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PERTURBATION RESULTS FOR PROJECTING A POINT ONTO A LINEAR MANIFOLD*

JIU DING[†]

Abstract. Some new results will be presented on the perturbation analysis for the orthogonal projection of a point onto a linear manifold. The obtained perturbation upper bound is with respect to the distance from the perturbed solution to the unperturbed manifold.

Key words. projection, least squares, generalized inverse

AMS subject classifications. 15A09, 65F20

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1. Introduction. Let $A \in R^{m \times n}$ be an $m \times n$ matrix, and let $b \in R^m$ be an m dimensional vector. We consider the following perturbation problem of the orthogonal projection of a point onto a linear manifold:

$$(1) \quad \min \|p - x\|, \text{ subject to } \|Ax - b\| = \min_{z \in R^n} \|Az - b\|,$$

where $p \in R^n$ is a fixed point and the norm is the usual Euclidean 2-norm. The collection of all vectors x satisfying the constraint in (1) will be called the feasible set and its elements will be called feasible solutions of (1).

Solving the problem (1) is important in many applications, which include the usual minimal norm least squares problem ($p = 0$) and interior point methods ($b = 0$) in which the main work is the projection of the negative gradient of some so-called potential function onto the null space of some scaled matrix (see, e.g., [8]).

The unique optimal solution to (1) is given by

$$x = x(A, b, p) = A^\dagger b + (I - A^\dagger A)p,$$

where $A^\dagger$ is the Moore–Penrose generalized inverse of A ; see, e.g., Theorem 3.6.2 of [3]. Thus, we have a useful expression

$$(2) \quad p - x = A^\dagger(Ap - b)$$

for the difference $p - x$ of the point p and its projection x . Such a relation between a point and its projection will be used several times in the remainder of this paper.

Since the above expression of the solution x to (1) involves the generalized inverse $A^\dagger$ of A , from the well-known discontinuity property of $A \rightarrow A^\dagger$, it is obvious that x is discontinuous at (A, b, p) with A rank-deficient. When the perturbation is rank-preserving, error estimates have been given in [1], [2], and [7] in the special case that $b \in R(A)$, and in [4] in the general case.

When the perturbation is arbitrary, that is, when it may be rank-increasing, the upper continuity property of the projection and a corresponding error estimate were obtained in [4]. However, because of a special and tedious construction, this upper bound contains an unusual “relative condition number” $\tilde{\kappa} = \|A\| \|B^\dagger\|$ of a full-ranked

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submatrix B of A with respect to A , where $\|A\|$ is the induced Euclidean matrix norm, and a general relation between $\tilde{\kappa}$ and the condition number $\kappa = \|A\|\|A^\dagger\|$ of A is unknown. In this paper, we shall get a better error bound without using the relative condition number approach. It seems that this bound is optimal since it is with regard to the minimal distance of the perturbed solution to the manifold of the unperturbed problem, and it also looks beautiful in format since it will be reduced to well-known results in special cases. The main result in the paper has two parts. Part one covers consistent linear systems and part two covers the general systems, and they will be presented in sections 2 and 3, respectively.

2. Error bound: $b \in R(A)$. Suppose the problem (1) is perturbed to

$$(3) \quad \begin{aligned} & \min \quad \|(p+q) - y\|, \\ & \text{subject to } \|(A+E)y - (b+e)\| = \min_{z \in R^n} \|(A+E)z - (b+e)\|. \end{aligned}$$

In the following, the optimal solutions to (1) and (3) will be denoted by x^* and y^* , respectively. In this section, we assume that both linear systems $Ax = b$ and $(A+E)y = b+e$ are consistent. We always assume that $x \neq 0$ whenever $\|x\|$ appears in the denominator.

THEOREM 2.1. *Suppose $b \in R(A)$ and $b+e \in R(A+E)$. If $\|A^\dagger E\| < 1$, then there is a feasible solution x to (1) such that*

$$(4) \quad \frac{\|y^* - x\|}{\|x\|} \leq \frac{\kappa}{1 - \|A^\dagger E\|} \left(\frac{\|e\|}{\|b\|} + \frac{\|E\|}{\|A\|} \right).$$

If, in addition, $\text{Rank}(A+E) = \text{Rank}(A)$ and $\|A^\dagger\|\|E\| < 1$, then

$$(5) \quad \begin{aligned} \frac{\|y^* - x^*\|}{\|x^*\|} & \leq \frac{\|q\|}{\|x^*\|} + \kappa \left[\left(\frac{\|p\|}{\|b\|} + \frac{1+\sqrt{5}}{2} \frac{\|A^\dagger\|}{1 - \|A^\dagger\|\|E\|} \right) \|E\| \right. \\ & \quad \left. + \frac{\|e\|}{(1 - \|A^\dagger\|\|E\|)\|b\|} \right]. \end{aligned}$$

Proof. Let x be the orthogonal projection of y^* onto the feasible set of (1). Then, replacing p with y^* in (2), we get

$$y^* - x = A^\dagger(Ay^* - b) = A^\dagger(e - Ey^*),$$

since $(A+E)y^* = b+e$. Hence,

$$(I + A^\dagger E)(y^* - x) = A^\dagger(e - Ex).$$

Now

$$0 < (1 - \|A^\dagger E\|)\|y^* - x\| \leq \|(I + A^\dagger E)(y^* - x)\|$$

by the assumption. Therefore,

$$\begin{aligned} \frac{\|y^* - x\|}{\|x\|} & \leq \frac{1}{1 - \|A^\dagger E\|} \|A^\dagger\| \frac{\|e - Ex\|}{\|x\|} \\ & \leq \frac{1}{1 - \|A^\dagger E\|} \|A^\dagger\| \frac{\|e\| + \|Ex\|}{\|x\|} \\ & = \frac{\kappa}{1 - \|A^\dagger E\|} \frac{\|e\| + \|Ex\|}{\|A\|\|x\|} \\ & \leq \frac{\kappa}{1 - \|A^\dagger E\|} \left(\frac{\|e\|}{\|b\|} + \frac{\|E\|}{\|A\|} \right). \end{aligned}$$

This proves (4). To prove (5), subtracting the equality

$$p + q - y^* = (A + E)^\dagger [(A + E)(p + q) - (b + e)]$$

from the equality $p - x^* = A^\dagger(Ap - b)$, we have

$$\begin{aligned} y^* - x^* &= q + A^\dagger Ap - (A + E)^\dagger(A + E)(p + q) + (A + E)^\dagger(b + e) - A^\dagger b \\ &= [I - (A + E)^\dagger(A + E)]q + [A^\dagger A - (A + E)^\dagger(A + E)]p \\ &\quad + [(A + E)^\dagger - A^\dagger]b + (A + E)^\dagger e. \end{aligned}$$

Since $I - (A + E)^\dagger(A + E)$ is an orthogonal projector,

$$\begin{aligned} \|y^* - x^*\| &\leq \|q\| + \|(A + E)^\dagger(A + E) - A^\dagger A\| \|p\| \\ (6) \quad &\quad + \|(A + E)^\dagger - A^\dagger\| \|b\| + \|(A + E)^\dagger\| \|e\|. \end{aligned}$$

Now, since $\text{Rank}(A + E) = \text{Rank}(A)$, from [6],

$$\begin{aligned} \|(A + E)^\dagger(A + E) - A^\dagger A\| &\leq \|A^\dagger\| \|E\|, \\ \|(A + E)^\dagger - A^\dagger\| &\leq \frac{1 + \sqrt{5}}{2} \|(A + E)^\dagger\| \|A^\dagger\| \|E\|, \end{aligned}$$

and

$$\|(A + E)^\dagger\| \leq \frac{\|A^\dagger\|}{1 - \|A^\dagger\| \|E\|}.$$

Thus, by (6), and noting that $\|A\| \|x^*\| \geq \|Ax^*\| = \|b\|$, we have

$$\begin{aligned} \frac{\|y^* - x^*\|}{\|x^*\|} &\leq \frac{\|q\|}{\|x^*\|} + \frac{\|A^\dagger\| \|E\| \|p\|}{\|x^*\|} + \frac{1 + \sqrt{5}}{2} \frac{\|A^\dagger\|^2 \|E\| \|b\|}{(1 - \|A^\dagger\| \|E\|) \|x^*\|} \\ &\quad + \frac{\|A^\dagger\| \|e\|}{(1 - \|A^\dagger\| \|E\|) \|x^*\|} \\ &\leq \frac{\|q\|}{\|x^*\|} + \frac{\kappa \|p\| \|E\|}{\|A\| \|x^*\|} + \frac{1 + \sqrt{5}}{2} \frac{\kappa \|A^\dagger\| \|E\| \|b\|}{(1 - \|A^\dagger\| \|E\|) \|A\| \|x^*\|} \\ &\quad + \frac{\kappa \|e\|}{(1 - \|A^\dagger\| \|E\|) \|A\| \|x^*\|} \\ &\leq \frac{\|q\|}{\|x^*\|} + \kappa \left[\left(\frac{\|p\|}{\|b\|} + \frac{1 + \sqrt{5}}{2} \frac{\|A^\dagger\|}{1 - \|A^\dagger\| \|E\|} \right) \|E\| \right. \\ &\quad \left. + \frac{\|e\|}{\|b\| (1 - \|A^\dagger\| \|E\|)} \right]. \quad \square \end{aligned}$$

Remark 2.1. In the special case that A^{-1} exists, the feasible set of (1) just consists of $\{x^*\}$, and (4) is reduced to

$$\frac{\|y^* - x^*\|}{\|x^*\|} \leq \frac{\kappa}{1 - \|A^{-1}E\|} \left(\frac{\|e\|}{\|b\|} + \frac{\|E\|}{\|A\|} \right),$$

which is a classic result in numerical linear algebra.

Remark 2.2. It was not endeavored to get an optimal upper bound for (5). As pointed out by one referee, using $p - x^* = A^\dagger(Ap - b)$, from (8) of [4] a bound slightly different and a little better than (5) can be obtained.

3. Error bound: $b \notin R(A)$. Now we drop the assumption that $b \in R(A)$ and $b + e \in R(A + E)$.

THEOREM 3.1. *Suppose $\|A^\dagger\|\|E\| < 1$. Then there is a feasible solution x to (1) such that*

$$(7) \quad \frac{\|y^* - x\|}{\|x\|} \leq \frac{\kappa}{1 - \|A^\dagger\|\|E\|} \left[\frac{\|A^\dagger\|\|Ax - b\|\|E\| + 2\|e\|}{\|A\|\|x\|} + 2\frac{\|E\|}{\|A\|} \right].$$

If, in addition, $\text{Rank}(A + E) = \text{Rank}(A)$ and $\|A^\dagger\|\|E\| < 1$, then

$$(8) \quad \begin{aligned} \frac{\|y^* - x^*\|}{\|x^*\|} &\leq \frac{\|q\|}{\|x^*\|} + \kappa \left[\left(\frac{\|p\|}{\|A\|\|x^*\|} + \frac{1 + \sqrt{5}}{2} \frac{\|A^\dagger\|\|b\|}{(1 - \|A^\dagger\|\|E\|)\|A\|\|x^*\|} \right) \|E\| \right. \\ &\quad \left. + \frac{\|e\|}{(1 - \|A^\dagger\|\|E\|)\|A\|\|x^*\|} \right]. \end{aligned}$$

Proof. Let x be the orthogonal projection of y^* onto the feasible set of (1). Denote

$$r = (A + E)y^* - (b + e).$$

Then

$$(9) \quad (A + E)^\dagger r = 0$$

and

$$(10) \quad \begin{aligned} \|r\| &= \|(A + E)y^* - (b + e)\| \leq \|(A + E)x - (b + e)\| \\ &\leq \|Ax - b\| + \|e - Ex\|, \end{aligned}$$

since y^* is a least squares solution of the system $(A + E)y = b + e$. Thus,

$$(11) \quad A^\dagger r = [A^\dagger - (A + E)^\dagger] r,$$

from which, together with

$$y^* - x = A^\dagger(Ay^* - b) = A^\dagger(r + e - Ey^*) = A^\dagger[r + e - E(y^* - x) - Ex],$$

$$(12) \quad \begin{aligned} (I + A^\dagger E)(y^* - x) &= A^\dagger r + A^\dagger(e - Ex) \\ &= [A^\dagger - (A + E)^\dagger] r + A^\dagger(e - Ex). \end{aligned}$$

On the other hand, from the decomposition (see Theorem 8.5 of [5])

$$\begin{aligned} A^\dagger - (A + E)^\dagger &= A^\dagger E(A + E)^\dagger - A^\dagger(A^\dagger)^T E^T [I - (A + E)(A + E)^\dagger] \\ &\quad - (I - A^\dagger A) E^T [(A + E)^\dagger]^T (A + E)^\dagger \end{aligned}$$

and (9), we obtain

$$[A^\dagger - (A + E)^\dagger] r = -A^\dagger(A^\dagger)^T E^T r,$$

from which it follows that

$$(13) \quad \|[A^\dagger - (A + E)^\dagger] r\| \leq \|A^\dagger\|^2 \|E\| \|r\|.$$

Therefore, by (12), (13), and (10),

$$\begin{aligned}
\frac{\|y^* - x\|}{\|x\|} &\leq \frac{\|A^\dagger\|}{1 - \|A^\dagger E\|} \frac{\|A^\dagger\| \|E\| \|r\| + \|e - Ex\|}{\|x\|} \\
&\leq \frac{\|A^\dagger\|}{1 - \|A^\dagger E\|} \frac{\|A^\dagger\| \|E\| (\|Ax - b\| + \|e - Ex\|) + \|e - Ex\|}{\|x\|} \\
&\leq \frac{\|A^\dagger\|}{1 - \|A^\dagger E\|} \frac{\|A^\dagger\| \|E\| (\|Ax - b\| + \|e\| + \|Ex\|) + \|e\| + \|Ex\|}{\|x\|} \\
&\leq \frac{\kappa}{1 - \|A^\dagger E\|} \left[\frac{\|A^\dagger\| \|E\| \|Ax - b\| + \|e\| (\|A^\dagger\| \|E\| + 1)}{\|A\| \|x\|} \right. \\
&\quad \left. + \frac{\|E\| (\|A^\dagger\| \|E\| + 1)}{\|A\|} \right] \\
&\leq \frac{\kappa}{1 - \|A^\dagger E\|} \left[\frac{\|A^\dagger\| \|Ax - b\| \|E\| + 2\|e\|}{\|A\| \|x\|} + 2 \frac{\|E\|}{\|A\|} \right].
\end{aligned}$$

This proves (7), and (8) can be proved in the same way as for (5). $\square$

Remark 3.1. Actually we have proved that, for any feasible solution y of (3), there is a feasible solution x of (1) such that

$$\frac{\|y^* - x\|}{\|x\|} \leq \frac{\kappa}{1 - \|A^\dagger E\|} \left[\frac{\|A^\dagger\| \|Ax - b\| \|E\| + 2\|e\|}{\|A\| \|x\|} + 2 \frac{\|E\|}{\|A\|} \right].$$

Remark 3.2. Another upper bound for the rank-preserving perturbation was given in [4] (see Theorem 3.1 in [4]).

COROLLARY 3.1. *If, in addition, $b \in R(A)$, then*

$$(14) \quad \frac{\|y^* - x\|}{\|x\|} \leq \frac{2\kappa}{1 - \|A^\dagger E\|} \left(\frac{\|e\|}{\|b\|} + \frac{\|E\|}{\|A\|} \right).$$

Remark 3.3. The difference between (14) and (4) is that here the perturbed system of linear equations in the problem (3) may not be consistent, while the perturbation in Theorem 2.1 keeps the system consistent.

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