5-2017

Solving the Yang-Baxter Matrix Equation

Mallory O. Jennings

The University of Southern Mississippi

Solving the Yang-Baxter Matrix Equation

by

Mallory Jennings

A Thesis
Submitted to the Honors College of
The University of Southern Mississippi
in Partial Fulfillment
of the Requirements for the Degree of
Bachelor of Science
in the Department of Mathematics

May 2017

Approved by

Jiu Ding, Ph. D., Thesis Advisor
Professor of Mathematics

Bernd Schroeder, Ph. D.,
Mathematics Department Chair

Ellen Weinauer, Ph.D., Dean
Honors College

Abstract

The Yang-Baxter equation is one that has been widely used and studied in areas such as statistical mechanics, braid groups, knot theory, and quantum mechanics. While many sets of solutions have been found for this equation, it is still an open problem. In this project, I solve the Yang-Baxter matrix equation that is similar in format to the Yang-Baxter equation. I try to solve the corresponding Yang-Baxter matrix equation, $AXA = XAX$, where X is an unknown nxn matrix, and $A = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}$, by using the Jordan canonical form to find infinitely many solutions.

Acknowledgements

I would like to thank my thesis adviser, Dr. Jiu Ding for all of his help with this project. I have known since freshman year that Dr. Ding is who I wanted to be my thesis adviser because of how great of a teacher and mathematician he is. He has pushed me to do my best and has taught me so much throughout this experience. I know that without his guidance I would not have been able to complete my thesis. His teaching and encouragement are inspiring to me, and I hope that one day I can be half as good of a mentor and teacher as he is. Because of him, I know that I am a better student, a better mathematician, and I will be a better teacher. I cannot thank him enough for all he has done for me over these past four years.

I would also like to thank the mathematics honors committee, Dr. Tian, Dr. Zhu, Dr. Kohl, and Dr. Schroeder for their help and encouragement as I defended my thesis to them. Their critiques were much appreciated.

Table of Contents

Chapter 1: Introduction

The parameter free Yang-Baxter equation is a nonlinear matrix equation discovered independently by both Yang in 1967, and Baxter in 1972. The solutions of this equation, dubbed the Yang-Baxter equation, are extensive and have led to discoveries and research in many fields. These areas include statistical mechanics, braid groups, knot theory, and quantum mechanics (Ding & Rhee 2012). While the set of solutions for this equation has been heavily researched, it is still an open problem as not all solutions have been found (Nichita, 2015). However, the list of areas for which this equation is applicable is continuing to grow.

One area that has not been studied much using the Yang-Baxter equation is in matrix theory. Even with recent research, it is difficult to find all nontrivial solutions of the matrix equation (Ding & Rhee 2014). The matrix equation AXA=XAX is what we will refer to as the Yang-Baxter matrix equation because it is so similar to the original Yang-Baxter equation. A and X both represent square matrices of the same size. The goal is to determine for a given square matrix A, which square matrices, X, will solve this equation.

As stated earlier, finding trivial solutions of this equation is not a difficult task. Two trivial solutions of this equation are X=0 and X=A. These both solve the equation easily. The more difficult task is finding nontrivial solutions of the equation. Since finding all matrices X that solve the equation is such an extensive task, I will only be finding a few specific solutions to the equation for some class of known matrix A. For my project, I was given a matrix A and had to find the matrix X for which the equation holds true. I used linear algebra methods, and more specifically, eigenvalues and eigenvectors, and the resulting Jordan form, to complete the project.

Chapter 2: Literature Review

The Yang-Baxter equation is a parameter free equation that was discovered in the late 1900s and has been proven to have many applications in both mathematics and physics (Ding & Rhee, 2014). In 1967, while trying to solve the difficult task of finding the eigenfunctions of a fermion gas problem, Yang discovered an essential matrix identity: ABA=BAB. Five years later in 1972, Baxter discovered the same equation through his research in physics. As more and more people began to research this new identity, the equation was eventually dubbed the Yang-Baxter equation (Zhang, 1993).

Several years after the initial discoveries of the equation, the Yang-Baxter equation began to be extensively studied in physics and mathematics. New developments and applications of the equation have continued to arise, which has led many to believe that this equation is actually fundamental in many aspects (Zhang, 1993).

This equation has been shown to have applications in numerous areas. These areas include braid groups, knot theory, statistical mechanics, quantum mechanics and many more (Chen, 2012), and these topics have been studied heavily in relation to the Yang-Baxter equation.

The equation is interesting to those who study quantum mechanics because of its unifying features. This property has allowed the equation to be used in the production and research of quantum computers (Nichita, 2015). In addition, the Yang-Baxter equation is very closely related to braid groups. Finding unitary braid group representations can aid in constructing quantum computers. Braid groups and the Yang-Baxter equation can make processing quantum information a much easier task (Chen, 2012).

While many solutions to this equation have been found, it still remains an open problem. Part of the reason for its openness is that the equation is extremely difficult to solve. For example, if the equation is in vector space V, then the Yang-Baxter equation has d^6 cubic equations and d^4 unknowns, where d is the dimension of vector space V. Because of this, the number of equations and unknowns increases exponentially as the vector space increases (Chen, 2012).

The Yang-Baxter equation has been studied heavily in many areas, but one area where it has previously been overlooked is in matrix theory. More recently, however, there have been significant developments in this area. In the matrix equation ABA=BAB, researchers have tried to find square matrices A and B that make the equation true. However, this task has proven to be extremely difficult. While solving the linear equation AX=XA is relatively easy, solving quadratic AXA=XAX is much more difficult because it is nonlinear (Ding & Rhee, 2012). There is no one method to find all solutions. However, many different methods have been used to find some solutions of the Yang-Baxter matrix equation (Ding & Rhee, 2012).

To find the matrices that solve this equation, it is assumed that one matrix, A, is given and the other matrix, X, is to be found. The two trivial solutions of this equation are X=0 and X=A. To find the nontrivial solutions, several different methods were used. The first method used was the Brouwer fixed point theorem to find solutions when A^{-1} is a stochastic matrix. Using this theorem, two nontrivial solutions of the Yang-Baxter matrix equation were found (Ding & Rhee, 2012). Later, more solutions to the Yang-Baxter matrix equation were found using spectral projectors, but only a general spectral solution was found (Ding & Rhee, 2014). Additionally, more specific solutions were found by putting matrix A in its Jordan canonical form. Using this tactic, several specific solutions were found to commute with A (Ding, Zhang, & Rhee 2013). This allowed for some commuting solutions to be found, but the task of finding all commuting solutions still remains open (Ding, Zhang, & Rhee, 2015).

While all of these different methods and theorems have proven helpful in finding some of the solutions to the Yang-Baxter matrix equation, all solutions to the equation have not been found. Research continues to be done in order to try and find all solutions to the equation.

For my project, I will be specifically studying the Yang-Baxter matrix equation. The purpose of my project is to potentially find several more nontrivial solutions to the equation AXA=XAX. I will complete this project using linear algebra methods such as eigenvalues, eigenvectors, and Jordan structures.

Chapter 3: Materials and Methods

In order to try to find nontrivial solutions to the Yang-Baxter matrix equation, I used methods from linear algebra. There are two main ways I tried to solve the equation AXA=XAX. First, I use eigenvalues and eigenvectors to try to solve the equation. I also be used Jordan structures to simplify and try to solve the matrix equation.

Eigenvalues and eigenvectors work together to turn matrix equations into the much simpler equation of the product of a scalar and a vector. If given an *n x n* matrix "A" such as the one in the equation AXA=XAX, the eigenvector of A is a nonzero vector such that $Ax = \lambda x$. In this equation, λ is called the eigenvalue. It is the scalar value corresponding to the eigenvector *x.* If we are able to find an eigenvector *x* and eigenvalue whose product gives a matrix *Ax,* then that is a much simpler way to study properties of an *n x n* matrix. By finding eigenvalues and eigenvectors that represent X or A in the AXA=XAX equation, it will be much easier to find solutions. In this method, either the eigenvector or the eigenvalue will need to be found first. Once this has been determined, the other part (either eigenvalue or eigenvector) can also be found.

The second method will involve the use of Jordan structures and matrix similarity to find solutions of AXA=XAX. If we have matrices A and P with $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a, b, c, and d are real entries, I want to find the matrix *P* such that $P^{-1}AP = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$. In other words, I want a matrix *P* such that we get the diagonal matrix out of the previous equation. To begin using this method, I would first need to find P^{-1}. From linear algebra methods, we know that the inverse of a matrix is the product of the reciprocal of the determinant and the companion matrix. For example, in the earlier matrix *P,* the determinant would be *ad-bc.* So P^{-1} would be

$$\frac{1}{(ad-bc)} * \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

To check that this is indeed the inverse of matrix P, we could multiply $P^{-1} * P$, and would get $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ which is the identity matrix. Finally, the complete product would be,

$$P^{-1}AP = \frac{1}{(ad-bc)} * \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} * \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} * \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If this method works, I should end up with a matrix $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. Since the eigenvalues of a diagonal matrix fall on the diagonal, λ_1 and λ_2 would be our eigenvalues. Doing this computation and finding that simple matrix would again make solving the Yang-Baxter matrix equation much easier to do.

In general, the simplest form of a given square matrix may not be diagonal as the above example shows. However, in linear algebra we know that any square matrix is similar to its Jordan canonical form that is a block diagonal matrix consisting of the Jordan blocks of various size. A Jordan block is an upper triangular matrix of all diagonal entries λ, all super-diagonal entries 1, and all other entries 0. For example, the 2 x 2 matrix $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ is a Jordan block. After simplifying the given matrix A to its Jordan form, I can solve the simplified Yang-Baxter matrix equation. That is the strategy to solve this quadratic matrix equation.

While these are only two of many different techniques used in linear algebra, these two will prove to be most essential and helpful during this process. Hopefully, through proper use and manipulation of each of these techniques, further nontrivial solutions of the Yang-Baxter like matrix equation can be found.

Chapter 4: Solutions and Description of Processes

As stated earlier, if A is an nxn matrix, a real number λ is called an eigenvalue of matrix A if Ax = λx for some nonzero column x in R^n. The condition Ax = λx can be rewritten as (λI – A)x = 0 where I is an nxn identity matrix. To find the eigenvalues of a specific matrix, we take the determinant of (λI – A). To begin with a simple example, we use the 2x2 matrix A = $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. This means that the identity matrix we will use for this example will be I = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Substituting matrices A and I into the above equation, we get

$$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)=0$$

$$\det\left(\begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix}\right) = \lambda^2 - (-1) =$$

$$\lambda^2 + 1 = 0$$

$$\lambda^2 = -1$$

$$\lambda = \pm i.$$

In this same manner, we can find the eigenvalues for the matrix B=$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Again, the identity matrix will be the 2x2 matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Substituting B and I into the equation used previously, we get

$$\det(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})=0$$

$$\det\left(\begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix}\right)=\lambda^2 - 1 = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1.$$

Using this method, we have been able to easily find the eigenvalues for two simple 2x2 matrices, A and B. However, for this project our matrices will not be just 2x2 matrices. Instead we are looking at mxm block matrices that would be similar to A and B. For example, the 2x2 block identity matrix would look like

$$\begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix},$$

and the equation (λI – A) in block matrices would look like

$$\begin{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \end{bmatrix} - \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}.$$

Finally, we would need to find the determinant of the above matrix,

$$\det\left(\begin{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \end{bmatrix}\right).$$

This is just an example because in reality, we will need to use *mxm* block matrices to try to solve this problem.

We will let A be the block matrix $\begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}$. We already found the eigenvectors of the 2x2 version of A=$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to be 1 and -1. Since the block matrix of

A is of the same components, the eigenvalues will be the same. However, for the block matrix, we will have *m* sets of 1 and *m* sets of -1 for our eigenvalues. This gives us

$$\lambda = 1_1, 1_2, 1_3, \dots, 1_m, -1_1, -1_2, -1_3, \dots, -1_m.$$

Using these eigenvalues, we now need to use the equation Av=λv to find which vectors, v, this equation will hold true for. We begin by letting v = $\begin{bmatrix} x_m \\ y_m \end{bmatrix}$, where x_m and y_m are m-dimensional column vectors. Plugging in our block matrix for A, vector v, and our first eigenvalue 1, we get

$$\begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} \begin{bmatrix} X_m \\ Y_m \end{bmatrix} = 1 \begin{bmatrix} X_m \\ Y_m \end{bmatrix}$$

$$\begin{bmatrix} I_m Y_m \\ I_m X_m \end{bmatrix} = \begin{bmatrix} X_m \\ Y_m \end{bmatrix}$$

$$I_m Y_m = X_m$$

$$Y_m = X_m$$

and

$$I_m X_m = Y$$

$$X_m = Y_m$$

so $X_m = Y_m$.

This means that both X_m and Y_m are the same vector. This gives us vectors

$$V_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0_m \\ 1 \\ 0 \\ \vdots \\ 0_m \end{bmatrix},\ V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0_m \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0_m \end{bmatrix}, \dots\ V_m = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1_m \\ 0 \\ \vdots \\ 0 \\ 1_m \end{bmatrix}.$$

Doing the same thing with our second eigenvalue, -1, we get

$$\begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} \begin{bmatrix} X_m \\ Y_m \end{bmatrix} = -1 \begin{bmatrix} X_m \\ Y_m \end{bmatrix}$$

$$I_m Y_m = -X_m$$

$$Y_m = -X_m$$

and

$$I_m X_m = -Y_m$$

$$X_m = -Y_m.$$

This gives us vectors

$$V_1 = \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0_m \\ 1 \\ 0 \\ \vdots \\ 0_m \end{bmatrix},\ V_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ \vdots \\ 0_m \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0_m \end{bmatrix}, \dots\ V_m = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1_m \\ 0 \\ \vdots \\ 0 \\ 1_m \end{bmatrix}.$$

We have now found our eigenvectors when our eigenvalues are either 1 or -1.

Next, we will let A be the block matrix $\begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}$. We already found the eigenvectors of the 2x2 version of A=$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ to be *i* and -*i*. Since the block matrix of A is of the same components, the eigenvalues will be the same. However, for the block matrix, we will have *m* sets of *i* and *m* sets of -*i* for our eigenvalues. This gives us

$$\lambda = i_1, i_2, i_3, \dots, i_m, -i_1, -i_2, -i_3, \dots, -i_m.$$

We will use the equation $Av = \lambda v$ to find the eigenvectors for which this equation holds true. Using our block matrix for A, and the vector $\begin{bmatrix} X_m \\ Y_m \end{bmatrix}$ for v, we can find the eigenvector when our eigenvalue first is *i.* Plugging these into our equation, we get

$$\begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} \begin{bmatrix} X_m \\ Y_m \end{bmatrix} = i \begin{bmatrix} X_m \\ Y_m \end{bmatrix}$$

$$I_m Y_m = iX_m$$

$$-I_m X_m = iY_m$$

$$Y_m = iX_m$$

$$-X_m = iY_m$$

This gives us vectors

$$V_1 = \begin{bmatrix} -i \\ 0 \\ \vdots \\ 0_m \\ 1 \\ 0 \\ \vdots \\ 0_m \end{bmatrix}, \ V_2 = \begin{bmatrix} 0 \\ -i \\ 0 \\ \vdots \\ 0_m \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0_m \end{bmatrix}, \ \ldots \ V_m = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -i_m \\ 0 \\ \vdots \\ 0 \\ 1_m \end{bmatrix}.$$

Now, we go through the same method with our second eigenvalue, -i.

$$\begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} \begin{bmatrix} X_m \\ Y_m \end{bmatrix} = -i \begin{bmatrix} X_m \\ Y_m \end{bmatrix}$$

$$\begin{bmatrix} I_m Y_m \\ -I_m X_m \end{bmatrix} = \begin{bmatrix} -iX_m \\ -iY_m \end{bmatrix}$$

$$I_m Y_m = -iX_m$$

$$-I_m X_m = -iY_m$$

so

$$Y_m = -iX_m$$

and

$$X_m = iY_m.$$

Using this, we get eigenvectors

$$V_1 = \begin{bmatrix} i \\ 0 \\ \vdots \\ 0_m \\ 1 \\ 0 \\ \vdots \\ 0_m \end{bmatrix}, \ V_2 = \begin{bmatrix} 0 \\ i \\ 0 \\ \vdots \\ 0_m \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0_m \end{bmatrix}, \ \ldots \ V_m = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ i_m \\ 0 \\ \vdots \\ 0 \\ 1_m \end{bmatrix}.$$

We have now found our vectors for our four different eigenvalues that we could possibly have. However, because finding a solution to our equation at this point would still be difficult, we are going to limit our matrices to block diagonal matrices only. Therefore, we need to find a block matrix D that is similar to our matrix A. We know that A will be similar to D if there exists a nonsingular matrix U such that

$$D = U^{-1}AU.$$

To do this, we will look at two equations for the two different eigenvalues, 1 and -1.

$$A[v_1 \quad v_2 \quad \cdots \quad v_m] = [v_1 \quad v_2 \quad \cdots \quad v_m]\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

and

$$A[u_1 \quad u_2 \quad \cdots \quad u_m] = [u_1 \quad u_2 \quad \cdots \quad u_m]\begin{bmatrix} -1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{bmatrix}.$$

Combining the two, we get

$$A[v_1 \quad v_2 \quad \cdots \quad v_m \; u_1 \quad u_2 \quad \cdots \quad u_m] =$$

$$[v_1 \quad v_2 \quad \cdots \quad v_m \; u_1 \quad u_2 \quad \cdots \quad u_m]\begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{bmatrix}.$$

We know that the above block matrix, which would be D, is nonsingular because its determinant is $(-1)^m$. Solving the above equation for A, we get that $A = UDU^{-1}$ which tells us that A is similar to D. We have now found a diagonal matrix, D, that is similar to A and can attempt to solve the equation DYD=YDY for some unknown Y.

We need to solve the equation DYD=YDY using our known diagonal matrix for D. Plugging in our block matrix for D and our unknown matrix for Y, we get

$$\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}.$$

However, for this project, we only want to look at commuting matrices, so we will only look at cases where DY=YD. We define the individual terms in the block matrix Y to be

$$Y_{11} = \begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_m \end{bmatrix},$$

$$Y_{12} = \begin{bmatrix} y_1 & & & \\ & y_2 & & \\ & & \ddots & \\ & & & y_m \end{bmatrix},$$

$$Y_{21} = \begin{bmatrix} z_1 & & & \\ & \mathrm{z}_2 & & \\ & & \ddots & \\ & & & \mathrm{z}_m \end{bmatrix},$$

$$Y_{22} = \begin{bmatrix} w_1 & & & \\ & w_2 & & \\ & & \ddots & \\ & & & w_m \end{bmatrix}$$

So DY=YD is

$$\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

Solving for Y, we get

$$\begin{bmatrix} Y_{11} & Y_{12} \\ -Y_{21} & -Y_{22} \end{bmatrix} = \begin{bmatrix} Y_{11} & -Y_{12} \\ Y_{21} & -Y_{22} \end{bmatrix}.$$

So, $Y_{11} = Y_{11}$

$Y_{12} = -Y_{12}, 2Y_{12} = 0,$ so $Y_{12} = 0$

$Y_{21} = 0$

and

$-Y_{22} = -Y_{22}.$

Therefore we get

$$Y = \begin{bmatrix} Y_{11} & 0 \\ 0 & Y_{22} \end{bmatrix}$$

and we get

${Y_{11}}^2 = Y_{11}$

0= 0

0 = 0

${Y_{22}}^2 = -Y_{22}.$

Because ${Y_{11}}^2 = Y_{11}$ and ${Y_{22}}^2 = -Y_{22}$, we know that Y_{11} is a projection matrix and Y_{22} is a scewed projection matrix. Now we let Y_{11} and Y_{22} be 2x2 matrices.

If we let

$$Y_{11} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and

$$Y_{22} = \begin{bmatrix} e & f \\ g & h \end{bmatrix},$$

then we can solve for the equations ${Y_{11}}^2 = Y_{11}$ and ${Y_{22}}^2 = -Y_{22}$. Doing this, we find that for Y_{11},

$$a = a^2 + bc$$

$$b = ab + bd$$

$$c = ac + cd$$

and

$$d = d^2 + bc.$$

For Y_{22} we find that

$$-e = e^2 + fg$$

$$f = -ef - fh$$

$$g = -eg - gh$$

$$-h = h^2 + fg.$$

Now, using what we know from quadratics, we can solve these equations to find our variables.

Using quadratics to solve for *a, b, c,* and *d*, we get four cases:

Case 1: $a = \frac{1+\sqrt{1-4bc}}{2}$, $b = \frac{1}{4c}$, $c = \frac{1}{4b}$, $d = \frac{1+\sqrt{1-4bc}}{2}$.

Case 2: $a = \frac{1+\sqrt{1-4bc}}{2}$, $b = arbitrary$, $c = arbitrary$, $d = \frac{1-\sqrt{1-4bc}}{2}$.

Case 3: $a = \frac{1-\sqrt{1-4bc}}{2}$, $b = arbitrary$, $c = arbitrary$, $d = \frac{1+\sqrt{1-4bc}}{2}$.

Case 4: $a = \frac{1-\sqrt{1-4bc}}{2}$, $b = \frac{1}{4c}$, $c = \frac{1}{4b}$, $d = \frac{1-\sqrt{1-4bc}}{2}$.

We can use the same method to solve for *e, f, g,* and *h.* Doing this we again get four cases similar to the ones above.

Case 1: $e = \frac{1+\sqrt{1-4fg}}{2}, f = \frac{1}{4g},\ g = \frac{1}{4f}, \quad h = \frac{1+\sqrt{1-4fg}}{2}.$

Case 2: $e = \frac{1+\sqrt{1-4fg}}{2}, f = arbitrary,\ g = arbitrary, h = \frac{1-\sqrt{1-4fg}}{2}.$

Case 3: $e = \frac{1-\sqrt{1-4fg}}{2}, f = arbitrary,\ g = arbitrary, h = \frac{1+\sqrt{1-4fg}}{2}.$

Case 4: $e = \frac{1-\sqrt{1-4fg}}{2}, f = \frac{1}{4g},\ g = \frac{1}{4f}, h = \frac{1-\sqrt{1-4fg}}{2}.$

Additionally, we looked at cases where *b, c, f*, and *g* were equal to 0 using the same quadratic equations from before.

When *b = 0,* we got four possible matrices as solutions.

1. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, 2. $\begin{bmatrix} 0 & 0 \\ c & 1 \end{bmatrix}$, 3. $\begin{bmatrix} 1 & 0 \\ c & 0 \end{bmatrix}$, 4. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

When *c = 0,* we got

1. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, 2. $\begin{bmatrix} 0 & b \\ 0 & 1 \end{bmatrix}$, 3. $\begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix}$, 4. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

When *f = 0* we got

1. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, 2. $\begin{bmatrix} 0 & 0 \\ g & -1 \end{bmatrix}$, 3. $\begin{bmatrix} -1 & 0 \\ g & 0 \end{bmatrix}$, 4. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Finally, when *g = 0* we got

1. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, 2. $\begin{bmatrix} 0 & f \\ 0 & -1 \end{bmatrix}$, 3. $\begin{bmatrix} -1 & f \\ 0 & 0 \end{bmatrix}$, 4. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

These were all additional solutions to our equations.

Chapter 5: Conclusion

Using methods from linear algebra, I was able to find many more solutions to the Yang-Baxter matrix equation, *AXA=XAX.* The solutions I found were for commuting solutions of a specific diagonal matrix similar to *A.* The solutions were:

Case 1: $a = \frac{1+\sqrt{1-4bc}}{2}$, $b = \frac{1}{4c}$, $c = \frac{1}{4b}$, $d = \frac{1+\sqrt{1-4bc}}{2}$.

Case 2: $a = \frac{1+\sqrt{1-4bc}}{2}$, $b = arbitrary,\ c = arbitrary,\ d = \frac{1-\sqrt{1-4bc}}{2}$.

Case 3: $a = \frac{1-\sqrt{1-4bc}}{2}$, $b = arbitrary,\ c = arbitrary,\ d = \frac{1+\sqrt{1-4bc}}{2}$.

Case 4: $a = \frac{1-\sqrt{1-4bc}}{2}$, $b = \frac{1}{4c}$, $c = \frac{1}{4b}$, $d = \frac{1-\sqrt{1-4bc}}{2}$, when solving for a, b, c, and d.

The solutions when solving for e, f, g, and h, were

Case 1: $e = \frac{1+\sqrt{1-4fg}}{2}$, $f = \frac{1}{4g}$, $g = \frac{1}{4f}$, $h = \frac{1+\sqrt{1-4fg}}{2}$.

Case 2: $e = \frac{1+\sqrt{1-4fg}}{2}$, $f = arbitrary,\ g = arbitrary$, $h = \frac{1-\sqrt{1-4fg}}{2}$.

Case 3: $e = \frac{1-\sqrt{1-4fg}}{2}$, $f = arbitrary,\ g = arbitrary$, $h = \frac{1+\sqrt{1-4fg}}{2}$.

Case 4: $e = \frac{1-\sqrt{1-4fg}}{2}$, $f = \frac{1}{4g}$, $g = \frac{1}{4f}$, $h = \frac{1-\sqrt{1-4fg}}{2}$.

And the last four sets of solutions found were

When *b = 0,*

1. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, 2. $\begin{bmatrix} 0 & 0 \\ c & 1 \end{bmatrix}$, 3. $\begin{bmatrix} 1 & 0 \\ c & 0 \end{bmatrix}$, 4. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

When *c = 0,*

1. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, 2. $\begin{bmatrix} 0 & b \\ 0 & 1 \end{bmatrix}$, 3. $\begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix}$, 4. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

When $f = 0$,

$$1. \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad 2. \begin{bmatrix} 0 & 0 \\ g & -1 \end{bmatrix}, \quad 3. \begin{bmatrix} -1 & 0 \\ g & 0 \end{bmatrix}, \quad 4. \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

When $g = 0$,

$$1. \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad 2. \begin{bmatrix} 0 & f \\ 0 & -1 \end{bmatrix}, \quad 3. \begin{bmatrix} -1 & f \\ 0 & 0 \end{bmatrix}, \quad 4. \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Through this project, I was able to find many more additional solutions to the Yang-Baxter matrix equation, $AXA=XAX$. In my project, I only looked at diagonal commuting matrices, however, so there are many more solutions to be found. While this project did add to the number of known solutions to the Yang-Baxter matrix equation, it is most definitely still an open problem as all solutions have not yet been found since there is no one way to find all solutions. This problem will have to be studied further to find more solutions to this difficult matrix equation.

References

Chen, R. S. (2012). Generalized Yang–Baxter Equations and Braiding Quantum Gates. Journal of Knot Theory and Its Ramifications, 21(09), 1250087. Retrieved March 2, 2016.

Ding, J., & Rhee, N. H. (2012). A Nontrivial Solution to a Stochastic Matrix Equation. East Asian Journal on Applied Mathematics EAJAM, 2(4), 277-284. Retrieved March 2, 2016.

Ding, J., & Rhee, N. H. (2014). Spectral Solutions of the Yang-Baxter Matrix Equation. Retrieved March 2, 2016.

Ding, J., Zhang, C., & Rhee, N. H. (2013). Further Solutions of a Yang-Baxter-like Matrix Equation. East Asian Journal on Applied Mathematics EAJAM, 3(4), 352-362. Retrieved March 2, 2016.

Ding, J., Zhang, C., & Rhee, N. (2015). Commuting solutions of the Yang–Baxter matrix equation. Applied Mathematics Letters, 44, 1-4. Retrieved March 2, 2016.

Nichita, F. (2015). Yang–Baxter Equations, Computational Methods and Applications. Axioms, 4(4), 423-435. Retrieved March 2, 2016.

Zhang, D.Z. (1993). C.N. Yang and Contemporary Mathematics. The Mathematical Intelligencer, 15(4), 13-21. Retrieved March 2, 2016.