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# Induced matter: Curved N-manifolds encapsulated in Riemann-flat N+1 dimensional space

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## Abstract

Liko and Wesson have recently introduced a new 5-dimensional induced matter solution of the Einstein equations, a negative curvature Robertson-Walker space embedded in a Riemann-flat 5-dimensional manifold. We show that this solution is a special case of a more general theorem prescribing the structure of certain N+1 dimensional Riemann-flat spaces which are all solutions of the Einstein equations. These solutions encapsulate N-dimensional curved manifolds. Such spaces are said to “induce matter” in the sub-manifolds by virtue of their geometric structure alone. We prove that the N-manifold can be *any* maximally symmetric space.

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The concept of “induced matter”, was originally introduced by Wesson [1, 2]. While investigating 5D Kaluza-Klein theory, he recognized that a curved 4-space could be embedded in a Ricci-flat ( $R_{AB} = 0$ ;  $A, B, \dots \in \{0, 1, 2, 3, 4\}$ ) 5-space. This is a reflection of the Campbell-Magaard theorem [3] which, applied to 5D, states that it is always possible to embed a curved 4D manifold in a 5D Ricci-flat space. Seahra and Wesson [4] provide an overview and rigorous proof of the Campbell-Magaard theorem with applications to higher dimensions. Wesson takes “induced matter” to mean that the left-hand geometric side extra terms of the flat 5D Ricci-tensor provide the source terms in the 4D curved Ricci-tensor of the embedded space. A “weak” version of this concept utilizing an embedding of the Friedmann-Roberston-Walker(FRW) 4-space in a Minkowski 5-space has been used to visualize the big bang sectionally [5]. Here, the 5-space is Riemann-flat ( $R_{ABCD} = 0$ ) since it is Minkowski. There is no physics in the 4D subspace, except with reference to the original FRW coordinates. This simply provides a Euclidean embedding diagram.

More recently, Liko and Wesson have introduced a new 5D, Riemann-flat solution [6] which they found could “encapsulate” a 4D curved FRW space. We use the term “encapsulate” as distinct from embed since in this 5-space, the coordinates are not Minkowski. The 4D subspace is itself curved in the same 5D coordinates. It is true that a flattening transformation can be found to a 5D Minkowski space. However, this would simply produce another embedding diagram. The physics seems to lie in the encapsulating 5D metric. We shall use the term ”induced matter” to include a Riemann-flat 5D manifold encapsulating a curved 4D subspace. The Liko-Wesson induced matter solution goes on to describe an apparently inflationary universe as a negative curvature FRW space embedded in a special 5D universe. The RW space undergoes accelerated expansion subject to a repulsive “dark energy” ( $P = -\rho$ ). We will show in this paper that the Liko-Wesson solution is a special case of a more general class of maximally symmetric sub-manifolds embedded in Riemann-flat space. A detailed discussion of maximally symmetric sub-manifolds based on Poincarè metrics and their consequences can be found in ref.[7]. For convenience, we repeat some critical definitions and calculations.

Consider the Riemann manifold defined by

$$dS^2 = \tilde{g}_{ij} dx^i dx^j. \quad (1)$$

This space is said to be *maximally symmetric* if and only if it has constant *sectional curvature*  $\kappa = \kappa(i, j)$ , for any  $1 \leq i \neq j \leq N$ . In the plane spanned by the basis vectors  $(\hat{e}_i, \hat{e}_j)$  the sectional curvature is defined by,

$$\kappa(i, j) = \tilde{g}^{ii} R_{ijj}^i \quad (i, j \text{ not summed}). \quad (2)$$

For a maximally symmetric space  $R_{ijj}^j = \kappa \tilde{g}_{ii}$ ,  $j \neq i$ . For such a space,

$$R_{ii} = -\kappa(N-1)\tilde{g}_{ii}. \quad (3)$$

**Theorem:** Let  $\tilde{g}_{ij}$  represent a maximally symmetric space of sectional curvature  $\kappa$ . *The metric*

$$dS^2 = d\tau^2 - D\tau^2 \tilde{g}_{ij} dx^i dx^j, \quad i, j \dots, \in \{1, 2, \dots, N\}, \quad (4)$$

is *Riemann-flat* whenever  $D = -\kappa$ .

**Proof:** Consider the metric

$$dS^2 = d\tau^2 - f(\tau)^2 \tilde{g}_{ij} dx^i dx^j, \quad (5)$$

where  $\tilde{g}_{ij}$  denotes a maximally symmetric space. We compute the independent components of the curvature tensor (the overprime denotes differentiation in  $\tau$ ):

$$\begin{aligned} R^i{}_{0j0} &= -\frac{f''}{f} \delta^i{}_j \\ R^0{}_{i0j} &= -f f'' \tilde{g}_{ij} \\ R^k{}_{ikj} &= \tilde{R}_{ij} - (N-1) f'^2 \tilde{g}_{ij} \\ &= -\kappa(N-1) \tilde{g}_{ij} - (N-1) f'^2 \tilde{g}_{ij} \\ &= -(N-1)(f'^2 + \kappa) \tilde{g}_{ij}, \end{aligned} \quad (6)$$

where we have made use of the result Eq.(3). It is evident from Eq.(6) that the space will be Riemann-flat if and only if  $f'' = 0$  and  $f'^2 + \kappa = 0$ . Let

$f(\tau) = \sqrt{D}\tau$ . Then  $f'' = 0$  and  $f'^2 - D = 0$ . It follows that  $D = -\kappa$  and the proof is complete.

Liko and Wesson [6] introduce the line element (with overall sign of  $dS^2$  reversed from ours),

$$dS^2 = d\tau^2 - \frac{\tau^2}{L^2} \left[ dt^2 - L^2 \sinh^2\left(\frac{t}{L}\right) d\sigma_3^2 \right], \quad (7)$$

where,

$$d\sigma_3^2 = \left(1 + \frac{kr^2}{4}\right)^{-2} (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2), \quad (8)$$

is the Robertson-Walker 3-space with  $k = -1$ .

Define coordinates  $x^A = \{\tau, r, \theta, \phi, t\}$ ,  $A \in \{0, 1, 2, 3, 4\}$ .

We can then identify in Eq.(4),

$$\tilde{g}_{ii} = \{-f, -fr^2, -fr^2 \sin^2 \theta, 1\}, \quad f = L^2 \sinh^2(t/L) \left(1 + \frac{kr^2}{4}\right)^{-2}, \quad (9)$$

and  $D = 1/L^2$ . Eq.(7) is thus of the form (4) and will satisfy the theorem provided that the sectional curvature of the 4-space is  $\kappa = -1/L^2$ . Direct evaluation of the sectional curvature for two typical cases (by symmetry, the remaining cases are identical) results in,

$$\begin{aligned} \kappa(1, 2) &= \tilde{g}^{11} R_{121}^2 = -\frac{1}{L^2} \\ \kappa(4, 2) &= \tilde{g}^{44} R_{424}^2 = -\frac{1}{L^2} \end{aligned}$$

which show that the conditions (6) are met. That is, the 4-space has constant sectional curvature which then results in a Riemann-flat 5-space. We have thus shown that the new metric solution Eq.(7) introduced by Liko and Wesson is a special case of our more general theorem Eq.(4) which allows the N-space to be any maximally symmetric manifold.

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