

12-2014

Magic Surfaces

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The University of Southern Mississippi

MAGIC SURFACES

by

Brandi Crystal Moore

Abstract of a Thesis
Submitted to the Department of Mathematics
of The University of Southern Mississippi
in Partial Fulfillment of the Requirements
for the Degree of Bachelor of Science

December 2014

ABSTRACT

MAGIC SURFACES

by Brandi Crystal Moore

December 2014

Magic figures are discrete, two-dimensional (2-D) objects. We translated the definitions of magic squares and magic circles in an attempt to find three-dimensional (3-D) surfaces that adhere to the same definitions. This is what we defined as a magic surface.

First, we translated these definitions by thinking of the discrete integer entries in the magic squares and magic circles as volumes under the magic surfaces; these were evaluated by volume integrals. We then translated these definitions in a similar way to apply line integral constraints.

We found polynomial functions that satisfied these re-definitions of conditions for a magic surface. In the case over magic squares using volume integrals, we were able to form conjectures about the polynomial solutions and the systems of equations that formed them.

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A Thesis

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Approved:

Director

Chair, Department

December 2014

ACKNOWLEDGEMENTS

Thank you Dr. John Harris for giving me the concept of this research in the first place. And for poking your head in with funny stories and interesting opinions on my color choices.

Thank you Dr. Jeremy Lyle for making my life more difficult, for about a month. Your idea on making the function orthogonal to give us the uniqueness we were hoping to achieve was well-intended. Unfortunately, Sage and Mathematica were unable to solve the non-linear equations. Maybe we can explore it more in the future.

Thank you Mother for not letting your eyes glaze over during all my attempts to explain this project to you. And thank you for not yelling at me when the dishes did not get done because I was working on it.

Thank you Dawn, my sister, for letting me vent my frustrations about this project and paper while boosting my confidence.

Finally, thank you Dr. Karen Kohl, the other half of the "we", for holding my hand through this entire project. Thank you for teaching me parts of calculus my education lacked. Thank you for explaining the same concept three times before I finally understood it, and not getting visibly frustrated with me. Thank you for being available to communicate and meet numerous times throughout this process. And thank you for talking about our pets and puzzles and everything in-between.

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Chapter 1

INTRODUCTION

Calculus has been an integral part of the emergence of the modern world. Before tangent lines and areas under a curve interested mathematicians, recreational mathematics had been intriguing both math professionals and those with no mathematical training at all.

We will study discrete, 2-D magic figures: both magic squares and magic circles. Throughout the course of this research, we will bridge integral calculus and recreational mathematics.

1.1 Magic Figures

Magic figures will be the portion of recreational mathematics that we will study. These are two-dimensional (2-D), discrete structures that have certain properties that must be met that involve the placement of integers into various shapes. All magic figures have conditions on the placement of integers subject to the sums of those integers.

We will extend these properties and definitions to three-dimensional (3-D) continuous surfaces in all respects.

1.1.1 Magic Squares

Magic squares are just one of these magic figures. Anderson [1] explains magic squares' long and diverse history beginning around 2200 B.C. Magic squares have been used for astronomy, divination, alchemy, interpreting philosophy, and medicine in many cultures including Chinese, Islamic, Indian, West African, and European. The Arabic term for magic squares translates to "harmonious distribution of the numbers."

In general, they are an $n \times n$ square that meets certain conditions:

1. Entries are consecutive, unique integers that range from 1 to n^2 .
2. The sum of the entries in each row and column and diagonal are all equal.

This is an example of a 3×3 magic square:

Magic squares have been studied for over 4 millennia, so we know quite a bit about them. For instance, the smallest, non-trivial magic square is the 3×3 as there is no 2×2 . There are different types of magic squares aside from the normal ones we are using: panmagic, bimagic, multimagic, alphamagic. There are some algorithms that can generate the magic square by hand, such as the LUX method, lozenge method, and the Siamese method.

Molony [4] has written computer code that can generate these different types of magic squares. His code is broken into cases based on the sizes of magic squares: odd, singly even, and doubly even.

Definition 4. A **doubly even number** is a number divisible by 2 and 4.

Definition 5. A **singly even number** is a number divisible by 2 and not 4.

The number of distinct magic squares (those not generated through rotation and reflection) has been calculated for a few sized magic squares: 1, 880, 275305224 for 3×3 , 4×4 , and 5×5 , respectively. The number of all 6×6 magic squares is unknown, though it is estimated to be around 1.7745×10^{19} . Weisstein [7] also notes that "it is an unsolved problem to determine the number of magic squares of an arbitrary order." These open questions about magic squares make them all the more interesting.

1.1.2 Magic Circles

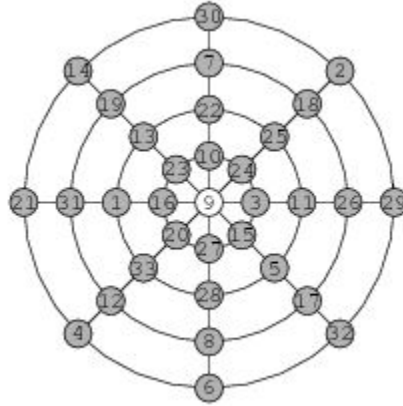
Another type of magic figure is the magic circle[6]. Although there are two types of magic circles, only one is interesting to our work. This type of magic circle has entries placed around a center in rings along 8 radial lines. Therefore, the number of rings must divide the number of entries less one by 8.

Definition 6. The **magic circle conditions** are:

1. Entries are consecutive, unique integers ranging from 1 to n
2. The sum of the entries around a ring sums to the magic circle constant
3. The sum of the entries across diameter lines sums to the magic circle constant, excluding the entry placed in the center

We denote the number of rings in a magic circle by $k = \frac{n-1}{8}$.

For example, this figure is a 4-ring magic circle with entries 1, 2, 3, ..., 33.



- $k = \frac{33-1}{8} = 4$
- The innermost ring sums to $3+24+10+23+16+20+27+15=138$
- The vertical diameter line sums to $30+7+22+10+27+28+8+6=138$

There are many magic circles of the same size; different entries in the center can lead to many more permutations of the entries. Just as with magic squares, no formula is known for the number of magic circles of a given size.

1.2 Magic Surfaces

Using these definitions of magic squares and circles, we will extend these two-dimensional, discrete objects into three-dimensions. We will adhere to the constraints about integer placement and summing of those entries by evaluating both volume and line integrals. We will search for continuous and smooth functions of 2 variables that satisfy a redefinition of the magic properties.

To do so, we will view $f(x,y)$ as a surface over a magic square placed in the first quadrant with the bottom left corner at the origin. In one case, we will consider each entry to be a quantity of the volume integral over a 1×1 subsquare of the xy -plane. Alternatively, we will view the entries in such subsquares as the evaluation of the line integral in both directions. We will also study functions over magic circles, with the center placed at the origin of the plane and using polar coordinates, by evaluating the volume and line integrals over sections set equal to the corresponding entries.

Chapter 2

Methodology

Throughout the course of this research the computer algebra system Sage was used to do the searching for solutions and graphing. The use of Mathematica's interface through Sage was considered briefly, but, in the end, Sage was able to solve the systems of equations using its own processes. In addition, the Sage server proved to be more user-friendly. After getting the solutions to the equations, we were able to substitute those solutions back into the equation and graph.

2.1 Overlay

To adhere to the constraint of unique integers being represented in a magic figure, we will view the surface over a magic figure in the xy -plane. For the square, we will place the function over the first quadrant of the Cartesian plane. In the case of the magic circles, we imagined the center of the circle at the origin.

2.2 Storage

We stored the magic squares and circles as matrices in Sage.

$$\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \quad (2.1)$$

is a 3×3 magic square, and

$$\begin{bmatrix} 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 24 & 10 & 23 & 16 & 20 & 27 & 15 \\ 11 & 25 & 22 & 13 & 1 & 33 & 28 & 5 \\ 26 & 18 & 7 & 19 & 31 & 12 & 8 & 17 \\ 29 & 2 & 30 & 14 & 21 & 4 & 6 & 32 \end{bmatrix} \quad (2.2)$$

is a 4-ring magic circle.

Here, note that Sage starts indexing matrices at 0. Therefore, $M[2, 1]$ for the magic square's matrix is 7.

2.3 Algorithm

We used a magic square generator written by Molony [4] that created a magic square of a given degree represented as a matrix using different algorithms for the different cases of sized squares. No algorithm was used for magic circle creation; they were input by hand.

To search for a magic surface function, we developed the following algorithm:

1. generate a polynomial of a given degree, p , with 2 specified variables and unknown coefficients using `create_poly`, a function we wrote
2. Set up equations by setting each integral evaluated over a certain region equal to the corresponding entry from that figure
3. Attempt to solve for the unknown coefficients in `create_poly`
4. If no solution was found, increase the degree of the polynomial, p , and repeat the algorithm

After the algorithm found a solution, we produced a graph of the solution.

Here is some sample output from `create_poly`:

```
k3*x^3 + k6*x^2*y + k9*x*y^2 + k12*y^3 + k2*x^2 + k5*x*y
+ k8*y^2 + k1*x + k4*y + k0
```

```
[k0, k4, k8, k12, k1, k5, k9, k2, k6, k3]
```

This includes the polynomial produced and the list of unknown coefficients.

Chapter 3

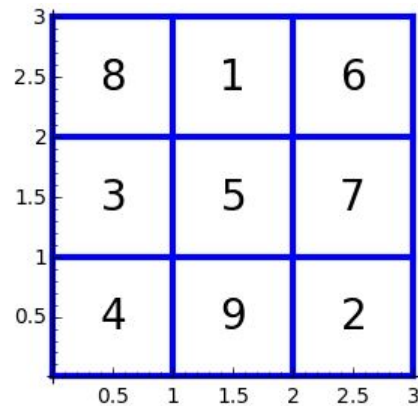
Magic Squares

3.1 Volume

To make the surface generated by the polynomial in x and y of degree p magic over a square, it must satisfy certain conditions:

1. The volume integral over the i^{th} horizontal row equals the magic constant: $H_i = M_2(n)$
2. The volume integral over the j^{th} vertical column equals the magic constant: $V_j = M_2(n)$
3. The volumes over subsquares are consecutive, unique integers

To accomplish this, we will start with a known $n \times n$ magic square. We imagine the integer entries as being the volume under the magic surface in that subsquare's specific region. To carry over the uniqueness of the integer entries in the magic square to the 3-D figure we hope to find, we will evaluate the volume integral over a subsquare and set it equal to the integer entry overlaid.



The constraint equations in step 2 of the algorithm will be of the form

$$\int_{n-i-1}^{n-i} \int_j^{j+1} f(x,y) dx dy = M[i,j] \quad (3.1)$$

where n is the dimension of the matrix that represents the square, and $0 \leq i, j < n$. Due to the additive interval (region) property of integration [3], this ensures the volume over a horizontal or vertical strip is equal to the magic constant.

There will be n^2 such equations will be of this form since we are evaluating the volume integrals over each subsquare and there are n^2 subsquares. For example, we want $\int_2^3 \int_0^1 f(x,y) dx dy = 8 = M[0,0]$. This is the top left subsquare of the 3×3 magic square. We evaluate this region first because we need it set equal to the $M[0,0]$ entry in the matrix: the first entry of the matrix lies in the top lefthand corner of the grid. We go through the rest of the subsquares and set them equal to their corresponding matrix entry to solve for the unknown coefficients.

3.1.1 3×3 Example

For example, we represent the magic square shown as the matrix below. Note that Sage/Python indexes rows and columns starting with 0; this determined the limits of integration in the constraint equation.

8	1	6
3	5	7
4	9	2

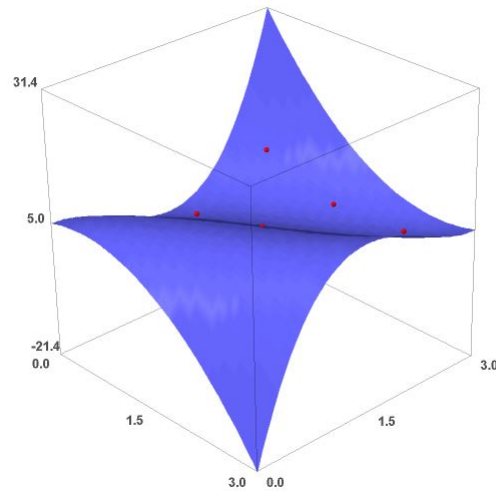
$$\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \quad (3.2)$$

We will search for a polynomial, $f(x,y)$, that when placed over the figure above, satisfies the volume integral constraint.

For the evaluation of this 3×3 magic square's volume,

- We start by assuming $f(x,y)$ is a degree 3 polynomial in variables x and y . The automatically generated polynomial is $f(x,y) = k_3x^3 + k_6x^2y + k_9xy^2 + k_{12}y^3 + k_2x^2 + k_5xy + k_8y^2 + k_1x + k_4y + k_0$, with unknown coefficients $[k_0, k_4, k_8, k_{12}, k_1, k_5, k_9, k_2, k_6, k_3]$.
- The system of 9 equations is set up by evaluating the volume integrals of the polynomial by looping through values of i and j .
- The solution of the system gave $f(x,y) = r_1x^3 + \frac{1}{11}(2r_2 + 45)y^3 - \frac{9}{2}(r_1 + 1)x^2 + 3x^2y - \frac{9}{22}(2r_2 + 23)y^2 - 6xy^2 + \frac{1}{2}(11r_1 + 9)x + r_2y + 9xy - \frac{3}{2}r_1 - \frac{3}{11}r_2 + \frac{247}{44}$ where r_1 and r_2 are free variables.
- After choosing to set $r_1 = 0$ and $r_2 = 0$, we have the solution given by $f(x,y) = \frac{45}{11}y^3 - \frac{9}{2}x^2 + 3x^2y - \frac{207}{22}y^2 - 6xy^2 + \frac{9}{2}x + 9xy + \frac{247}{44}$.

This is the graph of the solution:



The red dots on the surface are at the height of the entry of the subsquare in the center of the subsquare being represented. This shows how the height of the surface varies throughout the entire region being evaluated, with the average height of the subsquare being equal to the entry.

3.1.2 Results

Using the `search_mtx_sq_vol` function (Appendix A) for magic squares 3×3 through 12×12 , (Appendix B) we were able to notice some patterns. We assume the pattern holds for all magic squares of the same size. We used the magic square test function by Molony [4] to test all 3×3 magic squares; it would take an exceptionally long amount of time to try to generate the 4×4 magic squares, so we generalized from the 3×3 results. We confirmed that all magic squares of size 3×3 have the same results shown in the table below.

Square Size	Equations	Degree	Unknowns	Free Variables	Unknowns – Free Vars.
n	n^2	p	$\frac{(p+1)(p+2)}{2}$		
3	9	3	10	2	8
4	16	5	21	6	15
5	25	7	36	12	24
6	36	10	66	30	36
7	49	11	78	30	48
8	64	13	105	42	63
9	81	15	136	56	80
10	100	18	190	90	100
11	121	19	210	90	120
12	144	21	253	110	243

From the table of results, we formed several conjectures. Note that Conjecture 1 implies Conjecture 2.

Conjecture 1. The degrees of polynomials needed to solve a system of equations when evaluating the volume integrals over a $n \times n$ magic square increase in a pattern of 2, 2, 3, 1, 2, 2, 3, 1, ... starting at the degree of the 3×3 magic square, 3.

Conjecture 2. The degrees of polynomials are odd unless the size of the $n \times n$ magic square is singly even.

Conjecture 3. There only need be one less non-free unknown than equations in order to solve the system, unless the size of the $n \times n$ square is singly even.

These singly even magic squares are interesting, not only because they change these patterns, but because they also are built differently than the odd and doubly even magic squares. The most common, if not only, way to generate a singly even magic square is first by sectioning it half vertically and horizontally, then filling in these sections by the Siam method [5]. Finally, since this is not a magic square yet, elements in almost every column must be exchanged with another until the magic square is achieved.

We noticed that the graphs of solutions are exaggerated more than expected. Some fall below the z -axis much further away from the xy -plane than we had thought since we had pictured the surfaces above the magic square. The increase in the entries made an increase in the "waves" seen on the surface up until the 7×7 ; once the squares reached this size, they became what looked to be relatively flat with a sharp angle near one of the borders of the square.

3.1.3 Non-Magic Evaluation

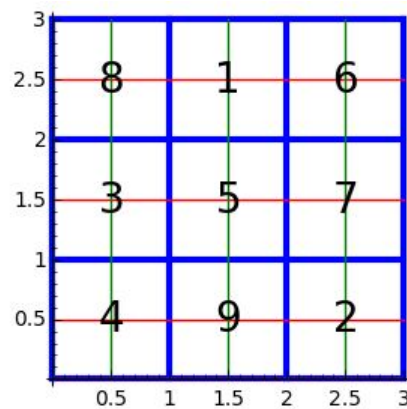
After noticing these patterns, we tested some non-magic squares for the same conditions. These squares merely contained consecutive, unique integers; the summing conditions of the entries were not met.

Square Size	Equations	Degree	Unknowns	Free Variables	Unk.-Free Vars.
n	n^2	$2(n-1)$	$n(2n-1)$	$n(n-1)$	
3	9	4	15	6	9
4	16	6	28	12	16
5	25	8	45	20	25
6	36	10	66	30	36

As shown in the table, these non-magic squares with the same sizes as the magic squares we evaluated do not have the same conjectured properties. The number of non-free unknowns is equal to the number of equations in each of these few cases, whereas the magic squares evaluated had one less non-free unknown than equations unless the size of the magic square was singly even. Another difference between the magic and non-magic squares evaluated is that all of the magic squares were given an odd degree of polynomial (unless, again, if the square's size was singly even) for the surface, but the non-magic squares' surfaces were produced with even degrees, regardless of size of square.

From these observations on a small sample size, we conclude that the surfaces produced and the patterns noted truly make for a magic surface in the volume integral constraint over a magic square. A more thorough study of surfaces over non-magic squares needs to be carried out.

3.2 Line Integrals



The other way we evaluated whether a surface has magic properties is by testing the line integrals, with respect to arc length, down the center of the row/column. The magic

conditions for the line integral case are:

1. The k^{th} horizontal line integral across a row equals the magic constant: $H_k = M_2(n)$
2. The l^{th} vertical line integral down a column equals the magic constant: $V_l = M_2(n)$
3. The diagonal and anti-diagonal equal the magic constant: $D_1 = D_2 = M_2(n)$
4. The line integrals over the subsquares are consecutive, unique integers

Just as with the volume integral case, we overlaid the surface above a magic square and set the evaluated integrals over a region equal to the entry in that subsquare.

Recall from [3] that the line integral of $f(x,y)$ with respect to arc length along a path C is denoted by

$$\int_C f(x,y) ds, \quad (3.3)$$

where $x(t)$ and $y(t)$ parameterize C , and

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad (3.4)$$

The horizontal line integrals over subsquares are given by

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + 0} dt \rightsquigarrow ds = dx \rightsquigarrow \int_a^b f(x,y) dx \quad (3.5)$$

This produced n^2 equations of the form

$$\int_j^{j+1} f(x, n-i-1/2) dx = M[i, j], \quad 0 \leq i, j \leq n \quad (3.6)$$

The limits of integration are the changing x -values as the y -value stays the same; the y -value is written as $n-i-\frac{1}{2}$ because we looped through the i 's until all y -values down the middle (where the $\frac{1}{2}$ comes from). These limits are as such since the first entry in the matrix is in the top left subsquare of the magic square.

The vertical line integrals over subsquares are given by

$$ds = \sqrt{0 + \left(\frac{dy}{dt}\right)^2} dt \rightsquigarrow ds = dy \rightsquigarrow \int_a^b f(x,y) dy \quad (3.7)$$

This again produced n^2 equations of this form

$$\int_{n-i-1}^{n-i} f(j+1/2, y) dy = M[i, j], \quad 0 \leq i, j \leq n \quad (3.8)$$

Here, the limits of integration are the changing y -values. The x -values are looped through going down the center of the subsquares, $j + \frac{1}{2}$. Again, because the matrix representing the magic square begins in the top.

Unlike the volume integrals, we adhere more strictly to the definition of a magic square by also evaluating the diagonals of each region and setting it equal to the entry in that subsquare.

We parameterize C for the main diagonal represented by the line $y = x$ by

$$x(t) = t, \quad y(t) = x = t \quad (3.9)$$

We parameterize C for the anti-diagonal represented by the line $y = -x + n$ by

$$x(t) = t, \quad y(t) = -t + n \quad (3.10)$$

So here, ds for the diagonals is

$$ds = \sqrt{\left(\frac{dt}{dt}\right)^2 + \left(\frac{dt}{dt}\right)^2} dt = \sqrt{(1)^2 + (\pm 1)^2} dt \rightsquigarrow ds = \sqrt{2}dt \quad (3.11)$$

For the diagonal, the n equations are of the form

$$\int_i^{i+1} f(t, -t + N)\sqrt{2}dt = M[i, i]. \quad (3.12)$$

Similarly, the n equations for the anti-diagonal are of the form

$$\int_{n-i-1}^{n-i} f(t, t)\sqrt{2}dt = M[i, n - i - 1]. \quad (3.13)$$

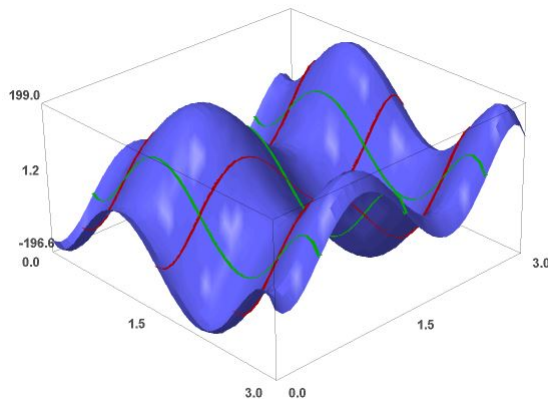
3.2.1 3×3 Example

We evaluated the same square for the line integral constraint as we did the volume integral constraint,

$$\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \quad (3.14)$$

A 6th degree polynomial was needed to solve these 24 equations; this output can be found in Appendix B.

This is the graph of the solution.



Here, the red lines drawn on the surface represent the vertical line integrals, and the green lines represent the horizontal integrals.

3.2.2 Results

Square Size	Equations	Degree	Unknowns	Free Variables	Unknowns-Free Variables
n	$2n(n+1)$	$2n$	$(2n+1)(n+1)$	$n+1$	
3	24	6	28	4	24
4	40	8	45	5	40
5	60	10	66	6	60
6	84	12	91	7	84

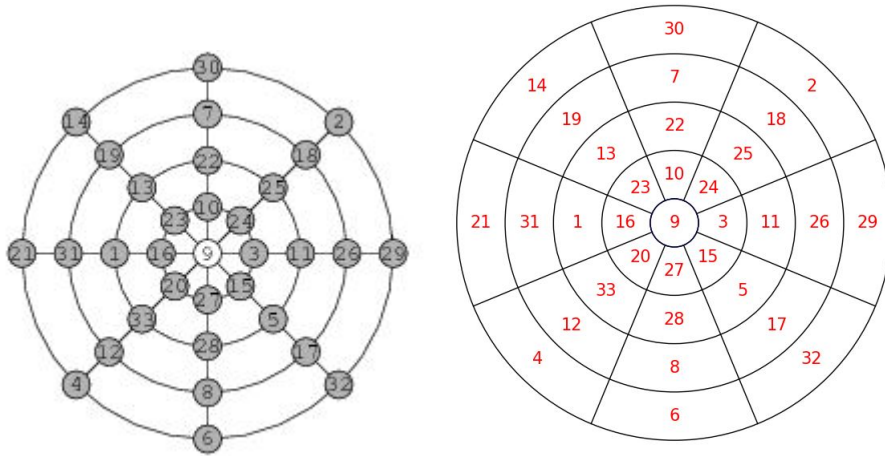
Unlike the volume integrals, it seems that the patterns from the volume case do not carry over. Here, the increase in degree is always by 2. Also, the number of unknowns minus the number of free variables is equal to the number of equations.

Chapter 4

Magic Circles

4.1 Volume

We will redraw magic circles so that the entries are now representing volumes of sections of the circle:



To make the surface in polar coordinates magic over a circle, it must satisfy certain conditions:

1. The volume integral of a wedge over a diameter line equals the magic constant
2. The volume integral of a washer over a ring equals the magic constant
3. The volumes over sections are consecutive, unique integers

As with the squares, we imagine the circle's entries as being the volume under the magic surface in that section's specific region. We stored these entries as a matrix with the center being the only entry in the top row; the rest of the matrix was filled in starting at the first ring on angle 0 and moving counter clockwise.

$$\begin{bmatrix} 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 24 & 10 & 23 & 16 & 20 & 27 & 15 \\ 11 & 25 & 22 & 13 & 1 & 33 & 28 & 5 \\ 26 & 18 & 7 & 19 & 31 & 12 & 8 & 17 \\ 29 & 2 & 30 & 14 & 21 & 4 & 6 & 32 \end{bmatrix} \tag{4.1}$$

The volume integral constraints over a section represented by an entry in the matrix have this form:

$$\int_{\frac{i\pi}{4}-\frac{\pi}{8}}^{\frac{i\pi}{4}+\frac{\pi}{8}} \int_{j-1/2}^{j+1/2} f(r, \theta) r dr d\theta = M[j, i] \quad (4.2)$$

We decided to keep the center for the volume case though we know there is a point of singularity there. We searched through many different types of polynomials, $f(r, \theta)$

- with terms $k_m r^i \sin^k h\theta$
- with terms $k_m r^i \cos^j h\theta$
- with terms produced by $(r + \theta)^p \times (\sin \theta + \cos \theta)$
- with terms $k_m r^i \theta^j$

It was finally decided to use regular polynomials in r and θ , $k_m r^i \theta^j$, since we were unable to produce solutions with the trigonometric functions included.

We were able to get solutions, but when they were graphed there was a discontinuity at $\theta = 0 = 2\pi$. To remedy this, we set the limit of the polynomial as $\theta \rightarrow 0$ equal to the limit of the polynomial as $\theta \rightarrow 2\pi$

$$\lim_{\theta \rightarrow 0^+} f(r, \theta) = \lim_{\theta \rightarrow 2\pi^-} f(r, \theta) \quad (4.3)$$

Then we looped through the derivatives of the polynomial and took the limits of them to make the function smooth.

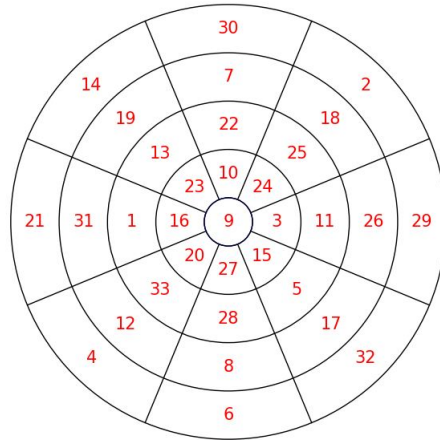
$$\lim_{\theta \rightarrow 0^+} f^{(d)}(r, \theta) = \lim_{\theta \rightarrow 2\pi^-} f^{(d)}(r, \theta) \quad (4.4)$$

where $1 \leq d \leq p$, and p is the degree of the polynomial.

We are matching the coefficients in these equations and appending these new equations into the system of equations of the volume integrals.

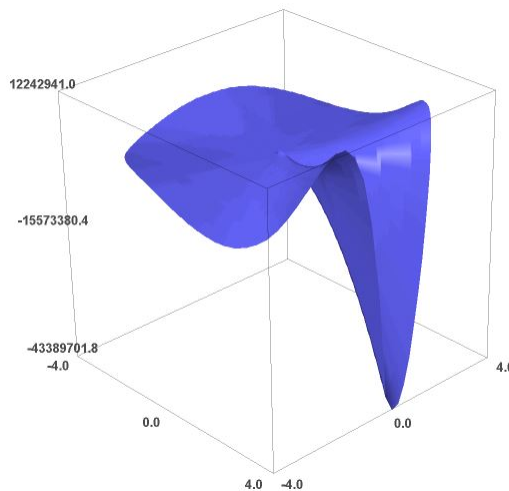
4.1.1 One Result

This is the result of evaluating volume integrals over sections set equal to the entries of the circle in



The solution required the polynomial $f(r, \theta)$ to be of degree 11. The number of equations, 99, far exceeded the number of unknowns, 78, needed to solve the system of equations. There are so many equations because not only are the volume integrals included in the system, but also all of the equations created when matching the coefficients in the limits of the derivatives. Since the degree of the polynomial is 11, we can take the derivative 11 times; this gives us 12 new equations (the 11 derivatives and the original function). There were 35 free variables, so the number of unknowns actually needed to solve the equation is $78-35=33$; this happens to be the number of entries, n . The full solution can be found in Appendix B.

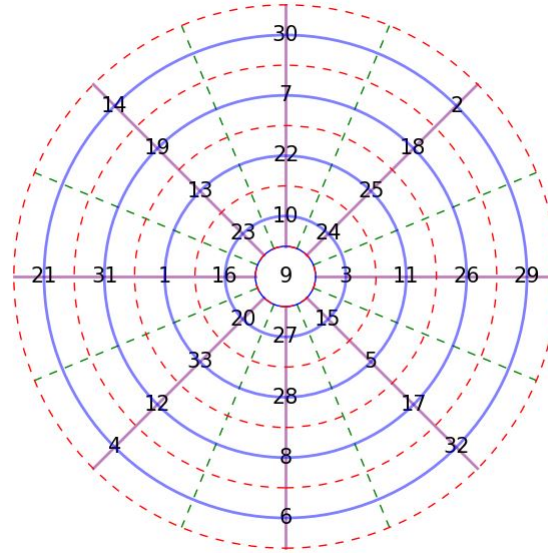
This is the graph of the solution:



Even after including the new constraints to make the function smooth and continuous at $\theta = 0 = 2\pi$, there is still an odd dip in the graph. There is a discontinuity at the origin, though we evaluated the volume integral in that region and set it equal to the central entry.

4.2 Line Integrals

This is our reimagining of a magic circle while taking the line integrals:



For example, if we were to look at the paths over 18, we would take the line integral from $\theta = \frac{\pi}{8}$ to $\theta = \frac{3\pi}{8}$ (between green dotted lines) along $r = 3$ (the blue arc) and set it equal to 18 then take the line integral from $r = 2.5$ to $r = 3.5$ (between red dotted lines) along the radial line $\theta = \frac{\pi}{4}$ (the purple line).

Recall that the line integral is given by

$$\int_C f(r, \theta) ds \quad (4.5)$$

where C is either a path along a radius or a path along an arc. In polar coordinates, we have

$$ds = \sqrt{dr^2 + r^2 d\theta^2} \quad (4.6)$$

and we will simplify this in the two cases of paths.

When r is constant, θ varies:

$$ds = \sqrt{0 + r^2 d\theta^2} \rightsquigarrow ds = rd\theta \quad (4.7)$$

so the line integral constraint along an arc segment of a ring is

$$\int_{(2j-1)\frac{\pi}{8}}^{(2j+1)\frac{\pi}{8}} f(i, \theta) rd\theta = M[j, i]. \quad (4.8)$$

When θ is constant, r varies:

$$ds = \sqrt{dr^2 + 0} \rightsquigarrow ds = dr \quad (4.9)$$

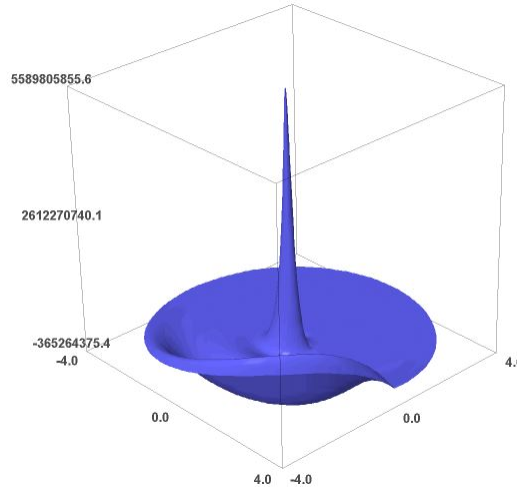
so the line integral constraint along a radial line is

$$\int_{(2j-1)\frac{1}{2}}^{(2j+1)\frac{1}{2}} f\left(r, \frac{i\pi}{4}\right) dr = M[j, i]. \quad (4.10)$$

4.2.1 One Result

We were able to find a solution to the line integral constraint over the same magic circle we evaluated the volume integral over. The degree of the polynomial needed was 12, which produced 91 unknowns. The system contained 155 equations, but there were only 15 free variables in the solution. This is the only case where there were more equations than non-free unknowns. The solution can be found in Appendix B.

This is the graph of the solution:



There is the discontinuity at the origin, just as with the volume integral case. In this line integral case, we did not evaluate over the central entry.

Chapter 5

Conclusion

5.1 Results

We were able to translate the definition of these discrete magic figures to generalize magic surfaces in these four cases. To do so, we wrote an algorithm to search for such surfaces over a magic figure of arbitrary size.

We were also able to notice patterns in the magic square cases; these included the degrees of the polynomials and the number of unknowns needed to solve the system dependent on the size of the square. The non-magic squares we tested did not have the same properties as the magic squares of the same sizes.

5.2 Future Work

The magic surfaces produced in this text have only been satisfying to some degree. There is still more work to be done; a lifetime could be spent exploring what we have not been able to and verifying what we have.

One priority would be to place additional constraints on the magic surfaces so that the surfaces produced would be strictly above the xy -plane or $r\theta$ -plane and bounded above. This would correct the appearance of "flat" areas in our graphs. We would want to be able to look closer at the curvature of the surface to see what other interesting properties we can identify.

We would also want to see if magic squares and circles exist that satisfy both the volume and line integral cases. Another option would be to relocate the discrete squares in an attempt to find surfaces that satisfy more than one than one magic square of different sizes.

Upon further research, we would want to test what different kinds of functions satisfy these magic constraints, especially those that are non-polynomial.

A final goal would be to further generalize these magic surfaces by getting away from these discrete magic figures while still retaining uniqueness in some other way.

Appendix A

Sage Functions

A.1 Common Functions

```

def create_vars(viter, varstr, num):
    #create the list num of variables using the iterator viter
    return [var(varstr + str(viter.next())) for i in range(num)]

def create_poly(deg, v1, v2):
    #creates a polynomial of degree deg in the 2 variables specified
    num=(1+deg)^2
    var('r T')
    K=[]
    citer = iter(range(1000))
    L=create_vars(citer, 'k', num)
    poly=0
    m=0
    for i in range(deg+1):
        for k in range(deg+1):
            if (i+k)<=deg:
                m+=1
                t=L[m]*v1**(i)*v2**(k)
                poly=t+poly
                K.append(L[m])

    return poly, K

def matcher(f, deg):
    #matches coefficients of degrees of r to better solve for k_i's
    #takes in the function and the highest degree of that function
    f(r, T)=f
    R=f(r, T).limit(T=0, dir='+')
    #takes the limit of the function as theta goes to 0 from the right
    L=f(r, T).limit(T=2*pi, dir='-')
    #takes the limit of the function as theta goes to 2*pi from the ←
    left
    #this is to remove the discontinuity on the circle where 0=2*pi
    eq = L==R

```

```

# loop through degree, match coefficients of powers of r to ←
  produce a list of new equations
eqlist = []
for i in range(deg+1):
#+1 because range goes from 0 to d-1
    eqn=L.coefficient(r,i)==R.coefficient(r,i)
    #sets the coefficients on either side of the equation equal to←
      each other
    eqlist.append(eqn) #adds the new equation to the list

return eqlist
#returns the list of new equations

def polar_plot3d(f):
#plots function in 3-d polar coordinates
f(r,T)=f
print f(r,T)
return parametric_plot3d((r*cos(T), r*sin(T), f(r,T)),(T,0,2*pi),(←
  r,0,4),mesh=True,dots=True,plot_points=[100,100])

```

A.2 Functions for volume over a square

```

def search_mtx_sq_vol(f,L,M,pr=False,solver='sage'):
    #searches for a function of given form f over a magic square ←
    #represented as a matrix M
    #L = list of unknown parameters
    f(x,y)=f
    N = M.dimensions()[0]
    EqnList = []
    for i in range(N):
        for j in range(N):
            h=integrate(integrate(f,x,j,j+1),y,N-i-1,N-i)
            #calculates subsquare volume
            if pr:
                print i,",",j," : ", h,"=", M[i,j]
            EqnList.append(h==M[i,j])
            #adds the volume constraint to the list

    # creates a dictionary of solutions
    if solver=='sage':
        sol = solve(EqnList,L,solution_dict=True)

        if sol ==[]:
            d = sol
            return "No solution"
        else:
            d=sol[0]
            #solve in Sage
    else:
        d = solvemeta(EqnList,L)
        # solve in Mathematica

    fsubs = f.subs_expr(d)
    #substitute into f the assignments given in dictionary d

    freevars = Set(fsubs.variables()).difference(Set([x,y]))
    # substitute that solution into f
    for freevar in list(freevars):
        var(freevar)

    print "# equations = ", len(EqnList)
    print "# unknowns = ", len(L)
    print "# free vars = ", len(freevars)
    return fsubs,freevars

def search_sq_vol_deg(M):
    #loops through degree of polynomial until solution is found
    deg=3

```

```

sol='No solution '
while (sol=='No solution ' and deg <=25):
    poly=create_poly(deg)

    sol=search_mtx_sq_vol(poly[0],poly[1],M)
    print ""
    deg+=1
print "deg=",deg-1
return sol

def conv_matrix_to_points(M):
    #converts the matrix M into points in the center of the subsquare
    N = M.dimensions()[0]
    L = []
    for i in range(N):
        for j in range(N):
            x = j + 1/2
            y = N-1-i+1/2
            z = M[i,j]
            L.append((x,y,z))
    return L

def solve_and_graph(N):
    #searches for a solution then graphs it
    M=make_magic_square(N)
    print M
    f=search_sq_vol_deg(M)
    print f[0]
    #function that has already been substituted with solutions
    print f[1]
    #list of free variables
    fvlist=f[1]
    d = {}
    #creates an empty dictionary
    for r in fvlist:
        #iterates through the list of free variables
        d[r]=0
        #sets each free variable in dictionary equal to zero
    print d
    func=f[0].subs_expr(d)
    #substitutes dictionary assignments into the function

    pts=point3d(conv_matrix_to_points(M),size=10,color='red')
    graph=plot3d(func,(x,0,N),(y,0,N))
    show(graph+pts)

```


A.3 Functions for line integrals over a square

```

def search_mtx_sq_line(f,L,M,pr=False,solver='sage'):
    #searches for a function of given form f over a magic square ←
    #represented as a matrix M
    #line integrals
    #L = list of unknown parameters
    f(x,y)=f
    var('t')
    print f
    N = M.dimensions()[0]
    EqnList = []

    for i in range(N):
        for j in range(N):
            h=integrate(f(x,N-i-1/2),x,j,j+1)
            v=integrate(f(j+1/2,y),y,N-i-1,N-i)
            if pr:
                print "Horiz:",i,",",j,": ", h,"=", M[i,j]
                print "Vert:",i,",",j,": ", v,"=", M[i,j]
            EqnList.append(h==M[i,j])
            EqnList.append(v==M[i,j])

    for i in range(N):
        # main diagonal
        d1 = integrate(f(t,-t+N)*sqrt(2),t,i,i+1)
        EqnList.append(d1==M[i,i])

        # anti-diagonal
        d2 = integrate(f(t,t)*sqrt(2),t,N-1-i,N-i)
        EqnList.append(d2==M[i,N-1-i])

    print "# equations = ", len(EqnList)
    print "# unknowns = ", len(L)

    # creates a dictionary of solutions
    if solver=='sage':
        sol = solve(EqnList,L,solution_dict=True)
        if sol==[]:
            return 'no solution'
        else:
            d=sol[0]
            #solve in Sage
    else:
        d = solvemeta(EqnList,L)
        # solve in Mathematica
    print "d =", d

    fsubs = f.subs_expr(d)
    #substitute into f the assignments given in dictionary d

```

```

freevars = Set(fsubs.variables()).difference(Set([x,y]))
# substitute that solution into f
for freevar in list(freevars):
    var(freevar)

return fsubs,freevars

def plot_li(f,d,N):
    #draws the line integrals on surface
    fsubs = f.subs_expr(d)
    #substitutes dictionary values into f
    fsubs(x,y) = fsubs
    print "fsubs=", fsubs
    g= plot3d(fsubs(x,y),(x,0,N),(y,0,N))

    width = .04
    # width of highlight on surface

    for c in range(N):
        ctr =c+.5
        below = ctr - width/2
        above = ctr + width/2
        vert=plot3d(fsubs(ctr,y),(x,below,above),(y,0,N),rgbcolor←
            =(1,0,0))
        g=g+vert
        horiz=plot3d(fsubs(x,ctr),(x,0,N),(y,below,above),rgbcolor←
            =(0,1,0))
        g=g+horiz
    show(g)

```

A.4 Functions for volume over a circle

```

def search_mtx_cir_vol(f,L,M,N,deg,solver='sage'):
    #searches for solution to volume integrals over a circle
    k=(N-1)/8
    f(r,T)=f
    rlines=pi/4
    #radial lines are at every multiple of pi/4
    split=pi/8
    #splits the difference between each radial line to get proper ←
        interval
    dist=1/2
    #how far up and down the radial line to go on either side of entry←
        for a complete interval of 1
    EqnList=[]
    for i in range (8):
        for j in range(1,k+1):
            v=integrate(integrate(f(r,T)*r,T,i*(pi/4)-(pi/8),i*(pi/4)←
                +(pi/8)),r,j-1/2,j+1/2)
            eq = v==M[j,i]
            if not eq:
                # if equation is not "False" (inconsistent), then append ←
                    it
                EqnList.append(eq)
    center=integrate(integrate(f(r,T)*r,r,0,1/2),T,0,2*pi) #i was just←
        testing to see what would happen without the center
    #print "center = ", center
    EqnList.append(center==M[0,0])

    # makes f continuous & smooth along theta=0 by matching ←
        coefficients of f(r,0) to f(r,2*pi) —several equations

    # matches derivatives by looping
    for d in range(deg+1):
        new_eqns = matcher(diff(f,r,d),deg-d)
        EqnList += new_eqns

    print "# equations = ", len(EqnList)
    print "# unknowns = ", len(L)

    # create a dictionary of solutions
    if solver=='sage':
        sol = solve(EqnList,L,solution_dict=True)

        if sol ==[] or sol=='No solution':
            d = sol
            return "No solution"
        else:
            d=sol[0]
            #solve in Sage
    else:

```

```

    d = solvemmma(EqnList,L)
    # solve in Mathematica
    print "MMA d =", d

    if str(d).replace(' ','')==("{}"):
        return "No solution"

fsubs = f.subs_expr(d)
#substitutes into f the assignments given in dictionary d
print "fsubs = ", fsubs

freevars = list(Set(fsubs.variables()).difference(Set([r,T])))
# substitute that solution into f
for freevar in freevars:
    var(freevar)
print "# freevars =", len(freevars)

return fsubs,freevars

def search_cir_vol_deg(M,N,solver='sage',p=8):
    #loops through degree of polynomial until solution is found
    deg=3
    sol='No solution'
    while (sol=='No solution' and deg <=15):
        #loops through degrees of polynomials until a solution is found
        poly=create_poly(deg,r,T)
        #creates the polynomial of degree in loop
        print "deg=",deg
        sol=search_mtx_cir_vol(poly[0],poly[1],M,N,deg,solver)
        #searches for a solution using that polynomial
        print ""
        deg+=1
    return sol

def solve_and_graph_cv(M):
    #finds the solution to the system of equations then graphs it

    rows,cols=M.dimensions()
    #cols is always 8
    N = cols*(rows-1)+1

    f=search_cir_vol_deg(M,N,solver='sage',p=8)
    print f[0]
    #function that has already been substituted with solutions
    print f[1]
    #list of free variables
    fvlist=f[1]

```

```
d = {}  
#creates an empty dictionary  
for r in fvlist:  
#iterates through the list of free variables  
    d[r]=0  
    #sets each free variable in dictionary equal to zero  
print d  
func=f[0].subs_expr(d)  
#substitutes dictionary assignments into the function  
  
graph=polar_plot3d(func)  
show(graph)
```

A.5 Functions for line integrals over a circle

```

def search_mtx_cir_li(f,L,M,N,deg,solver='sage'):
    #searches for a solution for line integrals over a circle
    k=(N-1)/8
    f(r,T)=f
    rlines=pi/4
    #radial lines are at every multiple of pi/4
    split=pi/8
    #splits the difference between each radial line to get proper ←
        interval
    dist=1/2
    #how far up and down the radial line to go on either side of entry←
        for a complete interval of 1
    EqnList=[]
    for i in range (8):
        for j in range(1,k+1):
            line_T=integrate(f(i,T)*r,T,(2*j-1)*pi/8,(2*j+1)*pi/8)
            #integrals along arc segments
            eq = line_T==M[j,i]
            if not eq:
                # if equation is not "False" (inconsistent), then append ←
                    it
                EqnList.append(eq)

    for m in range (8):
        for p in range (1,k+1):
            line_r=integrate(f(r,m*pi/4),r,(2*p-1)/2,(2*p+1)/2)
            #integrals down radial lines
            eq1 = line_r==M[p,m]
            if not eq1:
                # if equation is not "False" (inconsistent), then append ←
                    it
                EqnList.append(eq1)

    # makes f continuous & smooth along theta=0 by matching ←
        coefficients of f(r,0) to f(r,2*pi) —several equations

    # matches derivatives by looping
    for d in range(deg+1):
        new_eqns = matcher(diff(f,r,d),deg-d)
        EqnList += new_eqns

    print "# equations = ", len(EqnList)
    print "# unknowns = ", len(L)

    # create a dictionary of solutions
    if solver=='sage':
        sol = solve(EqnList,L,solution_dict=True)

        if sol ==[] or sol=='No solution':

```

```

        d = sol
        return "No solution"
    else:
        d=sol[0]
        #solve in Sage

fsubs = f.subs_expr(d)
#substitutes into f the assignments given in dictionary d
print "fsubs = ", fsubs

freevars = list(Set(fsubs.variables()).difference(Set([r,T])))
# substitutes that solution into f
for freevar in freevars:
    var(freevar)
print "# freevars =", len(freevars)

return fsubs,freevars

def search_cir_li_deg(M,N,solver='sage',p=8):
    #loops through degrees of polynomials until solution is found
    deg=3
    sol='No solution'
    while (sol=='No solution' and deg <=15):
        #loops through degrees of polynomials until a solution is ←
        found
        poly=create_poly(deg,r,T)
        #creates the polynomial of degree in loop
        print "deg=",deg
        sol=search_mtx_cir_li(poly[0],poly[1],M,N,deg)
        #searches for a solution using that polynomial
        print ""
        deg+=1
    return sol

def solve_and_graph_cli(M):
    #searches for a solution then graphs it

    rows,cols=M.dimensions()
    #cols is always 8
    N = cols*(rows-1)+1

    f=search_cir_li_deg(M,N,solver='sage',p=8)
    print f[0]
    #function that has already been substituted with solutions
    print f[1]
    #list of free variables
    fvlist=f[1]

```

```
d = {}  
#creates an empty dictionary  
for r in fvlist:  
#iterates through the list of free variables  
    d[r]=0  
    #sets each free variable in dictionary equal to zero  
print d  
func=f[0].subs_expr(d)  
#substitutes dictionary assignments into the function  
  
graph=polar_plot3d(func)  
show(graph)
```

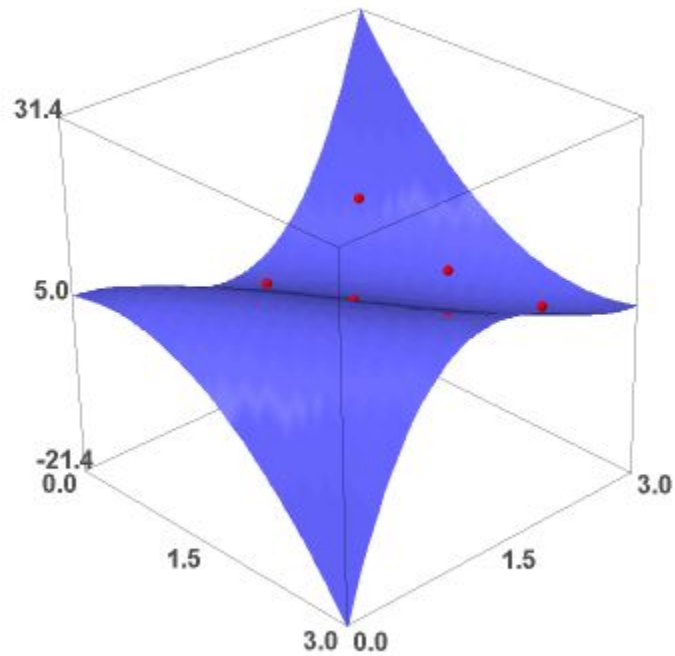

Appendix B

Output

B.1 Solutions for volume over a square

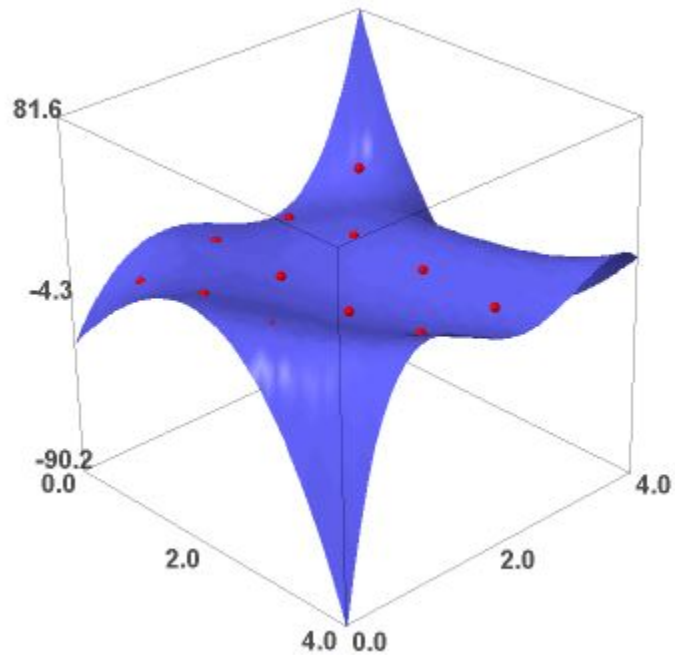
B.1.1 3×3

$$(x, y) \mapsto 3x^2y - 6xy^2 + \frac{45}{11}y^3 - \frac{9}{2}x^2 + 9xy - 207/22y^2 + \frac{9}{2}x + \frac{247}{44}$$



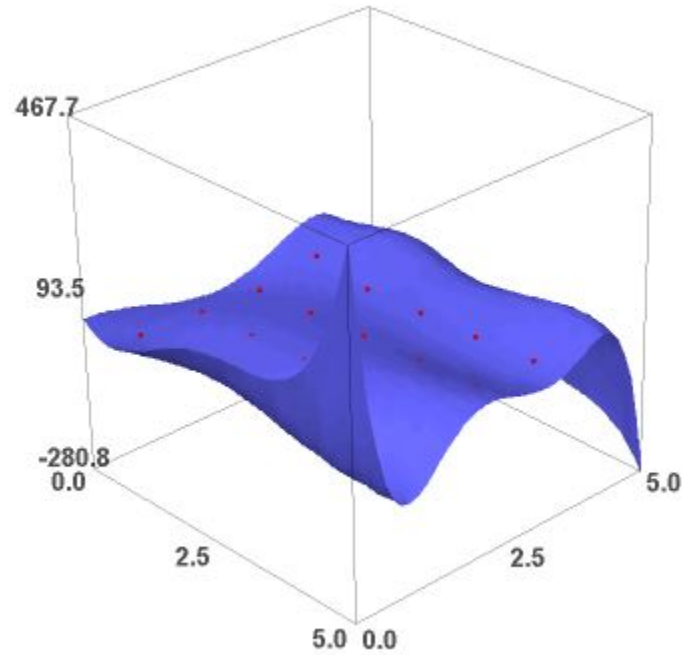
B.1.2 4×4

$$(x, y) \mapsto -x^3y^2 + 4x^2y^3 - 2xy^4 + 4x^3y - 18x^2y^2 + \frac{4}{3}y^4 - \frac{8}{3}x^3 + 18x^2y + \frac{87}{2}xy^2 - 4x^2 - 86xy - 31y^2 + \frac{212}{5}x + \frac{196}{3}y - \frac{251}{10}$$



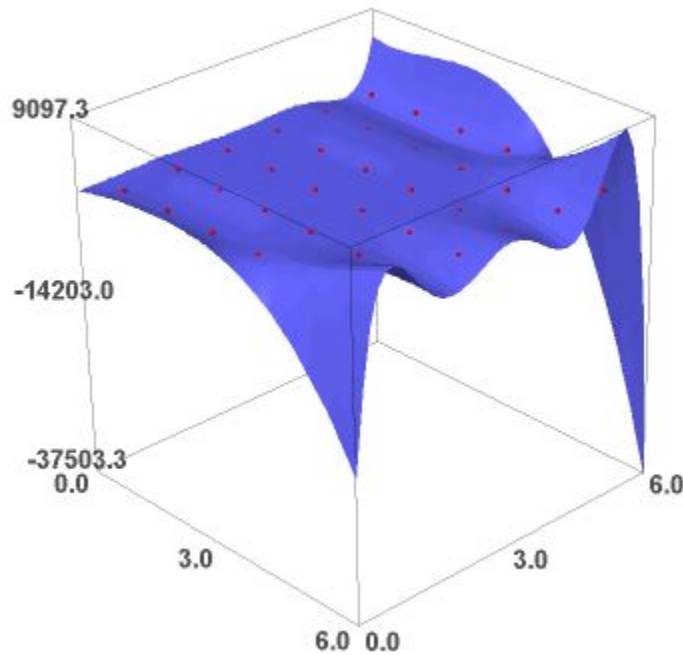
B.1.3 5×5

$$\begin{aligned}
 (x, y) \mapsto & -25/36*x^4*y^3 + 25/36*x^3*y^4 - 5/12*x^2*y^5 - \\
 & 3343/234630*y^7 + 125/24*x^4*y^2 - 75/274*x*y^5 - 845/72*x^4*y - \\
 & 115/4*x^3*y^2 + 85/18*x^2*y^3 + 75785/4932*x*y^4 + 5959/62568*y^5 + \\
 & 275/36*x^4 + 175/2*x^3*y + 50*x^2*y^2 - 28225/274*x*y^3 - 2375/36*x^3 - \\
 & 1910/9*x^2*y + 51935/274*x*y^2 + 259319/21330*y^3 + 550/3*x^2 - \\
 & 409297/10428*y^2 - 1367375/9864*x + 3109409/62568
 \end{aligned}$$



B.1.4 6×6

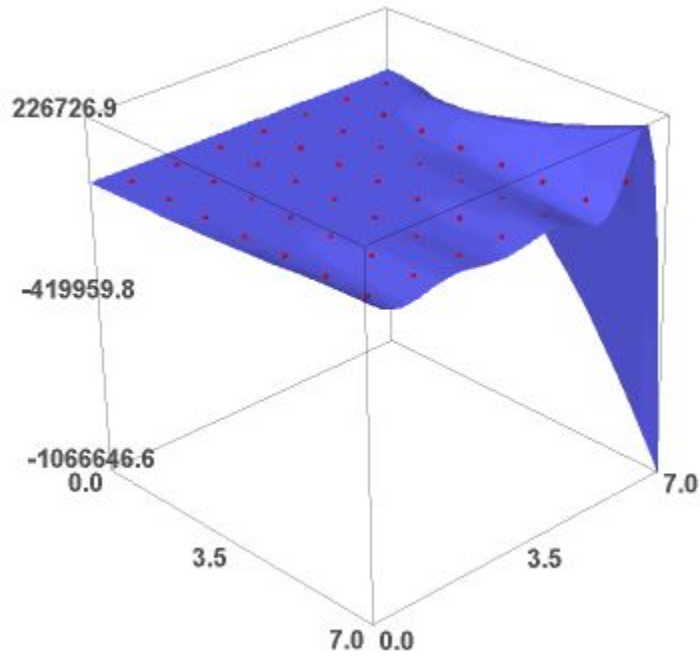
$$\begin{aligned}
 (x, y) \mapsto & -9/200*x^5*y^5 + 23/480*x^4*y^6 - 14873/36575*x^3*y^7 + \\
 & 63/80*x^5*y^4 + 1074399/209000*x^3*y^6 + 45732/36575*x^2*y^7 - \\
 & 21/4*x^5*y^3 - 829/96*x^4*y^4 - 3716337/209000*x^3*y^5 - \\
 & 406827/26125*x^2*y^6 + 17330637021/21943846400*y^8 + 1287/80*x^5*y^2 + \\
 & 593/8*x^4*y^3 + 5528529/104500*x^2*y^5 - 4433/2400*x*y^6 - \\
 & 913658751/78370880*y^7 - 2031/100*x^5*y - 19697/80*x^4*y^2 + \\
 & 182385237827/3134835200*y^6 + 279/50*x^5 + 5125/16*x^4*y + \\
 & 12953538/26125*x^3*y^2 + 136819/480*x*y^4 - 31489222203/313483520*y^5 - \\
 & 50517/560*x^4 - 1614474777/1463000*x^3*y - 144034443/104500*x^2*y^2 - \\
 & 87799/40*x*y^3 + 108707391/292600*x^3 + 4420728393/1463000*x^2*y + \\
 & 2579027/400*x*y^2 - 307366119/292600*x^2 - 601757/80*x*y + \\
 & 175199678597/685745200*y^2 + 6027117/2800*x - 63680388517/685745200
 \end{aligned}$$



B.1.5 7×7

$$\begin{aligned}
 (x, y) \mapsto & 1813/86400*x^6*y^5 - 49/3200*x^5*y^6 + 7/640*x^4*y^7 - \\
 & 1152216777509/2002350733344*x^2*y^9 - 12691/34560*x^6*y^4 - \\
 & 343/2880*x^5*y^5 - 2459/60480*x^3*y^7 + \\
 & 424229067376793/33372512222400*x^2*y^8 + \\
 & 148154691517/4768032501504*x*y^9 + 61397/25920*x^6*y^3 + \\
 & 9751/1920*x^5*y^4 + 4165/6912*x^4*y^5 - 4291/5760*x^3*y^6 - \\
 & 20369238101933/192533724360*x^2*y^7 - 29780121959/7946720835840*y^9 - \\
 & 26411/3840*x^6*y^2 - 169099/4320*x^5*y^3 - 208201/6912*x^4*y^4 + \\
 & 215861/17280*x^3*y^5 + 159308290039896599/400470146668800*x^2*y^6 - \\
 & 31886405085913/2980020313440*x*y^7 + 1109381/129600*x^6*y + \\
 & 589939/4800*x^5*y^2 + 26591803/103680*x^4*y^3 - \\
 & 487257997230893/834312805560*x^2*y^5 + \\
 & 237598679630501/1862512695900*x*y^6 + 1555574697943/993340104480*y^7 - \\
 & 34349/10800*x^6 - 691439/4320*x^5*y - 218099/256*x^4*y^2 - \\
 & 28076159/51840*x^3*y^3 - 4705041986147531/7946720835840*x*y^5 - \\
 & 200191392376031/9933401044800*y^6 + 442813/7200*x^5 + \\
 & 11930555/10368*x^4*y + 13910939/5760*x^3*y^2 + \\
 & 3195109801524151/2980020313440*x*y^4 +
 \end{aligned}$$

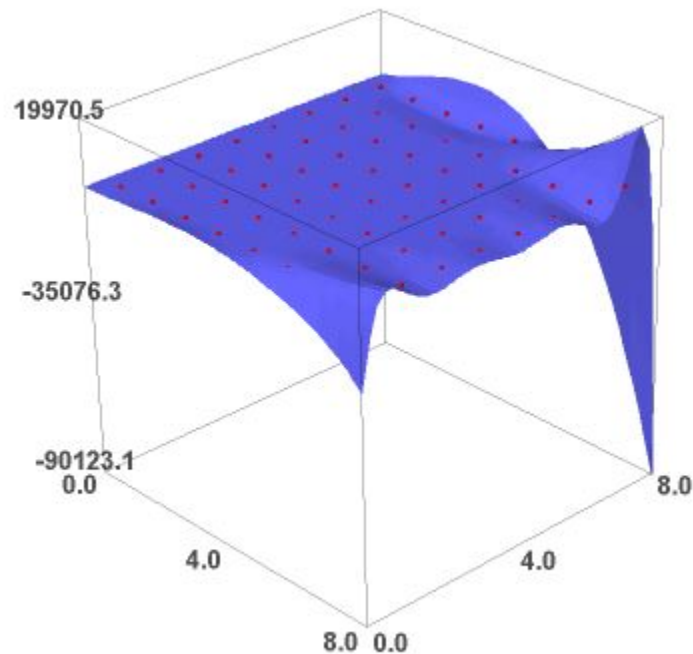
$$\begin{aligned}
& 1321227231308929/13244534726400*y^5 - 982499/2160*x^4 - \\
& 665470871/181440*x^3*y + 532115415438729829/400470146668800*x^2*y^2 - \\
& 11815380238207/62083756530*y^4 + 1335821/864*x^3 - \\
& 7660815562425389/4966700522400*x*y^2 - \\
& 35462255394170437/33372512222400*x^2 + 965065055275153/3311133681600*y^2 \\
& + 733442813831413/827783420400*x - 20742857285153/137963903400
\end{aligned}$$



B.1.6 8×8

$$\begin{aligned}
(x, y) \mapsto & -1/8100*x^7*y^6 + 2/2025*x^6*y^7 - 1/1350*x^5*y^8 - \\
& 591893/2736228600*x^3*y^{10} + 2/675*x^7*y^5 - 49/2025*x^6*y^6 + \\
& 4643/343980*x^4*y^8 + 96203/147987000*x^2*y^{10} + \\
& 602752/1867167315*x*y^{11} - 8254/10854718875*y^{12} - 43/1620*x^7*y^4 + \\
& 151/675*x^6*y^5 + 1007/3240*x^5*y^6 - 11330/51597*x^4*y^7 + \\
& 949931/30402540*x^3*y^8 - 117184775749/21173677352100*x*y^{10} + \\
& 44/405*x^7*y^3 - 391/405*x^6*y^4 - 21/5*x^5*y^5 - 5767/45675*x^2*y^8 + \\
& 176707/986792625*y^{10} - 133/675*x^7*y^2 + 3929/2025*x^6*y^3 + \\
& 76229/3240*x^5*y^4 + 49115/2457*x^4*y^5 + 51519374791/130701712050*x*y^8 \\
& + 248/2025*x^7*y - 1048/675*x^6*y^2 - 2866/45*x^5*y^3 - \\
& 4513919/29484*x^4*y^4 - 3052031/86175*x^3*y^5 + \\
& 100533763/7047000*x^2*y^6 - 98564/5221125*y^8 - 32/4725*x^7 + \\
& 584/2025*x^6*y + 64253/810*x^5*y^2 + 3535033/7371*x^4*y^3 +
\end{aligned}$$

$$\begin{aligned}
& 180837258499/547245720*x^3*y^4 - 40696/435*x^2*y^5 - \\
& 31723186349969/1008270350100*x*y^6 + 32/675*x^6 - 1492/45*x^5*y - \\
& 33415345/51597*x^4*y^2 - 135977698/120645*x^3*y^3 + \\
& 540328139/2959740*x^2*y^4 + 73231869440/373433463*x*y^5 + \\
& 234033161/140970375*y^6 - 9488/4725*x^5 + 13348000/51597*x^4*y + \\
& 333216251497/228019050*x^3*y^2 - 430192025563381/1058683867605*x*y^4 - \\
& 82176/7735*y^5 + 73024/1755*x^4 - 432722908/1809675*x^3*y + \\
& 350369137/2055375*x^2*y^2 + 4434531292/197358525*y^4 - \\
& 21720226096/59719275*x^3 - 1841944/1305*x^2*y + \\
& 412299974784368/588157704225*x*y^2 + 1572028256/1507275*x^2 - \\
& 6886639168/172297125*y^2 - 964919754032128/2156578248825*x + \\
& 2488659577/42664050
\end{aligned}$$

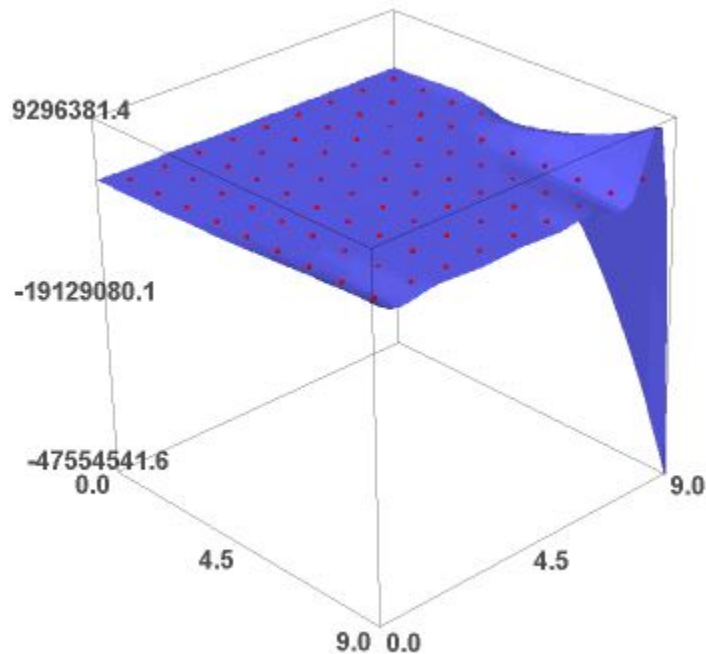


B.1.7 9×9

$$\begin{aligned}
(x, y) \mapsto & -227/1881600*x^8*y^7 + 199/1881600*x^7*y^8 - \\
& 199/2419200*x^6*y^9 + 9285205465/24872695981056*x^4*y^{11} - \\
& 49072489175075/42404830518403584*x^3*y^{12} + 681/179200*x^8*y^6 + \\
& 3/5600*x^7*y^7 + 7591/7257600*x^5*y^9 - 3723341587/358913361920*x^4*y^{10} \\
& + 10581236792557/446178772289600*x^3*y^{11} + \\
& 142426942329895336919/16687311162160730685440*x^2*y^{12} -
\end{aligned}$$

$$\begin{aligned}
& 8663/179200*x^8*y^5 - 32129/403200*x^7*y^6 - 133/57600*x^6*y^7 + \\
& 1835877693527/22611541800960*x^4*y^9 - \\
& 7358155626026948459201/25599852351042030028800*x^2*y^11 - \\
& 3624407839388419/2824795051414237440*x*y^12 + 1629/5120*x^8*y^4 + \\
& 14307/11200*x^7*y^5 + 10137/12800*x^6*y^6 - 517/1890*x^5*y^7 - \\
& 6986886834794099/2141658106990080*x^3*y^9 + \\
& 872567149522349606407/232198207265687347200*x^2*y^10 + \\
& 11365017920571191/412713562706625600*x*y^11 + \\
& 28851392356349/562120670186432000*y^12 - 617929/537600*x^8*y^3 - \\
& 298777/32256*x^7*y^4 - 1399861/96000*x^6*y^5 - 407/768*x^5*y^6 + \\
& 267264373182929267/10708290534950400*x^3*y^8 - \\
& 1416007114813076222516731/61439645642500872069120*x^2*y^9 - \\
& 15378840102551/14053016754660800*y^11 + 198243/89600*x^8*y^2 + \\
& 56461/1600*x^7*y^3 + 58073/512*x^6*y^4 + 36899263/576000*x^5*y^5 - \\
& 12940358817453/358913361920*x^4*y^6 + \\
& 439301735172330309566333/7585141437345786675200*x^2*y^8 - \\
& 5307988814555484233/1155597975578551680*x*y^9 - 3733367/1881600*x^8*y - \\
& 6196249/88200*x^7*y^2 - 1090885337/2419200*x^6*y^3 - 15443/24*x^5*y^4 + \\
& 27633744458949373/113057709004800*x^4*y^5 - \\
& 2368693798153612073/3441950529091200*x^3*y^6 + \\
& 144968208239455679/3079131296505600*x*y^8 + \\
& 714991301836277/4818177173026560*y^9 + 174471/313600*x^8 + \\
& 180749/2800*x^7*y + 1478349/1600*x^6*y^2 + 1033119847/362880*x^5*y^3 + \\
& 8838518388608775809/3569430178316800*x^3*y^5 - \\
& 17533497893710548509/103178390676656400*x*y^7 - \\
& 71410513934275113/70265083773304000*y^8 - 726387/39200*x^7 - \\
& 249932917/288000*x^6*y - 1584071/256*x^5*y^2 - \\
& 10546904785383203/2261154180096*x^4*y^3 - \\
& 112616887825535401031430479/102399409404168120115200*x^2*y^5 + \\
& 1141351/4480*x^6 + 47569292/7875*x^5*y + \\
& 1420240053149937/89728340480*x^4*y^2 - \\
& 36597753308989660409/2677072633737600*x^3*y^3 + \\
& 975636269939802108487/825427125413251200*x*y^5 - 4108957/2240*x^5 - \\
& 844685518536534691/44415528537600*x^4*y + \\
& 3365475335461191441241/176686793826681600*x^3*y^2 + \\
& 486081833516550206567746963/38399778526563045043200*x^2*y^3 +
\end{aligned}$$

$$\begin{aligned}
& 13579345850284010311/56212067018643200*y^5 + \\
& 4575299031428299/690908221696*x^4 - \\
& 159102058718947727386048049/7822177107262842508800*x^2*y^2 - \\
& 9491700599768581929541/962998312982126400*x*y^3 - \\
& 26993452693425414971/16060590576755200*y^4 - \\
& 765655727700947739/118373959995200*x^3 + \\
& 20788867269632604283217/1412397525707118720*x*y^2 + \\
& 1888507345344640837/430194390448800*y^3 + \\
& 2695770302196901835507913/325924046135951771200*x^2 - \\
& 4774313147302122125/1124241340372864*y^2 - \\
& 24127512673251150201/4374206173573600*x + \\
& 55207158100046568199/45672304452647600
\end{aligned}$$

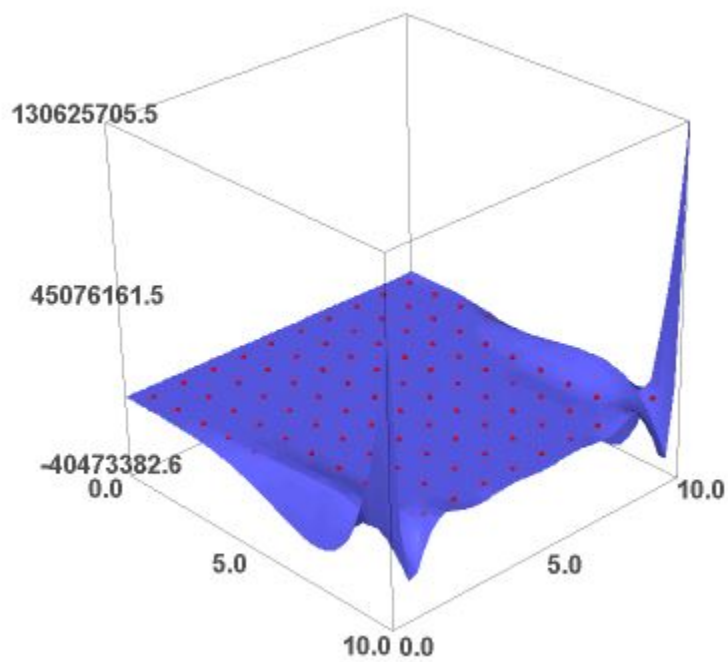
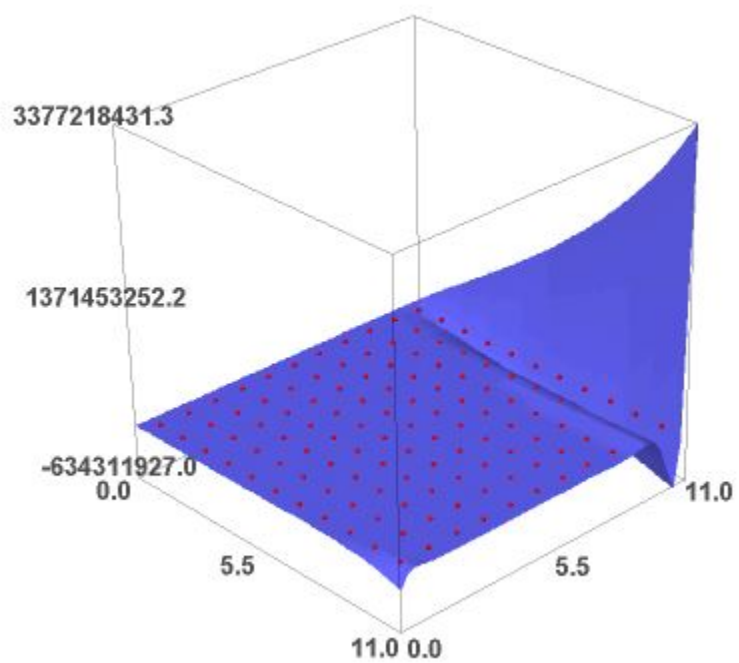


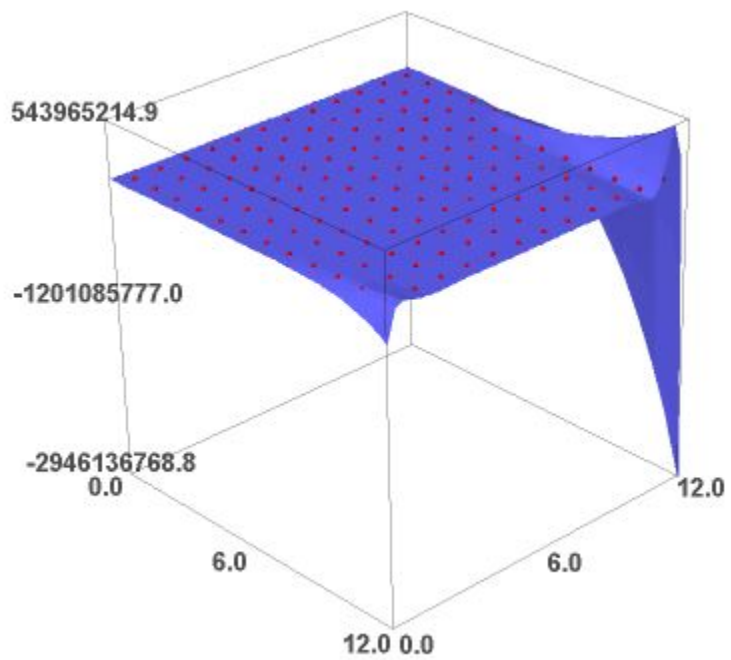
B.1.8 10×10

$$\begin{aligned}
(x, y) \mapsto & 3275/1316818944*x^9*y^9 - 4301/2090188800*x^8*y^{10} + \\
& 1674671/152613828864*x^7*y^{11} - 17275/146313216*x^9*y^8 - \\
& 352429451/1046494826496*x^7*y^{10} - 17906593/174415804416*x^6*y^{11} + \\
& 129565/54867456*x^9*y^7 + 436391/162570240*x^8*y^8 + \\
& 11010559801/3662731892736*x^7*y^9 + 6622782529/2092989652992*x^6*y^{10} + \\
& 355399567/639524616192*x^5*y^{11} - 811375/31352832*x^9*y^6 -
\end{aligned}$$

$$\begin{aligned}
& 579881/8128512*x^8*y^7 - 29662151237/1046494826496*x^6*y^9 - \\
& 24050692789/1395326435328*x^5*y^10 - \\
& 1312920035239/248236047360000*x^4*y^11 + \\
& 10647819290969322369078665/3024023442003306992172417564672*y^15 + \\
& 3538387/20901888*x^9*y^5 + 3228997/3686400*x^8*y^6 + \\
& 108270069017/697663217664*x^5*y^9 + \\
& 449688275290507/2383066054656000*x^4*y^10 + \\
& 14997350741/5036256352512*x^3*y^11 - \\
& 363017177554495/551921264298657792*x^2*y^12 - \\
& 22702646006971801469569858753/256257986492724681410758940295168*y^14 - \\
& 42335675/62705664*x^9*y^4 - 23541179/3870720*x^8*y^5 - \\
& 986891102497/348831608832*x^7*y^6 - \\
& 171710293396513/79435535155200*x^4*y^9 - \\
& 4179467076509/43952782712832*x^3*y^10 + \\
& 10955969885841391/482931106261325568*x^2*y^11 + \\
& 264575345/164602368*x^9*y^3 + 525085691/20901888*x^8*y^4 + \\
& 217720130975093/6104553154560*x^7*y^5 + \\
& 2670133986569/99666173952*x^6*y^6 + \\
& 19239087132697/21976391356416*x^3*y^9 - \\
& 36024477018358565/131708483525816064*x^2*y^10 + \\
& 24787790595188217373034023687/1152008930286974092256159072256*y^12 - \\
& 29227375/13716864*x^9*y^2 - 141938303/2322432*x^8*y^3 - \\
& 102234006951785/523247413248*x^7*y^4 - \\
& 588270903813841/1744158044160*x^6*y^5 - \\
& 9801224327969/66444115968*x^5*y^6 + \\
& 7695387275990911/35462292480000*x^4*y^7 + \\
& 177761461600399633/150523981172361216*x^2*y^9 - 2006567/914457600*x*y^10 \\
& - 1145831322016708204222399220515/4032031256004409322896556752896*y^11 + \\
& 12242647/9144576*x^9*y + 100139840497/1219276800*x^8*y^2 + \\
& 511091341712363/915682973184*x^7*y^3 + \\
& 1927701262218415/1046494826496*x^6*y^4 + \\
& 2154119238077749/1162772029440*x^5*y^5 - \\
& 243622609899002017/113479335936000*x^4*y^6 + \\
& 376347808205122469116315897392731/241921875360264559373793405173760*y^10 \\
& - 150085/571536*x^9 - 705647837/13547520*x^8*y - \\
& 6020663189057/7267325184*x^7*y^2 - 2740442799091157/523247413248*x^6*y^3
\end{aligned}$$

$$\begin{aligned}
& - 7022606093035135/697663217664*x^5*y^4 + \\
& 3971339384423017829/397177675776000*x^4*y^5 - \\
& 1774629388967329/2092989652992*x^3*y^6 + 590957/3386880*x*y^8 - \\
& 40335485192230355465946457285775/12096093768013227968689670258688*y^9 + \\
& 6017855/580608*x^8 + 20413120348289/36336625920*x^7*y + \\
& 64197985825711/8305514496*x^6*y^2 + 309537666100801/10900987776*x^5*y^3 \\
& - 1193853963572428007/47661321093120*x^4*y^4 + \\
& 384404807676492821/36627318927360*x^3*y^5 + 20076785/1016064*x*y^7 - \\
& 1029436509683/8478546048*x^7 - 108374733628361/20763786240*x^6*y - \\
& 114909377548043/2768504832*x^5*y^2 + \\
& 2711368195090581079/82745349120000*x^4*y^3 - \\
& 1217923552164376415/21976391356416*x^3*y^4 - \\
& 2289115685900587501717/1756113113677547520*x^2*y^5 - \\
& 18942231577/43545600*x*y^6 + 790280753981/692126208*x^6 + \\
& 1058940190154653/38066941440*x^5*y - \\
& 87448083339333127/4728305664000*x^4*y^2 + \\
& 821664155838501461/5494097839104*x^3*y^3 + \\
& 10028817592660310697805/1053667868206528512*x^2*y^4 + \\
& 1707823739/483840*x*y^5 - 55050768482191/8882286336*x^5 - \\
& 4499309168727277/21801975552*x^3*y^2 - \\
& 876213034624285879861/37630995293090304*x^2*y^3 - \\
& 1858076107/142884*x*y^4 + \\
& 61366652985458005310678320800905/252001953500275582681034797056*y^5 + \\
& 127236590835157/27581783040*x^4 + 315495409400235913/2398217310720*x^3*y \\
& + 848146842824167090735/45993438691554816*x^2*y^2 + \\
& 5510586071/290304*x*y^3 - \\
& 2638523024112239103307265318552755/4158032232754547114237074151424*y^4 - \\
& 18049237172069611/559584039168*x^3 - \\
& 368686702420475398601/114983596728887040*x^2*y + \\
& 21413081563/50803200*x*y^2 + 43700859290646910393/4529656840834944*x^2 - \\
& 26894770183/1693440*x*y + \\
& 622043084185718750288547468045563/1365010581459826072855605150720*y^2 - \\
& 882502211/620928*x + \\
& 1547404439098024312557762802261/1276795611909701258623440192
\end{aligned}$$

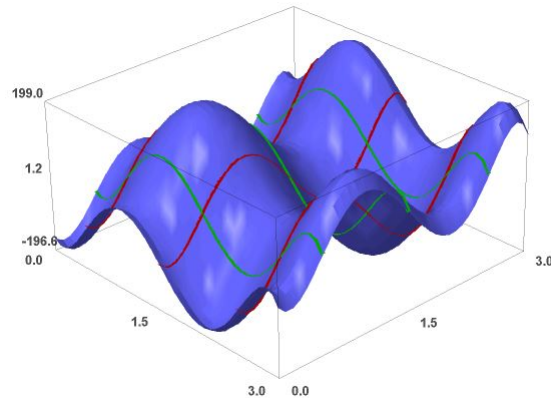
**B.1.9** 11×11 

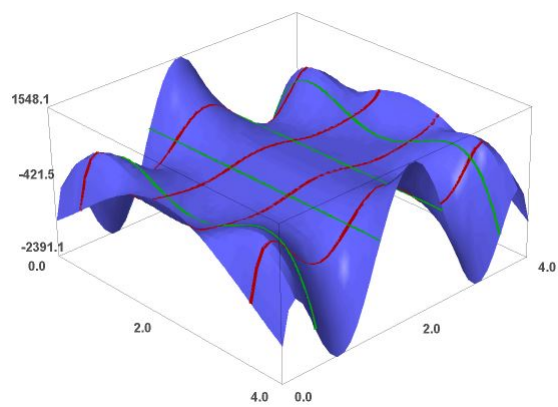
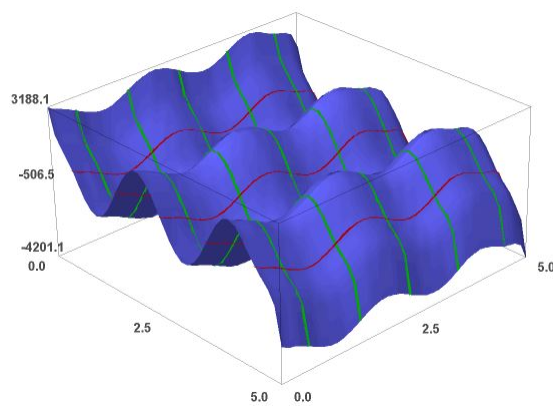
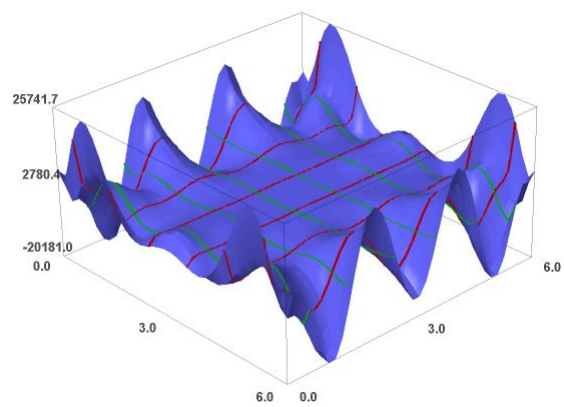
B.1.10 12×12 

B.2 Solutions for line integrals over a square

B.2.1 3×3

$$\begin{aligned}
 (x, y) \mapsto & -860/157437*x^3*y^3*(244*\sqrt{2} - 917) + \\
 & 464/52479*x*y^5*(244*\sqrt{2} - 917) + 11200/549*x^6*(\sqrt{2} - 2) + \\
 & 11200/549*y^6*(\sqrt{2} - 2) + 2/17493*x^2*y^3*(65788*\sqrt{2} - 243803) + \\
 & 2/17493*x^3*y^2*(45796*\sqrt{2} - 173831) - 4/5831*x*y^4*(24142*\sqrt{2} - \\
 & 90587) + 8/21*x^4*y*(2*\sqrt{2} - 7) + 2/157437*x*y^3*(2570684*\sqrt{2} - \\
 & 9707593) - 2/3843*y^4*(1434602*\sqrt{2} - 2867557) - \\
 & 4/3843*x^4*(718948*\sqrt{2} - 1439543) - 2/52479*x^3*y*(213940*\sqrt{2} - \\
 & 788543) - 9/5831*x^2*y^2*(19708*\sqrt{2} - 73493) + \\
 & 2/254065*x^3*(311636947*\sqrt{2} - 624369942) + \\
 & 2/254065*y^3*(310918306*\sqrt{2} - 621473601) + \\
 & 1/17493*x^2*y*(595172*\sqrt{2} - 2097403) - \\
 & 1/1524390*y^2*(3727375024*\sqrt{2} - 7458339959) - \\
 & 1/762195*x^2*(1866758069*\sqrt{2} - 3739916929) - \\
 & 2/5831*x*y*(94552*\sqrt{2} - 316631) + 1/32012190*y*(13902375286*\sqrt{2} \\
 & - 27830909141) + 1/10670730*x*(4644397109*\sqrt{2} - 9212597044) + \\
 & 5773446719/21341460*\sqrt{2} - 833150771/1524390
 \end{aligned}$$

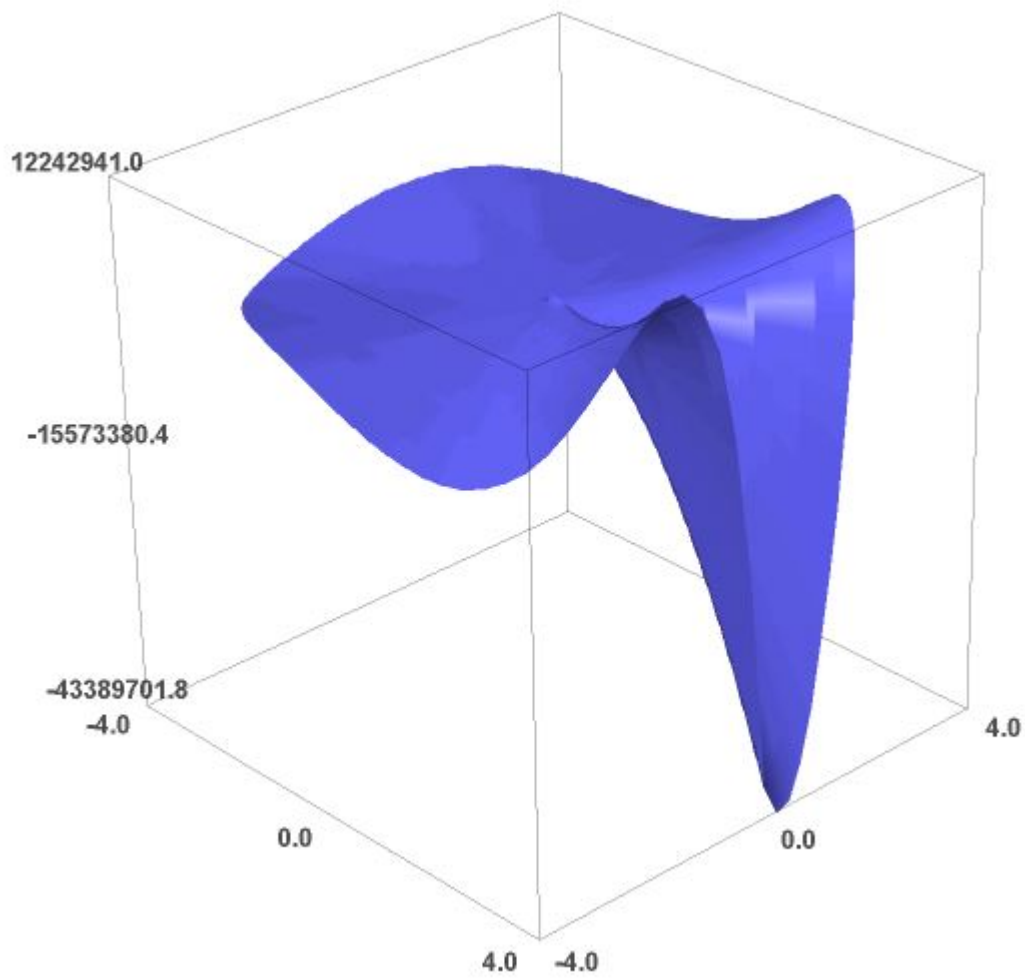


B.2.2 4×4 **B.2.3** 5×5 **B.2.4** 6×6 

B.3 Solution for volume over a circle

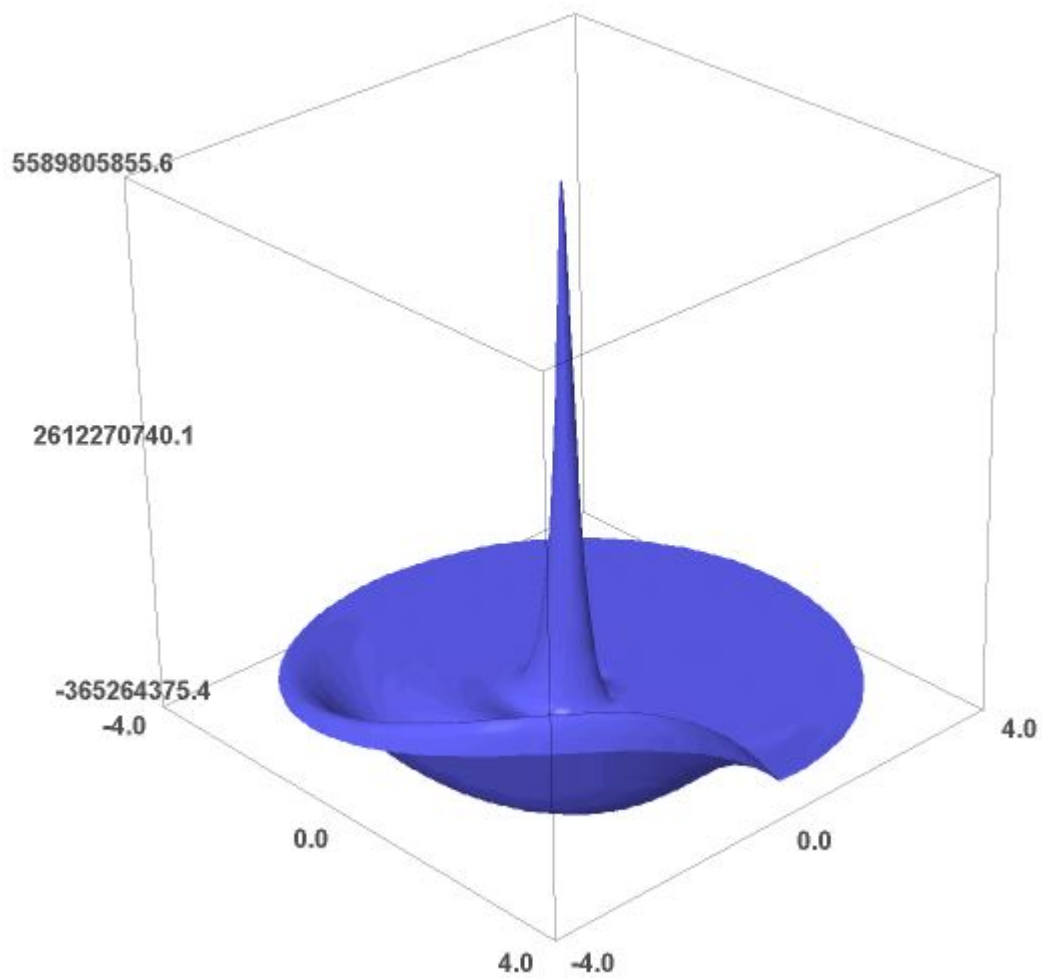
$$\begin{aligned}
& -725110537030224393188290344252529/682670175962816821923333980771520*r^4\backslash \\
& /pi - \\
& 1816344198150318412011493278392/20186548011797388851819339955*T*r^4/pi^2 \\
& + \\
& 784777562187011393923473581019758/723258363848099816044466406675*T^3*r^4\backslash \\
& /pi^4 - \\
& 8693749420287366160917192792936448/130909763856506066704048419608175*T^7\backslash \\
& *r^4/pi^8 + \\
& 120245502615175637802726789773296594851695/39378278073775300852828110866\backslash \\
& 00081830656*r^3/pi + \\
& 46866480915604496356452058366172260586/485172982593103988437386371568248\backslash \\
& 85*T*r^3/pi^2 - 8895890936711/2271257415*T^2*r^3/pi^3 + \\
& 4564976394637654835962001121672366796544/3476626289631247917145249303447\backslash \\
& 61745*T^3*r^3/pi^4 - 1292250084832/21631023*T^4*r^3/pi^5 + \\
& 7962491015168/108155115*T^5*r^3/pi^6 - \\
& 15579428098048/324465345*T^6*r^3/pi^7 + \\
& 148650410203621850645737184799546710491136/89895622631893696143327160560\backslash \\
& 57410835*T^7*r^3/pi^8 - 1581960331264/757085805*T^8*r^3/pi^9 - \\
& 252794010545435109302353936859759252885166022541/58146252369824891073347\backslash \\
& 71993657776433991754240*r^2/pi - \\
& 83806663267582804818647948035429055658969826/358204980878365516015521970\backslash \\
& 10939791609575*T*r^2/pi^2 + \\
& 282028709619621697/67075105011957*T^2*r^2/pi^3 - \\
& 863167217738955847198502086380600718109893733/51336117149087301024544971\backslash \\
& 848827524549855*T^3*r^2/pi^4 + \\
& 1354854223626451472/9582157858851*T^4*r^2/pi^5 - \\
& 9835312974643392896/79851315490425*T^5*r^2/pi^6 + \\
& 107579239095808342214537136424444825439663581184/18583674407969602970885\backslash \\
& 27980927556388704751*T^7*r^2/pi^8 - \\
& 738345193419931648/22358368337319*T^8*r^2/pi^9 + \\
& 621332324736827392/111791841686595*T^9*r^2/pi^10 - \\
& 8275028289473842532761107168168028052624433913/1938208412327496369111590\backslash \\
& 664552592144663918080*r/pi + \\
& 9364964187711456996997210979761055208445706/1194016602927885053385073233\backslash
\end{aligned}$$

$$\begin{aligned}
& 6979930536525 * T * r / \pi^2 + 22153982806023791 / 3194052619617 * T^2 * r / \pi^3 + \\
& 360275714269500057283738153439787682219298524 / 17112039049695767008181657 \backslash \\
& 282942508183285 * T^3 * r / \pi^4 - 807316029646392560 / 3194052619617 * T^4 * r / \pi^5 \\
& + 6384318233538350336 / 26617105163475 * T^5 * r / \pi^6 - \\
& 54037861150955871093355121495760844913719967744 / 442468438284990546925839 \backslash \\
& 995458941997310655 * T^7 * r / \pi^8 + \\
& 372241275299790848 / 5323421032695 * T^8 * r / \pi^9 - \\
& 443785954896904192 / 37263947228865 * T^9 * r / \pi^{10} + \\
& 5717346601461109617441191721311744 / 9694258138588628534123134485 * T^6 / \pi^7 \\
& - \\
& 61310065455980249771496234746078464 / 37699892761177999854923300775 * T^7 / \pi \backslash \\
& ^8 + \\
& 2025025581758006609163118597259264 / 1077139793176514281569237165 * T^8 / \pi^9 \\
& - \\
& 5044845067290146275387176962490368 / 4523987131341359982590796093 * T^9 / \pi^{10} \backslash \\
& 0 + \\
& 3247074696180182124139484563898368 / 9694258138588628534123134485 * T^{10} / \pi \backslash \\
& 11 - \\
& 654037165692554899635184666148864 / 16157096897647714223538557475 * T^{11} / \pi \backslash \\
& 12
\end{aligned}$$



B.4 Solution for line integrals over a circle

The solution was degree 12, and the output was too long to present here.



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