, and $|y| \leq h^{1+\delta}(x) = (n+k+1)^{1+\delta}$, it follows from (8) that

$$\sup_{|y|\leq h^{1+\delta}(x)} \left| \frac{\overline{F}(x+y)}{\overline{F}(x)} - 1 \right| \leq \frac{1}{n+k+1} \leq \frac{1}{n} \to 0, \quad \text{as } n \to \infty,$$

i.e., $\overline{F}$ is $h^{1+\delta}$-insensitive, which of course implies that $\overline{F}$ is h-insensitive. Since $x_{n+1} - x_n \geq x_n \geq x_n - x_{n-1}, n \geq 1$, $h'(x)$ is a nonincreasing function on $\cup_{n=1}^{\infty}(x_{n-1}, x_n)$, which implies that $h(x)$ is a concave function on $[0, \infty)$. The concavity of $h(x)$ and the fact $h(0) = 0$ lead to $h(\frac{x}{c}) = h\big(\frac{1}{c}x + (1-\frac{1}{c})0\big) \geq \frac{1}{c}h(x) + (1-\frac{1}{c})h(0) = \frac{1}{c}h(x)$, i.e., $h(x) \leq ch(\frac{x}{c})$, for any $x > 0, c > 1$.

Hence, $\frac{h(x)}{c} \leq h\big(\frac{x}{c}\big) \leq h^{1+\delta}\big(\frac{x}{c}\big)$ for $1 \leq c \leq b(x)$. Note that $\frac{h(x)}{c} \leq \frac{h(x)}{a(x)} = h^{1+\delta}(x) \leq h^{1+\delta}(\frac{x}{c})$ for $a(x) \leq c \leq 1$. The monotonicity of $\overline{F}$ yields $\overline{F}\big(\frac{x}{c} + h^{1+\delta}(\frac{x}{c})\big) \leq P\big(cX > x \pm h(x)\big) = \overline{F}\big(\frac{x}{c} \pm \frac{h(x)}{c}\big) \leq \overline{F}\big(\frac{x}{c} - h^{1+\delta}(\frac{x}{c})\big)$ for $a(x) \leq c \leq b(x)$. The uniform asymptotic relation (7) follows from the inequalities

$$\begin{aligned} \frac{\overline{F}\big(\frac{x}{c} + h^{1+\delta}(\frac{x}{c})\big)}{\overline{F}\big(\frac{x}{c}\big)} - 1 &\leq \frac{P(cX > x \pm h(x))}{P(cX > x)} - 1 = \frac{\overline{F}\big(\frac{x}{c} \pm \frac{h(x)}{c})\big)}{\overline{F}\big(\frac{x}{c}\big)} - 1 \\ &\leq \frac{\overline{F}\big(\frac{x}{c} - h^{1+\delta}(\frac{x}{c})\big)}{\overline{F}\big(\frac{x}{c}\big)} - 1, \quad a(x) \leq c \leq b(x), \end{aligned}$$

and the fact that $\overline{F}$ is $h^{1+\delta}$-insensitive. $\square$

Remark 5.1. It is easy show that $\frac{h(x)}{x} \searrow 0$ for $h(x)$ in the proof of Lemma 5.1.

Proof of Theorem 3.1. Assume that $\overline{F}_i$ is h_i-insensitive, where $h_i(x) = h(x; F_i)$ is given in Lemma 5.1, $1 \le i \le n$. Let $h(x) := h(x; F_1, \cdots, F_n) = \min\{h_i(x), 1 \le i \le n\} = o(x)$. Then all $\overline{F}_i$'s are h-insensitive and $h(x) \le ch(\frac{x}{c})$, $c \ge 1$, by Lemma 5.1. The uniform asymptotic relation (6), which is essentially the case of $n = 2$ in proof, will be proved by induction. It is obviously true for $n = 1$ by Lemma 5.1. Since distribution functions are nondecreasing, (6) is equivalent to

$$\lim_{x\to\infty} \inf_{a(x)\le c_i\le b(x),1\le i\le n} \frac{P\big(\sum_{i=1}^n c_i X_i > x + h(x)\big)}{P\big(\sum_{i=1}^n c_i X_i > x\big)} \ge 1, \tag{9}$$

and

$$\lim_{x\to\infty} \sup_{a(x)\le c_i\le b(x),1\le i\le n} \frac{P\big(\sum_{i=1}^n c_i X_i > x - h(x)\big)}{P\big(\sum_{i=1}^n c_i X_i > x\big)} \le 1. \tag{10}$$

Write $A + B + C$ for the union of disjoint sets A, B, C. The fact that $\big\{\sum_{i=1}^n c_i X_i > x \pm h(x)\big\} = \big\{\sum_{i=1}^n c_i X_i > x + h(x), c_n X_n \le \frac{x+h(x)}{2}\big\} + \big\{\sum_{i=1}^n c_i X_i > x + h(x), \sum_{i=1}^{n-1} c_i X_i \le \frac{x+h(x)}{2}\big\} + \big\{\sum_{i=1}^{n-1} c_i X_i > \frac{x+h(x)}{2}, c_n X_n > \frac{x+h(x)}{2}\big\}$ and independence of X_i's yield

$$\begin{aligned} P\Big(\sum_{i=1}^n c_i X_i > x + h(x)\Big) \quad \ge \quad & \int_{-\infty}^{x/2} P\Big(\sum_{i=1}^{n-1} c_i X_i > x + h(x) - t\Big) dP(c_n X_n \le t) \\ & + \int_{-\infty}^{x/2} P\big(c_n X_n > x + h(x) - t\big) dP\Big(\sum_{i=1}^{n-1} c_i X_i \le t\Big) \\ & + P\Big(\sum_{i=1}^{n-1} c_i X_i > \frac{x + h(x)}{2}\Big) P\Big(c_n X_n > \frac{x + h(x)}{2}\Big). \end{aligned} \tag{11}$$

The induction assumption with $b(x)$ replaced by $2b(x)$ implies that

$$\begin{aligned} & P\Big(\sum_{i=1}^{n-1} c_i X_i > \frac{x + h(x)}{2}\Big) P\Big(c_n X_n > \frac{x + h(x)}{2}\Big) \\ = \ & P\Big(\sum_{i=1}^{n-1} 2c_i X_i > x + h(x)\Big) P\Big(2c_n X_n > x + h(x)\Big) \\ \sim \ & P\Big(\sum_{i=1}^{n-1} 2c_i X_i > x\Big) P\Big(2c_n X_n > x\Big) = P\Big(\sum_{i=1}^{n-1} c_i X_i > \frac{x}{2}\Big) P\Big(c_n X_n > \frac{x}{2}\Big) \end{aligned} \tag{12}$$

holds uniformly for $a(x) \le c_i \le b(x), 1 \le i \le n$.
Use monotonicity of any distribution function and the inequality $h(x) \le 2h(\frac{x}{2})$ to obtain

$$1 \ge \inf_{t\le x/2} \frac{\overline{F}(x + h(x) - t)}{\overline{F}(x - t)} \ge \inf_{t\le x/2} \frac{\overline{F}\big(x - t + 2h(\frac{x}{2})\big)}{\overline{F}(x - t)} \ge \inf_{u=x-t\ge x/2} \frac{\overline{F}(u + 2h(u))}{\overline{F}(u)} \sim 1 \tag{13}$$

provided $\overline{F}$ is h-insensitive. It follows from the induction assumption and Lemma 5.1 that the tail distribution of $\sum_{i=1}^{n-1} c_i X_i$ and the tail distribution of $c_n X_n$ are h-insensitive. The asymptotic relation (12) and the inequality (11) imply

$$\begin{aligned}
&P\big(\sum_{i=1}^{n} c_iX_i > x + h(x)\big)\\
\geq\ & \Big(\int_{-\infty}^{x/2} P\Big(\sum_{i=1}^{n-1} c_iX_i > x-t\Big)dP(c_nX_n \leq t) + \int_{-\infty}^{x/2} P\big(c_nX_n > x-t\big)dP\Big(\sum_{i=1}^{n-1} c_iX_i \leq t\Big)\\
&+P\Big(\sum_{i=1}^{n-1} c_iX_i > \frac{x}{2}\Big)P\Big(c_nX_n > \frac{x}{2}\Big)\Big)(1+o(1))\\
=\ & (1+o(1))P\Big(\sum_{i=1}^{n} c_iX_i > x\Big),
\end{aligned}$$

where the term $o(1)$ goes to 0 uniformly for $a(x) \leq c_i \leq b(x)$, $1 \leq i \leq n$. This complete the proof of (9).

The other uniform asymptotic relation (10) can be obtained by substituting $+h(x)$, $+2h(\frac{x}{2})$, $\geq$, inf with $-h(x)$, $-2h(\frac{x}{2})$, $\leq$, sup, respectively, in the proof of (9). □

Proof of Theorem 3.2. The idea is from the proof of Theorem 2.1 of Chen et al. [3]. Let $\big\{\Omega_K = \{X_i \geq 0 \text{ for all } i \in K, X_j < 0 \text{ for all } j \in \{1,\cdots,n\}\backslash K\}, K \subseteq \{1,\cdots,n\}\big\}$ be a finite partition of the whole space Ω. Obviously, $P\big(\sum_{i=1}^{n} c_iX_i > x, \Omega_K\big)$ is not less than

$$\begin{aligned}
&P\Big(\sum_{i\in K} c_iX_i > x + h(x), \sum_{j\notin K} c_jX_j > -h(x), \Omega_K\Big)\\
=\ & P\Big(\sum_{i=1}^{n} c_iX_i^+ > x + h(x), \Omega_K\Big) - P\Big(\sum_{i\in K} c_iX_i > x + h(x), \sum_{j\notin K} c_jX_j \leq -h(x), \Omega_K\Big), \quad (14)
\end{aligned}$$

where, due to the independence of X_i's, the second term equals

$$P\Big(\sum_{i\in K} c_iX_i > x + h(x), \bigcap_{i\in K}\{X_i \geq 0\}\Big)P\Big(\sum_{j\notin K} c_j(-X_j) \geq h(x), \bigcap_{j\notin K}\{X_j < 0\}\Big).$$

and it is at most $P\big(\sum_{i=1}^{n} c_iX_i^+ > x + h(x)\big)P\big(\sum_{j=1}^{n} c_jX_j^- \geq h(x)\big)$, where $x^- = \max\{-x, 0\}$. Note that $\{\sum_{j=1}^{n} c_jX_j^- \geq h(x)\} \subseteq \bigcup_{j=1}^{n}\{c_jX_j^- \geq \frac{h(x)}{n}\} = \bigcup_{j=1}^{n}\{c_jX_j \leq -\frac{h(x)}{n}\}$, whose probability is at most $\sum_{j=1}^{n} P\Big(X_j \leq -\frac{h(x)}{nb(x)}\Big) = o(1)$ provided $b(x) = o(h(x))$. Therefore, uniformly for $0 < a \leq c_i \leq b(x)$, $1 \leq i \leq n$, the second term in (14) is $o\big(P\big(\sum_{i=1}^{n} c_iX_i^+ > x + h(x)\big)\big)$ and

$$P\Big(\sum_{i=1}^{n} c_iX_i > x, \Omega_K\Big) \geq P\Big(\sum_{i=1}^{n} c_iX_i^+ > x + h(x), \Omega_K\Big) + o\Big(P\Big(\sum_{i=1}^{n} c_iX_i^+ > x + h(x)\Big)\Big).$$

Sum it over all K's to get

$$P\Big(\sum_{i=1}^{n} c_iX_i > x\Big) \geq P\Big(\sum_{i=1}^{n} c_iX_i^+ > x + h(x)\Big) + o\Big(P\Big(\sum_{i=1}^{n} c_iX_i^+ > x + h(x)\Big)\Big).$$

Clearly, $X_i^+ \sim F_i^+(x) = F_i(x)I_{\{x\geq 0\}} \in \mathcal{L}, 1 \leq i \leq n$. Choose $h(x)$ such that (6) holds with F_i substituted by F_i^+. The desired result follows from Theorem 3.1 and the simple fact that $\sum_{i=1}^{n} c_iX_i \leq \max_{1\leq k\leq n}\sum_{i=1}^{k} c_iX_i \leq \sum_{i=1}^{n} c_iX_i^+$. □

Proof of Corollary 3.2. Recall that $\overline{F} \in \mathcal{S}$ if $\overline{F^+} \in \mathcal{S}$, i.e., $\overline{F^+ * F^+}(x) \sim 2\overline{F^+}(x)$ for $F^+(x) = F(x)I_{\{x\geq 0\}}$. Clearly, $F \in \mathcal{L}$ iff $F^+ \in \mathcal{L}$. If $F^+ \in \mathcal{S}$, the fact that $\mathcal{S} \subset \mathcal{L}$ implies $F \in \mathcal{L}$. Then it is equivalent to show that $\overline{F^+ * F^+}(x) \sim 2\overline{F^+}(x)$ iff $\overline{F * F}(x) \sim 2\overline{F}(x)$, i.e. $\overline{F^+ * F^+}(x) \sim \overline{F * F}(x)$ since $\overline{F^+}(x) = \overline{F}(x)$ for all $x > 0$. It is obviously true by Theorem 3.2. $\square$

The next two lemma can be easily checked from the definition of the class $\mathcal{C}$.

Lemma 5.2. *If X follows distribution $F \in \mathcal{C}$, then $\overline{F}(x)$ is h-insensitive provided $h(x) = o(x)$ and it holds that, uniformly for $0 < c < b(x) = o(x)$,*

$$P(cX > x \pm h(x)) \sim P(cX > x).$$

Lemma 5.3. *If $X_i \sim F_i \in \mathcal{C}, 1 \leq i \leq n$, are pQSAI random variables, it holds that, uniformly for $0 < c < b(x) = o(x)$,*

$$P\Big(c_jX_j > \frac{x}{n}, \max_{1\leq k\neq j\leq n} |c_kX_k| > b(x)\ln\Big(\frac{x}{b(x)}\Big)\Big) = o\big(P(c_jX_j > x)\big)$$

and consequently

$$P\Big(\bigcup_{j=1}^{n}\Big\{c_jX_j > \frac{x}{n}, \max_{1\leq k\neq j\leq n} |c_kX_k| > b(x)\ln\Big(\frac{x}{b(x)}\Big)\Big\}\Big) = o\Big(\sum_{j=1}^{n} P(c_jX_j > x)\Big).$$

Proof of Theorem 3.3. Let $h(x) = b(x)\ln\big(\frac{x}{b(x)}\big)$. The proof is similar to that of Theorem 3.4 and is omitted. $\square$

Proof of Corollary 3.3. Partition the range of the weights as $\{(c_1, \cdots, c_n) : 0 \leq c_i \leq b(x), 1 \leq i \leq n, \min_{i=1}^n c_i > 0\} = \bigcup_{K\subset\{1,\ldots,n\}}\{(c_1, \cdots, c_n) : 0 \leq c_i \leq b(x), i \in K, 0 < c_i \leq b(x), i \notin K\}$. The desired result follows from Theorem 3.3. $\square$

Lemma 5.4. *If $X_i \sim F_i \in \mathcal{D}, 1 \leq i \leq n$, are pSQAI random variables, $h(x) = o(x)$ and $h(x) \nearrow \infty$, it holds that, uniformly for $0 < a < c_i < b(x) = o(h(x)), 1 \leq i \leq n$,*

$$P\Big(c_jX_j > \frac{x}{n}, \max_{1\leq k\neq j\leq n} |c_kX_k| > h(x)\Big) = o\big(P(c_jX_j > x)\big)$$

and consequently

$$P\Big(\bigcup_{j=1}^{n}\{c_jX_j > \frac{x}{n}, \max_{1\leq k\neq j\leq n} |c_kX_k| > h(x)\}\Big) = o\Big(\sum_{j=1}^{n} P(c_jX_j > x)\Big).$$

Proof. The results follow from the fact that $F_i \in \mathcal{D}$ and $b(x) = o(h(x))$, the pSQAI property of X_i's and the elementary probability inequality $P(A \cap \cup_{i=1}^n B_i) \leq \sum_{i=1}^n P(AB_i)$. $\square$

If X_i is large, the pSQAI property of X_j's implies that other X_j's are relatively close to 0 and negligible compared with X_i. If $\sum_{i=1}^n c_iX_i > x$, there should be exactly one c_iX_i greater than $\frac{x}{n}$ and consequently Lemma 5.4 implies

$$P\Big(\sum_{i=1}^{n} c_iX_i > x\Big) \sim \sum_{j=1}^{n} P\Big(\sum_{i=1}^{n} c_iX_i > x, c_jX_j > \frac{x}{n}, \max_{1\leq k\neq j\leq n} |c_kX_k| \leq h(x)\Big).$$

It gives the idea of the proof of Theorem 3.4, which is simpler and more straightforward than the proof of Lemma 2.1 of Liu et al. [16] and Theorem 2.1 of Li [14].

Proof of Theorem 3.4. All asymptotic relations hold uniformly for $a(x) \le c_i \le b(x), 1 \le i \le n$, in the proof. By Lemma 5.1, there exists a positive nondecreasing function $h(x) := h(x, a; F_1, \cdots, F_n)$ satisfying $h(x) \nearrow \infty$ and $h(x) = o(x)$ such that (7) holds for $F = F_i, 1 \le i \le n$, respectively. Choose $b(x) = o(h(x))$ and $b(x) \nearrow \infty$. Note that

$$\begin{aligned}\Big\{\sum_{i=1}^n c_i X_i > x\Big\} &= \bigcup_{j=1}^n \Big\{\sum_{i=1}^n c_i X_i > x, c_j X_j > \frac{x}{n}\Big\} \\ &= \bigcup_{j=1}^n A_j \bigcup \Big\{\sum_{i=1}^n c_i X_i > x, \bigcup_{j=1}^n \{c_j X_j > \frac{x}{n}, \max_{1\le k\ne j\le n} |c_k X_k| > h(x)\}\Big\},\end{aligned}$$

where $A_j = \big\{\sum_{i=1}^n c_i X_i > x, c_j X_j > \frac{x}{n}, \max_{1\le k\ne j\le n} |c_k X_k| \le h(x)\big\}, 1 \le j \le n$, are mutually exclusive events provided $\frac{x}{n} > h(x)$. The elementary probability inequality $P(A) \le P(A \cup B) \le P(A) + P(B)$ and Lemma 5.4 lead to

$$P\Big(\sum_{i=1}^n c_i X_i > x\Big) = \sum_{j=1}^n P(A_j) + o\Big(\sum_{j=1}^n P(c_j X_j > x)\Big). \tag{15}$$

Lemma 5.1 and the fact that $c_j X_j$ is at least $x - (n-1)h(x)$ on A_j lead to

$$P(A_j) \le P\big(c_j X_j > x - (n-1)h(x)\big) = P(c_j X_j > x) + o\big(P(c_j X_j > x)\big), \quad 1 \le j \le n.$$

Since $\max_{1\le k\ne j\le n} |c_k X_k| \le h(x)$ on A_j, $c_j X_j > x + (n-1)h(x)$ implies $\sum_{i=1}^n c_i X_i > x$ on A_j for any $1 \le j \le n$. It follows from Lemma 5.1 and 5.4 that

$$\begin{aligned} P(A_j) &\ge P\big(c_j X_j > x + (n-1)h(x), \max_{1\le k\ne j\le n} |c_k X_k| \le h(x)\big) \\ &= P(c_j X_j > x + (n-1)h(x)) - P\big(c_j X_j > x + (n-1)h(x), \max_{1\le k\ne j\le n} |c_k X_k| > h(x)\big) \\ &= P(c_j X_j > x) + o\big(P(c_j X_j > x)\big), \quad 1 \le j \le n. \end{aligned}$$

Therefore, (15) can be written as

$$P\Big(\sum_{i=1}^n c_i X_i > x\Big) \sim \sum_{i=1}^n P(c_i X_i > x). \tag{16}$$

In the exactly same way, it can be proved that

$$P\Big(\sum_{i=1}^n c_i X_i^+ > x\Big) \sim \sum_{i=1}^n P(c_i X_i^+ > x) = \sum_{i=1}^n P(c_i X_i > x). \tag{17}$$

Note that $\sum_{i=1}^n c_i X_i \le \max_{1\le k\le n} \sum_{i=1}^k c_i X_i \le \sum_{i=1}^n c_i X_i^+$. The desired results follow from the uniform asymptotic relation (16) and (17). □

Remark 5.2. The proof of Theorem 3.4 also leads to Corollary 3.4.

Acknowledgments

The author would like to thank the anonymous referees for their comments and help in improving the paper.

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