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Homogeneous Products of Conjugacy Classes

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Abstract. Let G be a finite group and $a \in G$. Let $a^G = \{g^{-1}ag \mid g \in G\}$ be the conjugacy class of a in G. Assume that a^G and b^G are conjugacy classes of G with the property that $\mathbf{C}_G(a) = \mathbf{C}_G(b)$. Then $a^G b^G$ is a conjugacy class if and only if $[a,G] = [b,G] = [ab,G]$ and $[ab,G]$ is a normal subgroup of G.

1. Introduction

Let G be a finite group, $a \in G$ and $a^G = \{a^g \mid g \in G\}$ be the conjugacy class of a in G. Denote by $|a^G|$ the size of the set a^G. Given $a, g \in G$, set $[a,g] = a^{-1}a^g$. Also set $[a,G] = \{[a,g] \mid g \in G\}$. Let $\mathbf{C}_G(a) = \{g \in G \mid a^g = a\}$ be the centralizer of a in G and 1_G be the identity of G. Through this note, we will use the well known fact that $|a^G| = |G : \mathbf{C}_G(a)|$.

In Theorem A of [1], it is proved that if G is a finite nilpotent group and χ, ψ are faithful irreducible characters with the property that $\chi\psi$ is a multiple of an irreducible, then χ and ψ both vanish outside the center $\mathbf{Z}(G)$ of G, i.e. $\chi(g) = \psi(g) = 0$ for all $g \in G \setminus \mathbf{Z}(G)$. This note was motivated by wondering what would be the analogous result in conjugacy classes.

Let a^G and b^G be conjugacy classes such that the product $a^G b^G = \{xy \mid x \in a^G, y \in b^G\}$ is also a conjugacy class. We can check that $(ab)^G$ is a subset of $a^G b^G$ and thus if $a^G b^G$ is a conjugacy class, then $a^G b^G = (ab)^G$. Is there any relationship between a, b and G? The answer in general seems to be no. For instance, if we take any group G, any element a of G, then $a^G 1_G^G = (a1_G)^G$. But if we add the additional hypothesis that $\mathbf{C}_G(a) = \mathbf{C}_G(b)$, then we have the following

Theorem A. *Let G be a finite group, a^G and b^G be conjugacy classes of G. Assume that $\mathbf{C}_G(a) = \mathbf{C}_G(b)$. Then $a^G b^G = (ab)^G$ if and only if $[ab,G] = [a,G] = [b,G]$ and $[ab,G]$ is a normal subgroup of G. In particular, given any conjugacy class a^G of G, then $a^G a^G = (a^2)^G$ if and only if $[a,G]$ is a normal subgroup of G.*

We regard the hypothesis that a^G and b^G are conjugacy classes of G with $\mathbf{C}_G(a) = \mathbf{C}_G(b)$ as the "dual" to the hypothesis that two irreducible characters have the same kernel.

The following is a direct application of Theorem A.

Corollary. *Let G be a finite nonabelian simple group, a^G and b^G be conjugacy classes of G. Assume that $\mathbf{C}_G(a) = \mathbf{C}_G(b)$. Then $a^G b^G = (ab)^G$ if and only if $a = b = 1_G$. In particular, $a^G a^G = (a^2)^G$ if and only if $a = 1_G$.*

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Is it possible to find a finite group G and a conjugacy class a^G of G such that $a^Ga^G = (a^2)^G$ and $|a^G| = 2$? The answer is no, such group with such conjugacy class can not exist. In Proposition 3.4, we show that if G is a finite group, a^G and b^G are conjugacy classes such that $\mathbf{C}_G(a) = \mathbf{C}_G(b)$, then necessarily a^Gb^G is the union of exactly 2 distinct conjugacy classes. But then, is it possible to find a finite group G and a conjugacy class a^G of G such that $a^Ga^G = (a^2)^G$ and $|a^G|$ is a power of 2? If, in addition, we require that the group is supersolvable, then the answer is again no. More specifically, in Proposition 3.8 is shown that if G is a finite supersolvable group, a^G and b^G are conjugacy classes of G with $\mathbf{C}_G(a) = \mathbf{C}_G(b)$ and $|a^G| = 2^n$ for some integer $n > 0$, then a^Gb^G is the union of at least 2 distinct conjugacy classes. The author wonders if the answer remains no if we do not require that the group is supersolvable. On the other hand, in Proposition 4.4, given any odd integer $n > 0$, we provide an example of a nilpotent group G and a conjugacy class a^G such that $a^Ga^G = (a^2)^G$ and $|a^G| = n$.

We want to close this introduction by mentioning that there is a number of papers concerning products of conjugacy classes and finite groups. A very recent development is [3], where the authors classify all finite groups G such that the product of any two non-inverse conjugacy classes of G is always a conjugacy class of G.

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2. Proof of Theorem A

We will denote by 1_G the identity of the group G.

Lemma 2.1. *Let G be a finite group and $a, b \in G$. Then*

$$a^Gb^G = ab[a^{b^{-1}}, G][b, G].$$

Thus if $a^b = a$ then $a^Gb^G = ab[a, G][b, G]$.

Proof. Observe that

$$\begin{aligned} a[a, G]b[b, G] &= ab[a, G]^{b^{-1}}[b, G] \\ &= ab[a^{b^{-1}}, G^{b^{-1}}][b, G] \\ &= ab[a^{b^{-1}}, G][b, G]. \end{aligned}$$

□

Lemma 2.2. *Let G be a finite group and $c \in G$. If $[c, G]$ is a subgroup of G, then $[c, G]$ is a normal subgroup of G.*

Proof. Let $g \in G$ and $x \in [c, G]$. By definition, $(c)^g = cy$ for some $y \in [c, G]$, and $cx = (c)^h$ for some $h \in G$. Also $(c)^{hg} = cw$ for some $w \in [c, G]$. Observe that $(c)^{hg} = ((c)^h)^g = (cx)^g = (c)^gx^g = cyx^g$. Thus $yx^g = w$ and $x^g = y^{-1}w \in [c, G]$. We conclude that $[c, G]$ is a normal subgroup of G. □

Proof of Theorem A. Since $\mathbf{C}_G(a) = \mathbf{C}_G(b)$, we have that $ab = ba$. Observe that if $[a, G] = [b, G] = [ab, G]$ and $[ab, G]$ is a normal subgroup, then by Lemma 2.1, we have that $a^Gb^G = ab[a, G][b, G] = ab[ab, G] = (ab)^G$. We may assume now that

$a^G b^G = (ab)^G$ and we want to conclude that $[a, G] = [b, G] = [ab, G]$ and $[ab, G]$ is a normal subgroup of G.

Since $\mathbf{C}_G(a) = \mathbf{C}_G(b)$, we have that $|a^G| = |G : \mathbf{C}_G(a)| = |G : \mathbf{C}_G(b)| = |b^G|$, $\mathbf{C}_G(ab) \geq \mathbf{C}_G(a)$ and therefore $|(ab)^G| \leq |a^G|$. Because $a^G b^G = (ab)^G$, we have then that $|(ab)^G| = |a^G| = |b^G|$. Since $a^G b^G = (ab)^G$ and $ab = ba$, by Lemma 2.1 we have that $[a, G][b, G] = [ab, G]$. Thus $[a, G] = [b, G] = [ab, G]$ since $|(ab)^G| = |a^G| = |b^G|$ and 1_G is in both $[a, G]$ and $[b, G]$.

Since $[a, G] = [b, G] = [ab, G]$ and $[a, G][b, G] = [ab, G]$, then $[ab, G][ab, G] = [ab, G]$. Clearly $[ab, G]$ is nonempty since $1_G \in [ab, G]$. We conclude that $[ab, G]$ is a subgroup of G since $uv \in [ab, G]$ for any u, v in $[ab, G]$ and $[ab, G]$ is a nonempty finite set. The result then follows by Lemma 2.2. □

3. Further Results

Let X be a G-invariant subset of G, i.e. $X^g = \{x^g \mid x \in X\} = X$ for all $g \in G$. Then X can be expressed as a union of n distinct conjugacy classes of G, for some integer $n > 0$. Set $\eta(X) = n$.

Lemma 3.1. *Let G be a finite p-group and N be a normal subgroup of G. Let a and b be elements of G. If $(aN)^{G/N} \cap (bN)^{G/N} = \emptyset$ then $a^G \cap b^G = \emptyset$. Thus $\eta((aN)^{G/N}(bN)^{G/N}) \leq \eta(a^G b^G)$.*

Proof. See Lemma 2.1 of [2]. □

Proposition 3.2. *Let G be a group of odd order and a^G be the conjugacy class of a in G. Then*

$$\mathbf{Z}(G) \cap a^G a^G \neq \emptyset \tag{3.3}$$

if and only if $|a^G| = 1$. Thus if $|a^G| > 1$ and $b^G \subseteq a^G a^G$, then $|b^G| > 1$.

Proof. Suppose that there exist some $z \in \mathbf{Z}(G) \cap a^G a^G$. Then there exist some $g \in G$ such that $aa^g = z$. Thus $a^g = a^{-1}z$ and therefore $(a^{-1})^g = az^{-1}$. Observe that

$$a^{g^2} = (a^g)^g = (a^{-1}z)^g = (a^{-1})^g z = (az^{-1})z = a.$$

Thus $g^2 \in \mathbf{C}_G(a)$. Since G is of odd order, $g^2 \in \mathbf{C}_G(a)$ implies that $g \in \mathbf{C}_G(a)$. So $a^2 = z$ and $a \in \mathbf{Z}(G)$. We conclude that $|a^G| = 1$. □

Let E be an extraspecial group of order 3^3 and exponent 3. Let $a \in E \setminus \mathbf{Z}(E)$. Set $b = a^2$. We can check that $\mathbf{C}_G(a) = \mathbf{C}_G(b)$ and $a^G b^G = \mathbf{Z}(E)$. Thus given a finite group G of odd order, conjugacy classes a^G and b^G of G with $\mathbf{C}_G(a) = \mathbf{C}_G(b)$, $\mathbf{Z}(G) \cap a^G b^G \neq \emptyset$ may not imply that $|a^G| = 1$.

Let Q_8 be the quaternion group and $a \in Q_8$ be an element of order 4. We can check that $a^{Q_8} = \{a, a^{-1}\}$ and $a^{Q_8} a^{Q_8} = \mathbf{Z}(Q_8)$. Thus Proposition 3.2 may not remain true if the group G has even order.

Proposition 3.4. *Let G be a finite group, a^G and b^G be conjugacy classes with $\mathbf{C}_G(a) = \mathbf{C}_G(b)$ and $|a^G| = 2$. Then $\eta(a^G b^G) = 2$. In particular $\eta(a^G a^G) = 2$.*

Proof. Set $N = \mathbf{C}_G(a)$. Observe that N is a normal subgroup of G since $|G : N| = |G : \mathbf{C}_G(a)| = 2$. Since $\mathbf{C}_G(a) = \mathbf{C}_G(b)$, we have that $ab = ba$. Fix $g \in G \setminus N$. Since $|a^G| = |b^G| = 2$, $\mathbf{C}_G(a) = \mathbf{C}_G(b)$ and $g \in G \setminus N$, we have that $a^G = \{a, a[a, g]\}$ and $b^G = \{b, b[b, g]\}$. Also $[a, g]b = b[a, g]$ since $[a, g] \in N = \mathbf{C}_G(b)$. Therefore

$$a^G b^G = \{ab, ab[a, g], ab[b, g], ab[a, g][b, g]\}. \tag{3.5}$$

Since $|G:N|=2$, we have that $a^{g^2}=a$ and $a^{g^2}=(a[a,g])^g=a^g[a,g]^g=a[a,g][a,g]^g$. Thus $[a,g][a,g]^g=1_G$. Fix $n\in N$. Observe that

$$a^{gn}=(a^g)^n=(a[a,g])^n=a^n[a,g]^n=a[a,g]^n. \tag{3.6}$$

Also observe that

$$a^{ng[g,n]}=(a^n)^{g[g,n]}=a^{g[g,n]}=(a[a,g])^{[g,n]}=a^{[g,n]}[a,g]^{[g,n]}=a[a,g]^{[g,n]}. \tag{3.7}$$

Since $gn=ng[g,n]$, we have that $a^{gn}=a^{ng[g,n]}$. Thus by (3.6) and (3.7) we have that $[a,g]^n=[a,g]^{[g,n]}$ and so $[a,g]=[a,g]^{(n^{-1})^g}$. Thus $N^g\in \mathbf{C}_G([a,g])$. Since N is normal in G, we conclude that $[a,g]^n=[a,g]$ for any $n\in N$. Similarly, we can check that $[b,g]^n=[b,g]$ for any $n\in N$.

Since $[a,g][a,g]^g=1_G$, $[a,g]^n=[a,g]$ and $[b,g]^n=[b,g]$ for all $n\in N$, and $ab=ba$, we have

$$\begin{aligned}(ab[a,g])^g &= (ba[a,g])^g=b^ga^g[a,g]^g\\ &= b[b,g]a[a,g][a,g]^g=ba[b,g][a,g][a,g]^g\\ &= ab[b,g].\end{aligned}$$

Thus $(ab[a,g])^G=\{ab[a,g],ab[b,g]\}$ since $[a,g]^n=[a,g]$ and $[b,g]^n=[b,g]$ for all $n\in N$, $|G:N|=2$ and $g\in G\setminus N$.

Since $(ab)^G=\{ab,ab[a,g][b,g]\}$, $[a,g]\neq 1$ and $[b,g]\neq 1$, we conclude that $\{ab,ab[a,g]b,g]\}$ and $\{ab[a,g],ab[b,g]\}$ are two distinct conjugacy classes. By (3.5) we have then that $a^Gb^G=\{ab,ab[a,g][b,g]\}\cup\{ab[a,g],ab[b,g]\}$. Therefore $\eta(a^Gb^G)=2$. □

Proposition 3.8. *Let G be a finite supersolvable group, a^G and b^G be conjugacy classes of G with $\mathbf{C}_G(a)=\mathbf{C}_G(b)$ and $|a^G|=2^n$ for some integer $n>0$. Then $\eta(a^Gb^G)\geq 2$.*

Proof. Let G be a supersolvable group, a^G and b^G be conjugacy classes with $\mathbf{C}_G(a)=\mathbf{C}_G(b)$ and $|a^G|=2^n$ for some integer $n>0$. Assume that the statement is false and G, a^G and b^G are a minimal counterexample of the statement with respect to $|a^G|$, i.e. $\eta(a^Gb^G)=1$ and for any supersolvable group K, any conjugacy classes c^K and d^K of K such that $\mathbf{C}_K(c)=\mathbf{C}_K(d)$ and $|c^K|=2^m$, where $0<m<n$, then necessarily $\eta(c^Kd^K)\geq 2$. We are going to conclude that then $2\leq 1$ and thus the statement holds.

By Theorem A, the set $[ab,G]$ is a normal subgroup of G. Let N be a normal subgroup of G contained in $[ab,G]$ with $|N|=2$. Observe such subgroup exists since $[ab,G]$ is a normal subgroup of G and G is supersolvable. Consider now the group G/N. Since $N\leq[ab,G]$ and $|N|=2$, we can check that $|(aN)^{G/N}|=|a^G|/2$. Since $\eta(a^Gb^G)=1$, by Proposition 3.4 we have that $|a^G|=2^n>2$, that is $n>1$, and therefore $|(aN)^{G/N}|=2^{n-1}>1$, i.e. $n-1>0$. Observe that $\mathbf{C}_{G/N}(aN)=\mathbf{C}_{G/N}(bN)$ because $\mathbf{C}_G(a)=\mathbf{C}_G(b)$. Since $|(aN)^{G/N}|=2^{n-1}<|a^G|$ with $n-1>0$, we have that $\eta((aN)^{G/N}(bN)^{G/N})\geq 2$. By Lemma 3.1 we have that $\eta((aN)^{G/N}(bN)^{G/N})\leq\eta(a^Gb^G)=1$ and thus $2\leq 1$. □

Corollary 3.9. *Let G be a finite nilpotent group and a^G be a conjugacy class G. If $a^Ga^G=(a^2)^G$, then $|a^G|$ is an odd number.*

4. Examples

Lemma 4.1. *Let G and K be finite groups, a^G be the conjugacy class of a in G and b^K be the conjugacy class of b in K. Assume that $\eta(a^G a^G) = 1$ and $\eta(b^K b^K) = 1$. Let $G \times K$ be the direct product of G and K. Then $\eta((a,b)^{G\times K}(a,b)^{G\times K}) = 1$ and $|(a,b)^{G\times K}| = |a^G||b^K|$.*

Proof. By definition of direct product, we have that $(a,b)^{G\times K} = \{(x,y) \mid x \in a^G, y \in b^K\}$ and $(a,b)^{G\times K} = \{(xu,yv) \mid x,u \in a^G \text{ and } y,v \in b^K\}$. Thus $|(a,b)^{G\times K}| = |a^G||b^K|$ and $\eta((a,b)^{G\times K}(a,b)^{G\times K}) = 1$. □

Lemma 4.2. *Let p be a prime number and $n > 0$. Let E be an extraspecial group of order p^3 and exponent p. Let $G = E \times E \cdots \times E$ be the direct product of n copies of E. Fix $a = (e_1, e_2, \ldots, e_p) \in G$, where $e_i \in E \setminus \mathbf{Z}(E)$ for $i = 1, \ldots p$. Then*

$$a^G = \{az \mid z \in \mathbf{Z}(G)\}. \tag{4.3}$$

Thus $|a^G| = p^n$ and $a^G a^G = (a^2)^G$. Therefore given any prime p and any integer $n > 0$, there exist a p-group G and a conjugacy class a^G such that $a^G a^G = (a^2)^G$ and $|a^G| = p^l$.

Proof. Since G is the direct product of n copies of E, then $\mathbf{Z}(G) = \mathbf{Z}(E) \times \cdots \times \mathbf{Z}(E)$ and thus $|\mathbf{Z}(G)| = p^n$. We can check that given any $z \in \mathbf{Z}(G)$, there exist some $g \in G$ such that $a^g = az$. Also, given any $g \in G$, there exists some $z \in \mathbf{Z}(G)$ such that $a^g = az$. Thus (4.3) follows and the proof is now complete. □

Proposition 4.4. *Given any odd integer $n \geq 1$, there exist a nilpotent group G and a conjugacy class a^G such that $a^G a^G = (a^2)^G$ and $|a^G| = n$.*

Proof. It follows from Lemmas 4.1 and 4.2. □

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