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A Cacciopoli-Type Inequality to Prove Coercivity of a Bilinear Form Associated with Spatial Hysteresis Internal Damping for an Euler-Bernoulli Beam

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Abstract

We prove an inequality that resembles Cacciopoli inequalities in that it bounds the norm of the derivative of a function by using the norm of the function. Unlike in Cacciopoli inequalities, there is no restriction on the function, a fact made up for by adding an extra term to the norm of the function. The inequality arose in the proof that a bilinear form associated with spatial hysteresis internal damping for an Euler-Bernoulli beam is coercive.

Keywords: Cacciopoli inequality, Gagliardo-Nirenberg inequality, spatial hysteresis internal damping, Euler-Bernoulli beam, coercivity

2010 MSC: 35A02, 35A23, 35A24, 46N20, 93C20

1. Introduction

Let $L > 0$, let $p \in [1, \infty)$, let $W^{1,p}[0, L]$ denote the Sobolev space of real-valued p -integrable weakly differentiable functions with p -integrable weak derivative and let $W_0^{1,p}[0, L]$ be the subspace of $W^{1,p}[0, L]$ -functions that are zero on the boundary of the domain. The Poincaré inequality says that there

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is a $c_P > 0$ so that, for all $\phi \in W^{1,p}[0, L]$, we have that

$$\int_0^L |\phi(x) - \bar{\phi}|^p dx \leq c_P \int_0^L |\phi'(x)|^p dx,$$

where $\bar{\phi} = \frac{1}{L} \int_0^L \phi(x) dx$.

The Friedrichs inequality says that there is a $c_F > 0$ so that, for all $\phi \in W_0^{1,p}[0, L]$, we have that

$$\int_0^L |\phi(x)|^p dx \leq c_F \int_0^L |\phi'(x)|^p dx.$$

In a Cacciopoli inequality, the comparability in the Poincaré inequality or the Friedrichs inequality is reversed. Easy examples show that this reversal is not possible for all functions in $W^{1,p}[0, L]$ or $W_0^{1,p}[0, L]$, respectively.

In this note, we prove the following Cacciopoli-type inequality. (The interval $[0, L]$ is chosen for convenience only and can be replaced with any finite interval $[a, b]$. Scaling up to multidimensional domains, if possible at all, is more sophisticated than a simple application of Fubini's Theorem, because of the double integral on the left side.)

Theorem 1.1. *Let $L > 0$ and $p \geq 1$. There is a constant $C_{L,p} > 0$ so that, for all functions $\phi \in W^{1,p}[0, L]$, we have that*

$$\int_0^L \int_0^L |\phi'(x) - \phi'(\xi)|^p dx d\xi + \int_0^L |\phi(x)|^p dx \geq C_{L,p} \int_0^L |\phi'(x)|^p dx.$$

Unlike for the Cacciopoli inequalities we found in the literature, there are no restrictions on $\phi \in W^{1,p}[0, L]$. This freedom comes at the price of needing the extra term $\int_0^L \int_0^L |\phi'(x) - \phi'(\xi)|^p dx d\xi$ on the left side. Because the left side is a sum of two terms, the inequality could also be considered a relative of the Gagliardo-Nirenberg inequality. However, the Gagliardo-Nirenberg inequality involves L^p norms with four different values for p , whereas this inequality stays with one p . Consideration of straight lines $\phi(x) = ax$ shows that, just like the integral $\int_0^L |\phi(x)|^p dx$, the extra term is not solely responsible for the truth of the inequality in Theorem 1.1.

The inequality in Theorem 1.1 arose in the modeling of an Euler-Bernoulli beam. The governing equation for an Euler-Bernoulli beam is a fourth order partial differential equation in space. There are numerous types of damping that can reasonably be used in the modeling of different types of beams, depending on the application. For beams made of composite materials, such as a fiber material embedded in a matrix, spatial hysteresis internal damping, introduced by Russell (see [3]), has been shown to be the most appropriate form of internal damping (see [2]). Two of the authors (Evans, Walters) are considering this damping in an extension of work related to modeling and control of flexible wing micro aerial vehicles (see, for example, [4], [5], [6], [8], [9]). As this new research direction is being pursued, it is important that the model be well-posed so that existence of a unique solution that depends continuously on initial data is guaranteed. Otherwise, proceeding with numerical approximations and simulations, or, beyond that, control and other objectives, is fraught with peril.

A partial differential equation can be transformed into an ordinary differential equation involving differential and differential-integro operators, which are derived from bilinear forms. Some conditions that the bilinear forms must satisfy for a system to be well-posed are given in [1]. In the construction of said bilinear forms, for the problem two of the authors consider, the following terms arise.

Definition 1.2. *Let $L > 0$ and let $h \in L^2[0, L]^2$ be a kernel function. For all $\psi \in L^2[0, L]$, we define*

$$\nu[\psi](x) := \int_0^L h(x, \xi) d\xi \psi(x) \quad \text{and} \quad \mathcal{G}[\psi](x) := \int_0^L h(x, \xi)\psi(\xi) d\xi.$$

Clearly, for $\phi, \psi \in L^2[0, L]$, the L^2 -inner products $\langle \nu[\psi], \phi \rangle$ and $\langle \mathcal{G}[\psi], \phi \rangle$ are bilinear forms and so is their difference. For the bilinear form $\langle (\nu - \mathcal{G})[\psi], \phi \rangle$ associated with spatial hysteresis internal damping, the kernel function h in ν and \mathcal{G} is so that, for all $(x, \xi) \in [0, L]^2$, we have $h(x, \xi) = h(\xi, x)$, and, there are $\kappa, \mu > 0$ so that, for all $(x, \xi) \in [0, L]^2$, we have $\kappa \leq h(x, \xi) \leq \mu$. To assure that the bilinear form associated with spatial hysteresis internal damping for an Euler-Bernoulli beam is coercive with respect to the damping space $H^1[0, L]$ and the state space $L^2[0, L]$ (see [1] for more details), there must be a $C_{L,2} > 0$ so that, for all functions $\phi \in H^1[0, L] = W^{1,2}[0, L]$ that

satisfy $\phi(0) = 0$, we have that

$$\int_0^L (\nu - \mathcal{G})[\phi'](x)\phi'(x) dx + \int_0^L |\phi(x)|^2 dx \geq C_{L,2} \int_0^L |\phi'(x)|^2 dx.$$

The inequality above follows from Theorem 1.1 because Proposition 1.3 below shows that, for symmetric kernels, the first term above can safely be replaced with the simpler term we use in Theorem 1.1. Hence, the model under consideration is well-posed. To our knowledge, this is the first time a formal proof of the above inequality appears in the literature.

Proposition 1.3. *Let $L > 0$ and let $h \in L^2[0, L]^2$ be a kernel function so that, for all $(x, \xi) \in [0, L]^2$, we have $h(x, \xi) = h(\xi, x)$ and so that there are $\kappa, \mu > 0$ so that, for all $(x, \xi) \in [0, L]^2$, we have $\kappa \leq h(x, \xi) \leq \mu$. Then, for all $\psi \in L^2[0, L]$, we have that*

$$\begin{aligned} \frac{\kappa}{2} \int_0^L \int_0^L (\psi(x) - \psi(\xi))^2 dx d\xi &\leq \int_0^L (\nu - \mathcal{G})[\psi](x)\psi(x) dx \\ &\leq \frac{\mu}{2} \int_0^L \int_0^L (\psi(x) - \psi(\xi))^2 dx d\xi. \end{aligned}$$

Proof. First note the following.

$$\begin{aligned} &\int_0^L (\nu - \mathcal{G})[\psi](x)\psi(x) dx \\ &= \int_0^L \left(\int_0^L h(x, \xi) d\xi \psi(x) - \int_0^L h(x, \xi)\psi(\xi) d\xi \right) \psi(x) dx \\ &= \int_0^L \int_0^L h(x, \xi) ((\psi(x))^2 - \psi(\xi)\psi(x)) d\xi dx \\ &= \frac{1}{2} \int_0^L \int_0^L h(x, \xi) ((\psi(x))^2 - \psi(\xi)\psi(x)) d\xi dx \\ &\quad + \frac{1}{2} \int_0^L \int_0^L h(\xi, x) ((\psi(\xi))^2 - \psi(x)\psi(\xi)) dx d\xi \\ &= \frac{1}{2} \int_0^L \int_0^L h(x, \xi) ((\psi(x))^2 - \psi(\xi)\psi(x)) d\xi dx \\ &\quad + \frac{1}{2} \int_0^L \int_0^L h(x, \xi) (-\psi(\xi)\psi(x) + (\psi(\xi))^2) d\xi dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \int_0^L \int_0^L h(x, \xi) ((\psi(x))^2 - 2\psi(\xi)\psi(x) + (\psi(\xi))^2) d\xi dx \\
&= \frac{1}{2} \int_0^L \int_0^L h(x, \xi) (\psi(x) - \psi(\xi))^2 d\xi dx
\end{aligned}$$

The inequalities now follow from $0 < \kappa \leq h \leq \mu$. \square

Note that one, maybe even surprising, consequence of Proposition 1.3 is that $\int_0^L (\nu - \mathcal{G})[\psi](x)\psi(x) dx$ is nonnegative. For kernels without the symmetry condition $h(x, \xi) = h(\xi, x)$, this need not be the case as the functions

$$h(x, \xi) := \begin{cases} 2; & \text{for } \xi > x, \\ 1; & \text{for } \xi \leq x, \end{cases} \quad \text{and} \quad \psi(x) := \begin{cases} 1 & \text{for } 0 \leq x < \frac{1}{2}, \\ \frac{3}{2} & \text{for } \frac{1}{2} \leq x \leq 1, \end{cases}$$

show. For these functions, we have

$$\begin{aligned}
&\int_0^1 \int_0^1 h(x, \xi)\psi(x)(\psi(x) - \psi(\xi)) d\xi dx \\
&= \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} h(x, \xi)\psi(x)(\psi(x) - \psi(\xi)) d\xi dx + \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 h(x, \xi)\psi(x)(\psi(x) - \psi(\xi)) d\xi dx \\
&= \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} 1 \cdot \frac{3}{2} \left(\frac{3}{2} - 1 \right) d\xi dx + \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 2 \cdot 1 \left(1 - \frac{3}{2} \right) d\xi dx \\
&= \frac{1}{4} \cdot \frac{3}{4} + \frac{1}{4} (-1) = -\frac{1}{16}
\end{aligned}$$

Hence, in Proposition 1.3, the symmetry condition $h(x, \xi) = h(\xi, x)$ on the kernel cannot be omitted. This mathematical insight is mirrored physically in the fact that (see [3], p. 136) Newton's second law dictates the symmetry condition $h(x, \xi) = h(\xi, x)$.

Although we only needed Theorem 1.1 for $p = 2$ and with the additional boundary conditions $\phi(0) = 0$ and $\phi'(0) = 0$ (clamped beam) imposed, it is stated and proved for p^{th} powers and without boundary conditions. Usually, the generalization from second powers/Hilbert spaces to p^{th} powers/Banach spaces, if it is possible at all, requires significant extra work. However, in the case of Theorem 1.1, the generalization of this particular proof from 2 to p does not require any extra steps, so it is only prudent to provide the more general version. Similarly, because Lemma 2.3, which substantially shortens the final argument, does not require boundary conditions, we avoided the need for boundary conditions.

2. Lemmas

We will see in Section 3 that, when the first term on the left side of the inequality in Theorem 1.1 is “small,” most of the values of ϕ' can be constricted to a relatively narrow range. Hence, in this case, in most of its domain, the function ϕ will rise with a certain slope. Functions like $\phi'(x) := \sum_{k=1}^n (-1)^k \mathbf{1}_{[\frac{k-1}{n}, \frac{k}{n}]}$ show that positive and negative parts of a derivative can be interleaved in such a way that the norm of the derivative stays large, while the norm of the function ϕ can be made arbitrarily small. Although the interleaving for this example leads to a large first term on the left side of the inequality in Theorem 1.1, for estimates involving the second term, it is best to avoid such interleaving, which can be done with a one-dimensional version of the **nondecreasing rearrangement** of a function (see, for example, Chapter 10 of [7]). Rearrangements seem to arise on an “as needed” basis in the literature, so we will at least sketch the construction and also include the proof of Lemma 2.2. Because we will need the nondecreasing rearrangement of a derivative ϕ' , we choose the notation ψ_{\uparrow} for the rearrangement instead of the more common ψ^* , and we adopt the notation $\phi'_{\uparrow} = (\phi')_{\uparrow}$. As is customary, throughout this presentation, λ will denote Lebesgue measure.

Lemma 2.1. *Let $L > 0$. For every measurable function $\psi : [0, L] \rightarrow \mathbb{R}$, there is a nondecreasing function $\psi_{\uparrow} : [0, L] \rightarrow \mathbb{R}$ so that, for all $c \in \mathbb{R}$, we have that $\lambda \{x : \psi(x) < c\} = \lambda \{x : \psi_{\uparrow}(x) < c\}$.*

Proof. (Sketch.) For simple functions $\psi(x) = \sum_{k=1}^n a_k \mathbf{1}_{A_k}$, this is trivial.

For a measurable function ψ that is bounded above, approximate from above with a nonincreasing sequence of simple functions and use the limit of their rearrangements. For a general measurable function ψ , use the limit of the rearrangements of $\min\{\psi, n\}$. \square

Lemma 2.2. *Let $L > 0$ and let $\phi : [0, L] \rightarrow \mathbb{R}$ be an absolutely continuous function with $\phi(0) = 0$. Then, for all $x \in [0, L]$, we have $\phi(x) \geq \int_0^x \phi'_{\uparrow}(t) dt$.*

Proof. Let $x \in [0, L]$. For all $c \in \mathbb{R}$ we have the following.

$$\begin{aligned} \lambda \{t \in [0, x] : \phi'(t) < c\} &\leq \min \{x, \lambda \{t \in [0, L] : \phi'(t) < c\}\} \\ &= \min \{x, \lambda \{t \in [0, L] : \phi'_{\uparrow}(t) < c\}\} \\ &= \lambda \{t \in [0, x] : \phi'_{\uparrow}(t) < c\} \end{aligned}$$

Therefore, for all $y \in \mathbb{R}$, we also have the following inequality.

$$\begin{aligned}
\lambda \{t \in [0, x] : \phi'(t) > y\} &= x - \lambda \{t \in [0, x] : \phi'(t) \leq y\} \\
&= x - \lim_{n \rightarrow \infty} \lambda \left\{ t \in [0, x] : \phi'(t) < y + \frac{1}{n} \right\} \\
&\geq x - \lim_{n \rightarrow \infty} \lambda \left\{ t \in [0, x] : \phi'_\uparrow(t) < y + \frac{1}{n} \right\} \\
&= x - \lambda \{t \in [0, x] : \phi'_\uparrow(t) \leq y\} \\
&= \lambda \{t \in [0, x] : \phi'_\uparrow(t) > y\}
\end{aligned}$$

With these inequalities, we can prove the result.

$$\begin{aligned}
\phi(x) &= \int_0^x \phi'(t) dt = \int_0^x (\phi')^+(t) dt - \int_0^x (\phi')^-(t) dt \\
&= \int_0^\infty \lambda \{t \in [0, x] : \phi'(t) > y\} dy - \int_0^\infty \lambda \{t \in [0, x] : -\phi'(t) > y\} dy \\
&= \int_0^\infty \lambda \{t \in [0, x] : \phi'(t) > y\} dy - \int_0^\infty \lambda \{t \in [0, x] : \phi'(t) < -y\} dy \\
&\geq \int_0^\infty \lambda \{t \in [0, x] : \phi'_\uparrow(t) > y\} dy - \int_0^\infty \lambda \{t \in [0, x] : \phi'_\uparrow(t) < -y\} dy \\
&= \int_0^\infty \lambda \{t \in [0, x] : \phi'_\uparrow(t) > y\} dy - \int_0^\infty \lambda \{t \in [0, x] : -\phi'_\uparrow(t) > y\} dy \\
&= \int_0^x \phi'_\uparrow(t) dt.
\end{aligned}$$

□

The typical proof in Section 3 that uses the second term on the left side of the inequality in Theorem 1.1 occurs in a situation in which ϕ has a “large enough” positive derivative on a “large enough” subset of the interval. Lemma 2.3 below shows that, in this situation, there is a certain lower bound for $\int_0^L |\phi(x)|^p dx$, which we will prove to be large enough in all cases.

Lemma 2.3. *Let $L > 0$, $\varrho > 0$, $\tau \in (0, \frac{1}{2})$, let $\phi : [0, L] \rightarrow \mathbb{R}$ be an absolutely continuous function and let $k \in \{\lambda\{x : \phi'(x) \geq \varrho\}, \lambda\{x : \phi'(x) > \varrho\}\}$ be not*

equal to zero. If $\int_{\{x:\phi'(x)<0\}} |\phi'(x)| dx \leq \varrho\tau k$, then

$$\int_0^L |\phi(x)|^p dx \geq \frac{1}{p+1} \left(\frac{1}{2} - \tau\right)^{p+1} k^{p+1} \varrho^p.$$

Proof. Without loss of generality, assume that $k = \lambda\{x : \phi'(x) \geq \varrho\}$. First consider the case that

$$\lambda\{x : \phi'(x) \geq \varrho \wedge \phi(x) \geq 0\} \geq \frac{1}{2}\lambda\{x : \phi'(x) \geq \varrho\}.$$

Let $s := \inf\{x \in [0, L] : \phi(x) > 0\}$. Because ϕ is continuous, we have $\phi(s) \geq 0$. For all $x \in [0, L - s]$, let $\psi(x) := \phi(x + s)$. Then

$$\begin{aligned} \lambda\{x : \psi'(x) \geq \varrho \wedge \psi(x) \geq 0\} &= \lambda\{x \geq s : \phi'(x) \geq \varrho \wedge \phi(x) \geq 0\} \\ &= \lambda\{x : \phi'(x) \geq \varrho \wedge \phi(x) \geq 0\} \\ &\geq \frac{1}{2}\lambda\{x : \phi'(x) \geq \varrho\} \\ &= \frac{1}{2}k. \end{aligned}$$

Let $a := \lambda\{x : \psi'(x) < \varrho\} + \tau k$. Then

$$\begin{aligned} \int_0^a \psi'_\uparrow(x) dx &\geq \int_0^{\lambda\{x:\psi'(x)<0\}} \psi'_\uparrow(x) dx + \int_{\lambda\{x:\psi'(x)<0\}}^{\lambda\{x:\psi'(x)<0\}+\tau k} \psi'_\uparrow(x) dx \\ &\geq - \int_{\{x:\psi'(x)<0\}} |\psi'(x)| dx + \varrho\tau k \\ &\geq - \int_{\{x:\phi'(x)<0\}} |\phi'(x)| dx + \varrho\tau k \\ &\geq 0. \end{aligned}$$

Now, with $h := \left(\frac{1}{2} - \tau\right)k$, we have $a + h \leq L - s$. Moreover, for all $x \in [a, a + h]$, by Lemma 2.2, because $\psi(0) = \phi(s) \geq 0$, we have that

$$\begin{aligned} \psi(x) &\geq \int_0^x \psi'_\uparrow(t) dt \\ &= \int_0^a \psi'_\uparrow(t) dt + \int_a^x \psi'_\uparrow(t) dt \\ &\geq \varrho(x - a) \end{aligned}$$

Now

$$\begin{aligned}
\int_0^L |\phi(x)|^p dx &\geq \int_0^{L-s} |\psi(x)|^p dx \\
&\geq \int_a^{a+h} |\varrho(x-a)|^p dx \\
&\geq \varrho^p \int_0^h x^p dx \\
&= \varrho^p \frac{1}{p+1} h^{p+1} \\
&= \frac{1}{p+1} \varrho^p \left(\left(\frac{1}{2} - \tau \right) k \right)^{p+1} \\
&= \frac{1}{p+1} \left(\frac{1}{2} - \tau \right)^{p+1} k^{p+1} \varrho^p
\end{aligned}$$

Now consider the case that

$$\lambda\{x : \phi'(x) \geq \varrho \wedge \phi(x) \geq 0\} < \frac{1}{2} \lambda\{x : \phi'(x) \geq \varrho\}.$$

Then, with $\theta(x) := -\phi(L-x)$, we have $\theta'(x) = \phi'(L-x)$ and hence

$$\begin{aligned}
\lambda\{x : \theta'(x) \geq \varrho \wedge \theta(x) > 0\} &= \lambda\{x : \phi'(x) \geq \varrho \wedge \phi(x) < 0\} \\
&> \frac{1}{2} \lambda\{x : \phi'(x) \geq \varrho\} \\
&= \frac{1}{2} \lambda\{x : \theta'(x) \geq \varrho\},
\end{aligned}$$

which means the result follows from the first case applied to θ . \square

The proof of Lemma 2.3 highlights the final difficulty we will face in the proof of Theorem 1.1. To get a lower bound for $\phi(x)$, we need to use an integral of ϕ' . However, for the inequality in Theorem 1.1, we must ultimately connect with the (p^{th} power of the) L^p -norm of ϕ' and all integral conditions that arise in the proof will be stated in terms of the L^p -norm of ϕ' . There are no overall inequalities that relate L^1 -norms and L^p -norms, except that the integral of $|\psi|$ is bounded by the length of the interval plus the integral of $|\psi|^p$. (Another simple insight that we will use frequently.) To obtain workable estimates, it will be advantageous to scale the functions so that the

value ϱ in Lemma 2.3 is within a certain range. The simple Lemma 2.4 shows that, as long as all other conditions behave similarly (which will be indicated explicitly), we are allowed to scale the function ϕ , which means we can scale cutoff values for the function, too.

Lemma 2.4. *Let $L > 0$, $C_{L,p} > 0$ and let $\phi \in W^{1,p}[0, L]$. If there is an $M \neq 0$ so that the inequality in Theorem 1.1 holds for $M\phi$ with $C_{L,p}$ as the constant, then the inequality holds for ϕ with $C_{L,p}$ as the constant, too.*

Proof. This is trivial, because the factor M factors out of all terms as an $|M|^p$ and can thus be canceled. \square

3. Proof of Theorem 1.1

The idea for the proof of Theorem 1.1 is simple: After identifying situations in which the first term on the left side of the inequality in Theorem 1.1 suffices to establish the inequality (Cases 1-3 and “early” estimates in the hierarchy for Case 4), we are left with functions ϕ so that “many” values of ϕ' are restricted to a reasonably “small” range of positive numbers. (Case 5 handles the restriction to a range of negative numbers via Lemma 2.4.) This then allows us to use Lemma 2.3.

The constants we use throughout the proof are chosen to assure that the requisite steps can be established reasonably easily, and not to provide sharp estimates for $C_{L,p}$. Let $\alpha \in (0, \frac{1}{1000 \cdot 2^{2p}})$, $\beta \in (0, \frac{1}{1000000})$.

Case 1. $\lambda \{x : \phi'(x) < 0\} \leq \alpha L$ and $\lambda \{x : \phi'(x) > 0\} \leq \alpha L$.

Because it is not possible that $\int_{\{\xi: \phi'(\xi) < 0\}} |\phi'(\xi)|^p d\xi \leq \beta \int_0^L |\phi'(\xi)|^p d\xi$

and

$\int_{\{\xi: \phi'(\xi) > 0\}} |\phi'(\xi)|^p d\xi \leq \beta \int_0^L |\phi'(\xi)|^p d\xi$, by Lemma 2.4, we can assume without loss of generality that

$$\int_{\{\xi: \phi'(\xi) > 0\}} |\phi'(\xi)|^p d\xi > \beta \int_0^L |\phi'(\xi)|^p d\xi.$$

Hence, in this case, we have

$$\int_0^L \int_0^L |\phi'(x) - \phi'(\xi)|^p dx d\xi \geq \int_{\{\xi: \phi'(\xi)=0\}} \int_{\{x: \phi'(x) > 0\}} |\phi'(x) - \phi'(\xi)|^p dx d\xi$$

$$\begin{aligned}
&= \lambda \{ \xi : \phi'(\xi) = 0 \} \int_{\{x:\phi'(x)>0\}} |\phi'(x)|^p dx \\
&\geq (1 - 2\alpha)L\beta \int_0^L |\phi'(x)|^p dx,
\end{aligned}$$

which implies the desired inequality with $C_{L,p;1} := (1 - 2\alpha)L\beta$.

Case 2. $\lambda \{x : \phi'(x) < 0\} > \alpha L$ and $\int_{\{\xi:\phi'(\xi)>0\}} |\phi'(\xi)|^p d\xi > \beta \int_0^L |\phi'(\xi)|^p d\xi$.

In this case we note the following.

$$\begin{aligned}
\int_0^L \int_0^L |\phi'(x) - \phi'(\xi)|^p dx d\xi &\geq \int_{\{\xi:\phi'(\xi)>0\}} \int_{\{x:\phi'(x)<0\}} |\phi'(\xi) - \phi'(x)|^p dx d\xi \\
&\geq \int_{\{\xi:\phi'(\xi)>0\}} \int_{\{x:\phi'(x)<0\}} |\phi'(\xi)|^p dx d\xi \\
&= \lambda \{x : \phi'(x) < 0\} \int_{\{\xi:\phi'(\xi)>0\}} |\phi'(\xi)|^p d\xi \\
&> \alpha L \beta \int_0^L |\phi'(\xi)|^p d\xi
\end{aligned}$$

Hence the desired inequality holds with $C_{L,p;2} := \alpha L \beta$.

Case 3. $\lambda \{x : \phi'(x) > 0\} > \alpha L$ and $\int_{\{\xi:\phi'(\xi)<0\}} |\phi'(\xi)|^p d\xi > \beta \int_0^L |\phi'(\xi)|^p d\xi$.

By Lemma 2.4 with $M = -1$, it follows from Case 2 that, in Case 3, the desired inequality holds with $C_{L,p;3} := \alpha L \beta$.

The remaining cases. Obviously, the proof will be completed if we can prove the desired inequality for functions that do not satisfy any of the conditions in Cases 1-3. By negating and then combining the conditions in Cases 2 and 3, we obtain 4 pairs of conditions so that a function does not satisfy the conditions in Cases 2 and 3 iff it satisfies one of these four pairs of conditions. The pair of conditions $\lambda \{x : \phi'(x) < 0\} \leq \alpha L$ and $\lambda \{x : \phi'(x) > 0\} \leq \alpha L$ has been discussed in Case 1. In the discussion of Case 1, we have also already noted that the pair of conditions $\int_{\{\xi:\phi'(\xi)<0\}} |\phi'(\xi)|^p d\xi \leq \beta \int_0^L |\phi'(\xi)|^p d\xi$ and $\int_{\{\xi:\phi'(\xi)>0\}} |\phi'(\xi)|^p d\xi \leq \beta \int_0^L |\phi'(\xi)|^p d\xi$ cannot be satisfied. Hence, the

proof will be complete once we have established the desired inequality for functions that satisfy the pairs of conditions in Cases 4 and 5 below. Similar to Cases 2 and 3, by Lemma 2.4 with $M = -1$, Case 5 will follow from Case 4. Hence, the bulk of the proof focuses on Case 4.

Case 4. $\lambda \{x : \phi'(x) < 0\} \leq L\alpha$ and $\int_{\{\xi: \phi'(\xi) < 0\}} |\phi'(\xi)|^p d\xi \leq \beta \int_0^L |\phi'(\xi)|^p d\xi$.

(Note that, if $M > 0$, the defining conditions hold for ϕ iff they hold for $M\phi$. Hence we will be free to scale ϕ and use Lemma 2.4 when necessary.)

This case will have numerous subcases. Because the absolutely continuous functions are dense in $W^{1,p}[0, L]$, we can also assume, without loss of generality, that ϕ is absolutely continuous. Hence, we will be free to apply Lemma 2.3 when needed.

Consider the function

$$f_\phi(y) := \int_{\{x: \phi'(x) \geq y\}} |\phi'(x)|^p dx.$$

Clearly, the function f_ϕ is nonincreasing. Moreover, for every $d \in \mathbb{R}$, we have the following.

$$\begin{aligned} \lim_{y \rightarrow d^-} f_\phi(y) &= \lim_{y \rightarrow d^-} \int_{\{x: \phi'(x) \geq y\}} |\phi'(x)|^p dx = \int_{\bigcap_{y < d} \{x: \phi'(x) \geq y\}} |\phi'(x)|^p dx \\ &= \int_{\{x: \phi'(x) \geq d\}} |\phi'(x)|^p dx = f_\phi(d), \end{aligned}$$

that is, the function f_ϕ is left continuous. Additionally, for every $d \in \mathbb{R}$, we have

$$\begin{aligned} \int_{\{x: \phi'(x) > d\}} |\phi'(x)|^p dx &= \int_{\bigcup_{y > d} \{x: \phi'(x) \geq y\}} |\phi'(x)|^p dx \\ &= \lim_{y \rightarrow d^+} \int_{\{x: \phi'(x) \geq y\}} |\phi'(x)|^p dx \\ &= \lim_{y \rightarrow d^+} f_\phi(y). \end{aligned}$$

Finally,

$$\begin{aligned} f_\phi(0) &= \int_{\{x: \phi'(x) \geq 0\}} |\phi'(x)|^p dx = \int_0^L |\phi'(x)|^p dx - \int_{\{x: \phi'(x) < 0\}} |\phi'(x)|^p dx \\ &\geq (1 - \beta) \int_0^L |\phi'(x)|^p dx \geq (1 - \beta) \int_{\{x: \phi'(x) < 0\}} |\phi'(x)|^p dx. \end{aligned}$$

Depending on whether f_ϕ does or does not have a “large” discontinuity, we will be either in Case 4.1 or in Case 4.2 below. For the case distinction, fix $\delta := 101\beta$.

Case 4.1. *There is no $c > 0$ so that $\int_{\{x:\phi'(x)\geq c\}} |\phi'(x)|^p dx \geq \delta \int_0^L |\phi'(x)|^p dx$ and so that $\int_{\{x:\phi'(x)<c\}} |\phi'(x)|^p dx \geq \delta \int_0^L |\phi'(x)|^p dx$. That is, for every real $c > 0$, exactly one of the conditions $\int_{\{x:\phi'(x)\geq c\}} |\phi'(x)|^p dx < \delta \int_0^L |\phi'(x)|^p dx$ or $\int_{\{x:\phi'(x)<c\}} |\phi'(x)|^p dx < \delta \int_0^L |\phi'(x)|^p dx$ holds.*

(Note that, if $M > 0$, the defining condition holds for ϕ with c iff it holds for $M\phi$ with Mc . Hence we will be free to scale ϕ and c , and use Lemma 2.4 when necessary.)

In this case, the function f_ϕ has a jump discontinuity at some $d > 0$ so that, for $y \leq d$, we have

$$\begin{aligned} f_\phi(y) &= \int_{\{x:\phi'(x)\geq y\}} |\phi'(x)|^p dx \\ &= \int_0^L |\phi'(x)|^p dx - \int_{\{x:\phi'(x)<y\}} |\phi'(x)|^p dx \\ &> (1 - \delta) \int_0^L |\phi'(x)|^p dx \end{aligned}$$

and so that, for $y > d$, we have

$$f_\phi(y) = \int_{\{x:\phi'(x)\geq y\}} |\phi'(x)|^p dx < \delta \int_0^L |\phi'(x)|^p dx.$$

Note that, for any $M > 0$ and any $y > 0$, we have that

$$\begin{aligned} f_{M\phi}(y) &= \int_{\{x:(M\phi)'(x)\geq y\}} |(M\phi)'(x)|^p dx \\ &= M^p \int_{\{x:\phi'(x)\geq \frac{y}{M}\}} |\phi'(x)|^p dx \\ &= M^p f_\phi\left(\frac{y}{M}\right). \end{aligned}$$

Moreover, the condition that defines Case 4 holds for ϕ iff it holds for $M\phi$, and, the condition that defines Case 4.1 holds for ϕ with c iff it holds for $M\phi$ with Mc . By Lemma 2.4, establishing the desired inequality for $M\phi$ establishes it for ϕ , too. Hence, by replacing ϕ with an appropriate $M\phi$, we can assume, without loss of generality, that $d = \frac{1}{2}$.

Before we get to this point, we continue as follows.

$$\begin{aligned}
d^p \lambda\{x : \phi'(x) = d\} &= \int_{\{x:\phi'(x)=d\}} |\phi'(x)|^p dx \\
&= \int_{\{x:\phi'(x)\geq d\}} |\phi'(x)|^p dx - \int_{\{x:\phi'(x)>d\}} |\phi'(x)|^p dx \\
&= f_\phi(d) - \lim_{y \rightarrow d^+} f_\phi(y) \\
&> (1 - 2\delta) \int_0^L |\phi'(x)|^p dx
\end{aligned}$$

Let $\omega = 8\alpha$.

Case 4.1.1. $\lambda\{x : \phi'(x) = d\} < \omega L$.

In this case, first note that

$$\begin{aligned}
d^p \omega L &> d^p \lambda\{x : \phi'(x) = d\} \\
&> (1 - 2\delta) \int_0^L |\phi'(x)|^p dx.
\end{aligned}$$

Because $\int_{\{x:\phi'(x)>d\}} |\phi'(x)|^p dx = \lim_{y \rightarrow d^+} f(y) \leq \delta \int_0^L |\phi'(x)|^p dx$, we obtain

$$\begin{aligned}
d^p \omega L &> (1 - 2\delta) \int_0^L |\phi'(x)|^p dx \\
&\geq \delta \int_0^L |\phi'(x)|^p dx \\
&\geq \int_{\{x:\phi'(x)>d\}} |\phi'(x)|^p dx \\
&\geq d^p \lambda\{x : \phi'(x) > d\},
\end{aligned}$$

that is, $\lambda\{x : \phi'(x) > d\} < \omega L$, too.

Moreover,

$$\begin{aligned}
\int_{\{x: \frac{d}{2} < \phi'(x) < d\}} |\phi'(x)|^p dx &\leq \int_{\{x: \phi'(x) < d\}} |\phi'(x)|^p dx \\
&= \int_0^L |\phi'(x)|^p dx - \int_{\{x: \phi'(x) \geq d\}} |\phi'(x)|^p dx \\
&< \int_0^L |\phi'(x)|^p dx - (1 - \delta) \int_0^L |\phi'(x)|^p dx \\
&= \delta \int_0^L |\phi'(x)|^p dx
\end{aligned}$$

and so, because

$$\begin{aligned}
d^p \omega L &\geq (1 - 2\delta) \int_0^L |\phi'(x)|^p dx \\
&\geq \delta \int_0^L |\phi'(x)|^p dx \\
&> \int_{\{x: \frac{d}{2} < \phi'(x) < d\}} |\phi'(x)|^p dx \\
&\geq \frac{d^p}{2^p} \lambda \left\{ x : \frac{d}{2} < \phi'(x) < d \right\},
\end{aligned}$$

we have $\lambda \left\{ x : \frac{d}{2} < \phi'(x) < d \right\} < 2^p \omega L$.

We conclude that $\lambda \left\{ x : \phi'(x) \leq \frac{d}{2} \right\} > (1 - 2\omega - 2^p \omega) L$, which implies the following.

$$\begin{aligned}
&\int_0^L \int_0^L |\phi'(x) - \phi'(\xi)|^p dx d\xi \\
&\geq \int_{\{\xi: \phi'(\xi) = d\}} \int_{\{x: \phi'(x) \leq \frac{d}{2}\}} |\phi'(x) - \phi'(\xi)|^p dx d\xi \\
&\geq \int_{\{\xi: \phi'(\xi) = d\}} \int_{\{x: \phi'(x) \leq \frac{d}{2}\}} \left(\frac{d}{2}\right)^p dx d\xi \\
&= \lambda \left\{ x : \phi'(x) \leq \frac{d}{2} \right\} \frac{1}{2^p} d^p \lambda \{\xi : \phi'(\xi) = d\} \\
&> (1 - 2\omega - 2^p \omega) L \frac{1}{2^p} (1 - 2\delta) \int_0^L |\phi'(x)|^p dx.
\end{aligned}$$

Hence, in Case 4.1.1, the desired inequality holds with the constant $C_{L,p;4.1.1} := (1 - 2\omega - 2^p\omega) L \frac{1}{2^p} (1 - 2\delta)$.

Case 4.1.2. $\lambda\{x : \phi'(x) = d\} \geq \omega L$.

In this case, we want to apply Lemma 2.3. We will explicitly use, without loss of generality, that $d = \frac{1}{2}$, which means that $d^p \leq d$. Recalling that $\alpha = \frac{\omega}{8}$, we obtain the following.

$$\begin{aligned}
\int_{\{x:\phi'(x)<0\}} |\phi'(x)| dx &\leq \int_{\{x:\phi'(x)<0\}} 1 + |\phi'(x)|^p dx \\
&\leq \lambda\{x : \phi'(x) < 0\} + \int_{\{x:\phi'(x)<0\}} |\phi'(x)|^p dx \\
&\leq \alpha L + \beta \int_0^L |\phi'(x)|^p dx \\
&= \alpha L + \frac{\beta}{1-2\delta} (1-2\delta) \int_0^L |\phi'(x)|^p dx \\
&< \frac{1}{4} \cdot \frac{1}{2} \omega L + \frac{\beta}{1-2\delta} d^p \lambda\{x : \phi'(x) = d\} \\
&\leq \frac{1}{4} \cdot d \omega L + \frac{\beta}{1-2\delta} d \lambda\{x : \phi'(x) = d\} \\
&\leq \left(\frac{1}{4} + \frac{\beta}{1-2\delta} \right) d \lambda\{x : \phi'(x) = d\} \\
&\leq d \left(\frac{1}{4} + \frac{\beta}{1-2\delta} \right) \lambda\{x : \phi'(x) \geq d\}
\end{aligned}$$

With $\varrho := d$, $\tau := \frac{1}{4} + \frac{\beta}{1-2\delta} < \frac{3}{10}$ and $k := \lambda\{x : \phi'(x) \geq d\}$, we obtain the following via Lemma 2.3.

$$\begin{aligned}
\int_0^L |\phi(x)|^p dx &\geq \frac{1}{p+1} \left(\frac{1}{2} - \tau \right)^{p+1} k^{p+1} \varrho^p \\
&\geq \frac{1}{p+1} \left(\frac{1}{2} - \frac{3}{10} \right)^{p+1} k^p d^p \lambda\{x : \phi'(x) \geq d\} \\
&\geq \frac{1}{p+1} \left(\frac{1}{5} \right)^{p+1} (\omega L)^p (1-2\delta) \int_0^L |\phi'(x)|^p dx \\
&= \frac{\omega^p L^p (1-2\delta)}{(p+1) 5^{p+1}} \int_0^L |\phi'(x)|^p dx
\end{aligned}$$

Hence, in Case 4.1.2, the desired inequality holds with the constant $C_{L,p;4.1.2} := \frac{\omega^p L^p (1-2\delta)}{(p+1)5^{p+1}}$.

Case 4.2. *There is a $c > 0$ so that $\int_{\{x:\phi'(x) \geq c\}} |\phi'(x)|^p dx \geq \delta \int_0^L |\phi'(x)|^p dx$ and so that $\int_{\{x:\phi'(x) < c\}} |\phi'(x)|^p dx \geq \delta \int_0^L |\phi'(x)|^p dx$.*

(Note that, if $M > 0$, the defining condition holds for ϕ with c iff it holds for $M\phi$ with Mc . Hence we will be free to scale ϕ and c , and use Lemma 2.4 when necessary.)

In this case, let

$$\eta := \left(\frac{1}{2}\right)^{\frac{1}{p}} \in (0, 1)$$

be fixed.

$$\begin{aligned} & \int_0^L \int_0^L |\phi'(x) - \phi'(\xi)|^p dx d\xi \\ & \geq \int_0^L \int_{\{x:\phi'(\xi) \leq \eta\phi'(x) \wedge \phi'(x) > 0\}} |\phi'(x) - \phi'(\xi)|^p dx d\xi \\ & \geq \int_0^L \int_{\{x:\phi'(\xi) \leq \eta\phi'(x) \wedge \phi'(x) > 0\}} (1-\eta)^p |\phi'(x)|^p dx d\xi \\ & \geq \int_{\{\xi:\phi'(\xi) \leq c\eta\}} \int_{\{x:\phi'(x) \geq \frac{\phi'(\xi)}{\eta} \wedge \phi'(x) > 0\}} (1-\eta)^p |\phi'(x)|^p dx d\xi \\ & \quad \text{Now use that } c \geq \frac{\phi'(\xi)}{\eta} \\ & \geq \int_{\{\xi:\phi'(\xi) \leq c\eta\}} (1-\eta)^p \int_{\{x:\phi'(x) \geq c\}} |\phi'(x)|^p dx d\xi \\ & \geq \int_{\{\xi:\phi'(\xi) \leq c\eta\}} (1-\eta)^p \delta \int_0^L |\phi'(x)|^p dx d\xi \\ & = \lambda \{\xi : \phi'(\xi) \leq c\eta\} (1-\eta)^p \delta \int_0^L |\phi'(x)|^p dx \end{aligned}$$

Now we either have $\lambda \{\xi : \phi'(\xi) \leq c\eta\} > \alpha L$ or $\lambda \{\xi : \phi'(\xi) \leq c\eta\} \leq \alpha L$.

Case 4.2.1. $\lambda \{\xi : \phi'(\xi) \leq c\eta\} > \alpha L$.

In this case, the above derivation shows that

$$\int_0^L \int_0^L |\phi'(x) - \phi'(\xi)|^p dx d\xi \geq \alpha L (1 - \eta)^p \delta \int_0^L |\phi'(x)|^p dx,$$

which means that, in Case 4.2.1, the desired inequality holds with the constant $C_{L,p;4.2.1} := \alpha L (1 - \eta)^p \delta$.

Case 4.2.2. $\lambda \{\xi : \phi'(\xi) \leq c\eta\} \leq \alpha L$.

(Note that, if $M > 0$, the defining condition holds for ϕ with c iff it holds for $M\phi$ with Mc . Hence we will be free to scale ϕ and c , and use Lemma 2.4 when necessary.)

First note that we have $\lambda \{\xi : \phi'(\xi) > c\eta\} \geq (1 - \alpha)L$ and

$$\begin{aligned} c^p L &\geq \int_{\{x:0 < \phi'(x) < c\}} |\phi'(x)|^p dx \\ &= \int_{\{x:\phi'(x) < c\}} |\phi'(x)|^p dx - \int_{\{x:\phi'(x) < 0\}} |\phi'(x)|^p dx \\ &\geq \delta \int_0^L |\phi'(x)|^p dx - \beta \int_0^L |\phi'(x)|^p dx \\ &= 100\beta \int_0^L |\phi'(x)|^p dx \end{aligned}$$

Let $M > 0$. The condition that defines Case 4 holds for ϕ iff it holds for $M\phi$. The conditions that define Cases 4.2 and 4.2.2 hold for ϕ with c iff they hold for $M\phi$ with Mc . Hence, we can assume, without loss of generality, that

$$c := \eta = \left(\frac{1}{2}\right)^{\frac{1}{p}}.$$

As in Case 4.1.2, we first estimate the integral $\int_{\{x:\phi'(x) < 0\}} |\phi'(x)| dx$.

$$\begin{aligned} \int_{\{x:\phi'(x) < 0\}} |\phi'(x)| dx &\leq \int_{\{x:\phi'(x) < 0\}} 1 + |\phi'(x)|^p dx \\ &\leq \lambda\{x : \phi'(x) < 0\} + \int_{\{x:\phi'(x) < 0\}} |\phi'(x)|^p dx \\ &\leq \alpha L + \beta \int_0^L |\phi'(x)|^p dx \end{aligned}$$

$$\begin{aligned}
&\leq \alpha L + \frac{1}{100} c^p L \\
&= c\eta \frac{\alpha + \frac{1}{100} c^p}{c\eta(1-\alpha)} (1-\alpha)L \\
&\leq c\eta \frac{\alpha + \frac{1}{100} c^p}{c\eta(1-\alpha)} \lambda \{ \xi : \phi'(\xi) > c\eta \}
\end{aligned}$$

Now with $\varrho := c\eta$, $\tau := \frac{\alpha + \frac{1}{100} c^p}{c\eta(1-\alpha)} < \frac{1}{4}$ and $k := \lambda \{ \xi : \phi'(\xi) > c\eta \}$, we obtain the following via Lemma 2.3.

$$\begin{aligned}
\int_0^L |\phi(x)|^p dx &\geq \frac{1}{p+1} \left(\frac{1}{2} - \tau \right)^{p+1} k^{p+1} \varrho^p \\
&\geq \frac{1}{p+1} \left(\frac{1}{2} - \frac{1}{4} \right)^{p+1} (1-\alpha)^{p+1} L^{p+1} c^p \eta^p \\
&= \frac{1}{p+1} \left(\frac{1}{4} \right)^{p+1} (1-\alpha)^{p+1} L^p \eta^p c^p L \\
&\geq \frac{1}{p+1} \left(\frac{1}{4} \right)^{p+1} (1-\alpha)^{p+1} L^p \eta^p 100\beta \int_0^L |\phi'(x)|^p dx \\
&= \frac{(1-\alpha)^{p+1} L^p \eta^p 100\beta}{(p+1)4^{p+1}} \int_0^L |\phi'(x)|^p dx
\end{aligned}$$

Hence, in Case 4.2.2, the desired inequality holds with the constant $C_{L,p;4.2.2} := \frac{(1-\alpha)^{p+1} L^p \eta^p 100\beta}{(p+1)4^{p+1}}$.

Case 5. $\lambda \{ x : \phi'(x) > 0 \} \leq L\alpha$ and $\int_{\{\xi: \phi'(\xi) > 0\}} |\phi'(\xi)|^p d\xi \leq \beta \int_0^L |\phi'(\xi)|^p d\xi$.

By Lemma 2.4 with $M = -1$, it follows from Case 4 that, in Case 5, the desired inequality holds with $C_{L,p;5} := \min\{C_{L,p;4.1.1}, C_{L,p;4.1.2}, C_{L,p;4.2.1}, C_{L,p;4.2.2}\}$.

Overall, the inequality holds with

$$C_{L,p} := \min\{C_{L,p;1}, C_{L,p;2}, C_{L,p;3}, C_{L,p;4.1.1}, C_{L,p;4.1.2}, C_{L,p;4.2.1}, C_{L,p;4.2.2}\}. \quad \square$$

- [1] H. T. Banks and R. C. Smith and Y. Wang, Smart Material Structures, Masson and Wiley, Paris, 1996
- [2] H. T. Banks and D. J. Inman, On Damping Mechanisms in Beams, Institute for Computer Applications in Science and Engineering NASA Langley Research Center, September 1989

- [3] David L. Russell, On Mathematical Models for the Elastic Beam with Frequency-Proportional Damping, in: H. T. Banks, Control and Estimation in Distributed Parameter Systems, Frontiers in Applied Mathematics Series, SIAM, 1992, pp. 125–170
- [4] A. Chakravarthy and K.A. Evans and J. Evers, Sensitivities and Functional Gains for a Flexible Aircraft-Inspired Model, Proceedings of the 2010 American Control Conference (2010), 4893–4898
- [5] A. Chakravarthy and K.A. Evans and J. Evers and L. Kuhn, Target Tracking Strategies for a Nonlinear, Aircraft-Inspired Model, Proceedings of the 2011 American Control Conference (2011), 1783–1788
- [6] A. Chakravarthy and K.A. Evans and J. Evers and L. Kuhn, Nonlinear Controllers for Wing Morphing Trajectories of a Heave Dynamics Model, Proceedings of the 2011 Conference on Decision and Control (2011), 2788–2793
- [7] G. H. Hardy, J. E. Littlewood, and G. Pólya, Inequalities, Cambridge University Press, 1952, translated from the 1934 German original
- [8] L. Kuhn (August 2011), Modeling and Control for Heave Dynamics of a Flexible Wing Micro Aerial Vehicle Distributed Parameter System, Ph.D. thesis, Louisiana Tech University
- [9] M. Kumar and K. Rajagopal and S.N. Balakrishnan, Proper Orthogonal Decomposition Technique for Near-Optimal Control of Flexible Aircraft Wings, Proceedings of the 2013 AIAA Guidance, Navigation, and Control (GNC) Conference (2013), 10.2514/6.2013-4935